The Semi Discrete formulation of Galerkin and Galerkin-Conservation Finite Element Methods for two dimensional coupled Burgers ' Problem.

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Abstract

In this paper, the semi discrete formulation of Galerkin and Galerkin-Conservation (G and G-C) finite element methods are used to approximate the solution of the coupled Burgers' problem. The theoretical evidence proved that the property of the bilinear form $A(u,v)$ (v-elliptic and continuity) and the stability of both schemes, also we proved that the error estimate of these methods are of $O(h)$. We used the artificial diffusion method to improve the analytics solution of the problem and the approximate solution when $(\in< h)$. Numerical example is tested to illustrate these schemes and the numerical results by using ODE 15s, ordinary differential equation, solvers matlab are compared with the exact solution.

Mathematics: Subject Classification: 65NXX, 65N30

Key words: Galekin, Galekin-Conservation, error analysis, two dimensional Burgers' equation

1 – Introduction

Burgers' equation is a fundamental partial differential equation from fluid mechanics. It occurs in various areas of applied mathematics, such as modeling of dynamics, heat conduction, and acoustic waves, [Abazari 2010], [Alharbi and fahmy 2010]. Due to its wide range of applicability some researchers have been interested in studying its solution using various numerical techniques. Numerical techniques for the solution of Burgers' equation usually fall into the following classes: finite difference [Bahadir 2003], [liao 2008] [Srivaslava *et al*. 2011]), finite element [Smith 1997], [Pugh, 1995], [Volkwein 2003], [Qing Yang 2013]) and Decompostion method [Zhu, *et al*. 2010], [Zhu 2010]), [Abbasbandy and Darvishi 2005] and references cited therein.

In [Pugh, 1995] used G and G-C finite element methods in solving the homogeneous Burgers' equation in one dimension and noted that the solution obtained using 18 nodes was close to that obtained using 34 nodes, he compared G and G-C finite element methods and found in some examples with taking Re=120 and 240 however, the G solution grew exponentially in time despite the convergent behavior of the G and G-C solutions and determined that the G-C method was more accurate and computed more quickly than the G method for the Burgers' equation with Neumann boundary conditions. [Burns and Balogy 1998] showed that this was true for any initial condition, provided that the steady-state limit exists. [Smith 1997] showed that the G method produced solutions with slightly less error, but the G-C method required less time, and showed that both methods produced virtually identical results except for one case where the G solution grew exponentially in time despite the convergent behavior of the G- C solution, his results were similar to what was obtained by [Pugh 1995] and determined that the G– C method to give better results for the one dimensional Burgers' equation with Robin's boundary conditions and saw that sufficient accuracy is achieved at 18 nodes.

In this paper, we present the semi discrete G & G-C finite element methods for the two dimensional Burgers' equations to get system of ordinary differential equation. The theoretical evidence proved the property of the bilinear form $A(u,v)$ (v-elliptic and continuity) and the stability of both schemes are satisfied, also we proved that the error estimate of these methods are of order $O(h)$. We used the artificial diffusion method [Johnson(1987)] to improve the analytics solution of the problem and the approximate solution when ($\in < h$). Numerical example is tested to illustrate these schemes and the numerical results by using ODE 15s mat lab solver are compared with the exact solution for both cases ($\in > h \; and \; \in < h$).

2-Definitions and important lemma:

It is beneficial to mention the definitions of the vector space that we used during this study. The vector space $L^2(\Omega)$ is the space of square-integrable functions on $\Omega \subset R^n$, [Johnson, 2010]

$$L^2(\Omega) = \{v: \Omega \rightarrow R \; s.t. \; \int v^2 \, d\Omega \leq \infty\},$$

indeed $L^2(\Omega)$ is Hilbert space with respect to the following inner product

$(u,v) = \int_\Omega u(x) v(x) \, dx$ and norm $\|v\|_{L^2(\Omega)} = \left(\int_\Omega v^2 d\,\Omega\right)^{\frac{1}{2}}$. For $p = \infty$, $L^\infty(\Omega)$ denotes the space of all functions which are bounded for almost all $x \in \Omega$:

$$L^\infty(\Omega) = \left\{ u : |u(x)| < \infty \;\; for \;\; almost \;\; all \;\; x \in \Omega \right\},$$

this space is equipped with the norm

$$\|v\|_{L^\infty(\Omega)} = \{ess\ \sup\{|v(x)|: x \in R\}.$$

We introduce the Sobolev space

$$H^1(\Omega) = \left\{v \in\ L^2(\Omega): \frac{\partial v}{\partial x_i} \in L^2(\Omega), i = 1,2 \dots \dots . d\right\},$$

and the corresponding norm,

$\|v\|_{H^1(\Omega)} = \left(\int_\Omega (\mathrm{v}^2 + (\nabla \mathrm{v})^2) \mathrm{d}\Omega\right)^{\frac{1}{2}}$, also $H_0^1(\Omega) = \{v \in H^1(\Omega) : v = 0\ on\ \partial\Omega\}$,

with the same scalar product and norm as $H^1(\Omega)$.

We introduce the norm for both continuous time $t \in [0, T]$ and space Ω by:

$\|v\|_{L^\infty(H^r(\Omega)) = \max_{0 \le t \le T} \|v\|_r}$ and $\|v_t\|_{L^2(L^2(\Omega))} = \left(\int_0^t \|v_t\|^2\right)^{\frac{1}{2}}$.

Lemma(2.1)[kashkool 2002]: Let U be the approximate solution and u be the exact solution. if $\| U^n - u^n \|_{0,2,\Omega} \le Ch$, $(n = 1,2, \dots, N_\tau)$, then

$$\| \nabla U^n \|_{0,2,\Omega} \le C,$$

where C is constant independent on h and τ **.**

3- Time- dependent modeling problems.

We consider time- dependent nonlinear two dimensional coupled Burgers' problem.

$u_t - \epsilon\Delta u + u\, u_x + v\, u_y \ = f$, on $\Omega \times (0,T]$

(3.1.a)

$v_t - \epsilon\Delta v \ + uv_x + vv_y = g,$ on $\Omega \times (0,T]$

(3.1.b)

with boundary conditions

$$u\,(x\,,y,t)\ = 0 \quad on \quad \partial\Omega \times (0,T], v\,(x\,,y,t)\ = 0 \quad on \quad \partial\Omega \times (0,T],$$

and initial conditions $u\,(x\,,y,0)\ = u^0(x,y)$ and $v\,(x\,,y,0) = v^0(x,y)$,

where $\epsilon > 0$ is a viscosity constant, $\Omega \subset \mathrm{R}^2$ with boundary $\partial\Omega$, the exact solutions $u = u\,(x\,,y,t)$, $v\,(x\,,y,t)$, and the source terms $f, g \in L^2(\Omega)$.

The weak formulation analogue of equation (3.1). Letting $V = H_0^1(\Omega)$, multiplying the equations (3.1a) and (3.1b) for a given t by $\varphi \in V$, integrating over Ω and Green's formula, we get:

$(u_t,\varphi) \ + \ a(u\,,\varphi) \ + \ (u\,u_x,\varphi) + (vu_y,\varphi) = \ (f,\varphi), \quad \forall\, \varphi \in H_0^1(\Omega)$

(3.2a)

$(v_t,\varphi) \ + \ a(v\,,\varphi) \ + \ (u\,v_x,\varphi) + (v\,v_y,\varphi) = \ (g,\varphi), \ \forall\, \varphi \in H_0^1(\Omega),$

(3.2b)

$(u(x,y\,,0),\varphi) = \ (u^0,\varphi), \qquad (v(x,y\,,0),\varphi) = \ (v^0,\varphi),$

where $a(u\,,\varphi) \ = \ (\epsilon\,\nabla u, \nabla\varphi)$ and $a(v\,,\varphi) \ = \ (\epsilon\,\nabla v\,, \nabla\varphi)$.

The conservation form of the equations (3.1a) and (3.1b) were given by [Fletcher 1984]. Here the $u\,u_x$ and $v\,v_y$ terms are replaced by $\frac{1}{2}(u^2)_x$, $\frac{1}{2}(v^2)_y$ respectively, we get

$u_t \ - \epsilon\,\Delta u + \frac{1}{2}(u^2)_x + vu_y \ = f$,

(3.3a)

$v_t - \epsilon\,\Delta v + u v_x + \frac{1}{2}(v^2)_y = g.$

(3.3b)

The weak formulation of (3.3) is : find $u, v \in V = H_0^1(\Omega)$ such that:

$(u_t, \varphi) + a(u\,,\varphi) + (\frac{1}{2}(u^2)_x, \varphi) + (v\,u_y, \varphi) = (f\,,\varphi),\ \forall\,\varphi \in H_0^1(\Omega)$ (3.4a)

$(v_t, \varphi) + a(v, \varphi) + (u\,v_x\,, \varphi) + (\frac{1}{2}(v^2)_y, \varphi) = (g\,,\varphi),\ \forall\,\varphi \in H_0^1(\Omega)$

(3.4b)

4 - The semi-discrete approximation.

Let V_h be a finite-dimensional subspace of V with basis functions $\{\varphi_1, \varphi_2, \dots \dots \dots . \varphi_N\}$. For definiteness, we shall assume that Ω is polygonal convex domain and that V_h consists of piecewise linear functions on quasi-uniform triangulation of Ω with mesh size h and satisfying the minimum angle [Ciarlet 1978]. Replacing V by the finite – dimensional subspace V_h[Johnson 1987], we get the following semi-discrete analogue of the equations (3.2) and (3.4) respectively: Find an approximate solution u_h, $v_h \in V_h$ such that :

$(u_{h,t}\,,\varphi_h) + a(u_h,\varphi_h) + (u_h\,u_{h,x},\varphi_h) + (v_h u_{h,y},\varphi_h) = (f, \varphi_h), \forall\varphi_h \in V_h,$

(4.1a)

$(v_{h,t},\varphi_h) + a(v_h,\varphi_h) + (u_h\,v_{h,x},\varphi_h) + (v_h v_{h,y},\varphi_h) = (g, \varphi_h), \forall\varphi_h \in V_h,$

(4.1b)

$(u_{h,t}\,,\varphi_h) + a(u_h,\varphi_h) + (\frac{1}{2}(u_h)_x^2, \varphi_h) + (v_h\,u_{h,y}, \varphi) = (f, \varphi_h), \forall\varphi_h \in V_h$

(4.2a)

$\left(v_{h,t}, \varphi_h\right) + a(v_h, \varphi_h) + \left(u_h\, v_{h,x}, \varphi\right) + (\frac{1}{2}\left(v_h\right)_y^2, \varphi_h\right) = (g, \varphi_h), \quad \forall \varphi_h \in V_h,$ (4.2b)

5- Assumptions:

1-We can write Equation (4.1) in the form,

$(u_{h,t},\varphi_h) + (u_h,\varphi_h) = (f,\varphi_h),$ *and* $(v_{h,t},\varphi_h) + A(v_h,\varphi_h) = (g,\varphi_h), \forall \varphi_h \in V_h$ (5.1)

where,

$$A(u_h,\varphi_h) = \epsilon(\nabla u_h, \nabla\varphi_h) + (u_h\, u_{h,x},\varphi_h) + (v_h u_{h,y},\varphi_h)$$

$$A(v_h,\varphi_h) = \epsilon(\nabla v_h, \nabla\varphi_h) + (u_h\, v_{h,x},\varphi_h) + (v_h v_{h,y},\varphi_h)$$

and Equation (4.2) in the form,

$(u_{h,t},\varphi_h) + A(u_h,\varphi_h) = (f,\varphi_h)$ and $(v_{h,t},\varphi_h) + A(v_h,\varphi_h) = (g,\varphi_h), \forall \varphi_h \in V_h,$ (5.2)

where,

$$A(u_h,\varphi_h) = \epsilon(\nabla u_h, \nabla\varphi_h) + (\tfrac{1}{2}(u_h)_x^2,\varphi_h) + (v_h u_{h,y},\varphi_h)$$

$$A(v_h,\varphi_h) = \epsilon(\nabla v_h, \nabla\varphi_h) + (u_h\, v_{h,x},\varphi_h) + (\tfrac{1}{2}(u_h)_y^2,\varphi_h).$$

2- In the following, we assume that Equations (4.1) and (4.2) satisfies,

A1- There exists a constant α such that : $\epsilon \geq \alpha > 0$.

A2- There exists a constant β such that : $\beta_1 \leq \frac{1}{2}$.

6- Properties of the bilinear form $A(u, v)$.

Let V be Hilbert space with scalar product $(\cdot,\cdot)_V$ and corresponding norm $\|u\|_{H_0^1(\Omega)}$. suppose that $A(u, v)$ is bilinear form on $V \times V$. We prove some lemmas for the continuous and v-elliptic.

Lemma 6.1

$A(u_h,\varphi_h)$ and $A(v_h,\varphi_h)$ given by (5.1) are continuous and ν-elliptic.

Proof:

$1 - For\ continuity, we\ have$

$$|A(u_h,\varphi_h)| \le |\epsilon(\nabla u_h, \nabla\varphi_h)| + |(u_h\, u_{h,x},\varphi_h)| + |(v_h u_{h,y},\varphi_h)|$$

$$|A(v_h,\varphi_h)| \le |\epsilon(\nabla v_h, \nabla\varphi_h)| + |(u_h\, v_{h,x},\varphi_h)| + |(v_h v_{h,y},\varphi_h)|$$

$$|A(u_h,\varphi_h)| \le |\epsilon(\nabla u_h, \nabla\varphi_h)| + |(u_h\, u_{h,x},\varphi_h)| + |(u_h\, u_{h,y},\varphi_h)| + |(v_h\, u_{h,x},\varphi_h)| + |(v_h\, u_{h,y},\varphi_h)|$$

$$|A(v_h,\varphi_h)| \le |\epsilon(\nabla v_h, \nabla\varphi_h)| + |(u_h\, v_{h,x},\varphi_h)| + |(u_h\, v_{h,y},\varphi_h)| + |(v_h v_{h,x},\varphi_h)| + |(v_h\, v_{h,y},\varphi_h)|$$

$$|A(u_h,\varphi_h)| \le |\epsilon\,(\nabla u_h, \nabla\varphi_h)| + |(u_h\nabla u_h,\varphi_h)| + |(v_h\nabla u_h,\varphi_h)|$$

$$|A(v_h,\varphi_h)| \le |\epsilon\,(\nabla v_h, \nabla\varphi_h)| + |(u_h\nabla v_h,\varphi_h)| + |(v_h\nabla v_h, \varphi_h)|$$

Applying Cauchy-Schwartz inequality gives,

$$|A(u_h,\varphi_h)| \le |\epsilon|_{L^\infty} \|\nabla u_h\| \|\nabla\varphi_h\| + \|u_h\| \|\nabla u_h\| \|\varphi_h\| + \|v_h\| \|\nabla u_h\| \|\varphi_h\|$$

$$|A(v_h,\varphi_h)| \le |\epsilon|_{L^\infty} \|\nabla v_h\| \|\nabla\varphi_h\| + \|u_h\| \|\nabla v_h\| \|\varphi_h\| + \|v_h\| \|\nabla v_h\| \|\varphi_h\|$$

From Poincare's inequalities, we have

$$|A(u_h,\varphi_h)| \le |\epsilon|_{L^\infty} \|\nabla u_h\| \|\nabla\varphi_h\| + \|u_h\| \|\nabla u_h\| \|\varphi_h\| + C\|\nabla v_h\| \|\nabla u_h\| \|\varphi_h\|$$

$$|A(v_h,\varphi_h)| \le |\epsilon|_{L^\infty} \|\nabla v_h\| \|\nabla\varphi_h\| + C\|\nabla u_h\| \|\nabla v_h\| \|\varphi_h\| + \|v_h\| \|\nabla v_h\| \|\varphi_h\|$$

From lemma(2.1) , gives

$A(u_h,\varphi_h)| \leq |\epsilon|_{L^\infty} \|\nabla u_h\| \|\nabla\varphi_h\| + C_m \|u_h\| \|\varphi_h\| + C_m \|\nabla u_h\| \|\varphi_h\|$

$|A(v_h,\varphi_h)| \leq |\epsilon|_{L^\infty} \|\nabla v_h\| \|\nabla\varphi_h\| + C_m \|\nabla v_h\| \|\varphi_h\| + C_m \|v_h\| \|\varphi_h\|$

$|A(u_h,\varphi_h)| \leq N\{\|\nabla u_h\| \|\nabla\varphi_h\| + \|u_h\| \|\varphi_h\| + \|\nabla u_h\| \|\varphi_h\|\}$

$|A(v_h,\varphi_h)| \leq N\{\|\nabla v_h\| \|\nabla\varphi_h\| + \|\nabla v_h\| \|\varphi_h\| + \|v_h\| \|\varphi_h\|\}$

$A(u_h,\varphi_h)| \leq N\{(\|\nabla u_h\| , \|u_h\|, \|\nabla u_h\|).(\|\nabla\varphi_h\| , \|\varphi_h\| , \|\varphi_h\|)\}$

$A(u_h,\varphi_h)| \leq N\{(\|\nabla u_h\| , \|u_h\|, \|\nabla u_h\|).(\|\nabla\varphi_h\| , \|\varphi_h\| , \|\varphi_h\|)\}$

$A(u_h,\varphi_h)| \leq$

$N\sqrt{\|\nabla u_h\|^2 + \|u_h\|^2 + \|\nabla u_h\|^2}\sqrt{\|\nabla\varphi_h\|^2 + \|\varphi_h\|^2 + \|\varphi_h\|^2}$

$A(v_h,\varphi_h)| \leq N\sqrt{\|\nabla v_h\|^2 + \|\nabla v_h\|^2 + \|v_h\|^2}\sqrt{\|\nabla\varphi_h\|^2 + \|\varphi_h\|^2 + \|\varphi_h\|^2}$

$A(u_h,\varphi_h)| \leq N\sqrt{2\|\nabla u_h\|^2 + 2\|u_h\|^2}\sqrt{2\|\nabla\varphi_h\|^2 + 2\|\varphi_h\|^2}$

$A(v_h,\varphi_h)| \leq N\sqrt{2\|\nabla v_h\|^2 + 2\|v_h\|^2}\sqrt{2\|\nabla\varphi_h\|^2 + 2\|\varphi_h\|^2}$

$A(u_h,\varphi_h)| \leq N\|u_h\|_{H_0^1}\|\varphi_h\|_{H_0^1}$ and $A(v_h,\varphi_h)| \leq N_1\|v_h\|_{H_0^1}\|\varphi_h\|_{H_0^1}$

where $N = max\{|\epsilon|_{L^\infty} , C_m\}$

■

2- For v- elliptic, we have

$A(u_h,u_h) = \epsilon(\nabla u_h , \nabla u_h) + (u_h\, u_{h,x}, u_h) + (v_h u_{h,y}, u_h)$

$A(v_h,v_h) = \epsilon(\nabla v_h , \nabla v_h) + (u_h\, v_{h,x}, v_h) + (v_h v_{h,y}, v_h)$

$A(u_h,u_h) \geq \alpha\, (\nabla u_h , \nabla u_h) + \beta_1\, (u_h\, u_{h,x}, u_h) + \beta_1 (v_h u_{h,y}, u_h)$

$A(v_h,v_h) \geq \alpha\, (\nabla v_h , \nabla v_h) + \beta_1\, (u_u\, v_{h,x}, \mathrm{v_h}) + \beta_1 (v_h v_{h,y}, v_h)$

$A(u_h,u_h) \geq \alpha\, \|\nabla u_h\|^2 + \beta_1 \|u_h\|^2\, \|u_{h,x}\| + \beta_1 \|v_h\|\, \|u_{h,y}\|\, \|u_h\|$

$A(v_h,v_h) \geq \alpha\, \|\nabla v_h\|^2 + \beta_1 \|u_h\|\, \|v_{h,x}\|\, \|v_h\| + \beta_1 \|v_h\|^2\, \|v_{h,y}\|$

Since $\|v_h\| \geq 0$, $\|u_h\| \geq 0$, we get

$A(u_h,u_h) \geq \alpha\, \|\nabla u_h\|^2 + \beta_1 \|u_h\|^2\, \|u_{h,x}\| + \beta_1 \|u_{h,y}\|\, \|u_h\|$

$A(v_h,v_h) \geq \alpha\, \|\nabla v_h\|^2 + \beta_1 \|v_{h,x}\|\, \|v_h\| + \beta_1 \|v_h\|^2\, \|v_{h,y}\|$

From Poincare's inequalities, we get

$A(u_h,u_h) \geq \alpha\, \|\nabla u_h\|^2 + \beta_1 \|\mathrm{u_h}\|^2 \|\mathrm{u_{h,x}}\| + \frac{\beta_1}{C} \|\mathrm{u_h}\|^2$

$A(v_h,v_h) \geq \alpha\, \|\nabla v_h\|^2 + \frac{\beta_1}{C} \|v_h\|^2 + \beta_1 \|v_h\|^2 \|v_{h,y}\|$

By using lemma (2.1), we get,

$A(u_h,u_h) \geq \alpha\, \|\nabla u_h\|^2 + \beta_1 C_m \|u_h\|^2 + \frac{\beta_1}{C} \|u_h\|^2$

$A(v_h,v_h) \geq \alpha\, \|\nabla v_h\|^2 + \frac{\beta_1}{C} \|v_h\|^2 + \beta_1 C_m \|v_h\|^2$

$A(u_h,u_h) \geq M \left\{ \|\nabla u_h\|^2 + \|u_h\|^2 + \|u_h\|^2 \right\}$

$$A(v_h,v_h) \geq M\left\{\|\nabla v_h\|^2 + \|v_h\|^2 + \|v_h\|^2\right\}$$

$$A(u_h,u_h) \geq M\left\{\|\nabla u_h\|^2 + 2\|u_h\|^2\right\} = M_1\|u_h\|^2_{H_0^1}$$

$$A(v_h,v_h) \geq M\left\{\|\nabla v_h\|^2 + 2\|v_h\|^2\right\} = M_1\|v_h\|^2_{H_0^1}$$

where $M = \min\left\{\alpha, \beta_1 C_m, \frac{\beta_1}{C}\right\}$ and $M_1 = \min\{M, 2M\}$.

■

Lemma (6.2).

$A(u_h,\varphi_h)$ and $A(v_h,\varphi_h)$ given by (4.2) are continuous and v-elliptic.

Proof: Similarly as the proof of lemma (6.1)

7-Stability.

Lemma (7.1): Let u_h, v_h are the solutions of equations (4.1), there exist a constant C>0 such that,

$$\|u_h(T)\|^2 \leq e^{-2CT}\|u_h^0\|^2 + \frac{1}{2C}\|f\|^2_{(C_1:(0,T))}$$

$$\|v_h(T)\|^2 \leq e^{-2CT}\|v_h^0\|^2 + \frac{1}{2C}\|g\|^2_{(C_1:(0,T))}$$

Proof: Choosing $\varphi_h = u_h$ in (4.1a) and $\varphi_h = v_h$ in (4.1b) gives,

$$(u_{h,t}, u_h) + a(u_h, u_h) + (u_h u_{h,x}, \varphi_h) + (v_h u_{h,y}, \varphi_h) = (f, u_h)$$

$$(v_{h,t}, v_h) + a(v_h, \varphi_h) + (u_h v_{h,x}, \varphi_h) + (v_h v_{h,y}, \varphi_h) = (g, v_h)$$

$(u_{h,t},u_h) = \frac{1}{2}\int_\Omega \frac{d}{dt} u_h^2\, dx, = \frac{1}{2}\frac{d}{dt}\|u_h\|^2$ and $(v_{h,t},v_h) = \frac{1}{2}\int_\Omega \frac{d}{dt} v_h^2\, dxdy = \frac{1}{2}\frac{d}{dt}\|v_h\|^2$

from [Abazari and Borhanifar, 2010], $a(u_h,u_h) \geq \alpha \|u_h\|^2$ and $a(v_h,v_h) \geq \alpha \|v_h\|^2$,

since $|, B(u,v,w)| \leq \beta \|u\| \|v\| \|w\|$ and by using Young's inequality, we get,

$(f, u_h) \leq \|f\| \|u_h\| \leq \frac{1}{4C}\|f\|^2 + C\|u_h\|^2$ and $(g,u_h) \leq \|g\| \|v_h\| \leq \frac{1}{4C}\|g\|^2 + C\|v_h\|^2$

$(uu_x,u_h) \leq \beta\|u_h\|^2 \|u_h\| \leq \frac{\beta}{4c_1}\|u_h\|^4 + \beta c_1\|u_h\|^2$

$(uu_y,u_h) \leq \beta\|u_h\|^2 \|v_h\| \leq \frac{\beta}{4c_1}\|u_h\|^4 + \beta c_1\|v_h\|^2$

$(vv_x,v_h) \leq \beta\|v_h\|^2 \|u_h\| \leq \frac{\beta}{4c_1}\|v_h\|^4 + \beta c_1\|u_h\|^2$

$(vv_y,v_h) \leq \beta\|v_h\|^2 \|v_h\| \leq \frac{\beta}{4c_1}\|v_h\|^4 + \beta c_1\|v_h\|^2$

Then,

$$\frac{1}{2}\frac{d}{dt}\|u_h\|^2 + a\|u_h\|^2 + \frac{\beta}{2c_1}\|u_h\|^4 + \beta c_1(\|u_h\|^2 + \|v_h\|^2) \leq \frac{1}{4C}\|f\|^2 + C\|u_h\|^2$$

$$\frac{1}{2}\frac{d}{dt}\|v_h\|^2 + a\|v_h\|^2 + \frac{\beta}{2c_1}\|v_h\|^4 + \beta c_1(\|u_h\|^2 + \|v_h\|^2) \leq \frac{1}{4C}\|g\|^2 + C\|u_h\|^2$$

Putting $\alpha = \beta c_1 = C$ $a$we get,

$$\frac{d}{dt}\|u_h\|^2 + 2C\|u_h\|^2 + 2C\|v_h\|^2 + \frac{\beta^2}{C}\|u_h\|^4 \leq \frac{1}{2C}\|f\|^2$$

$$\frac{d}{dt}\|v_h\|^2 + 2C\|u_h\|^2 + 2C\|v_h\|^2 + \frac{\beta^2}{C}\|v_h\|^4 \leq \frac{1}{2C}\|g\|^2$$

Since, $2C\|u_h\|^2$ *and* $2C\|v_h\|^2$ are non-negative , we have

$$\frac{d}{dt}\|u_h\|^2 + 2C\|u_h\|^2 + \frac{\beta^2}{C}\|u_h\|^4 \leq \frac{1}{2C}\|f\|^2$$

$$\frac{d}{dt}\|v_h\|^2+2C\|v_h\|^2+\frac{\beta^2}{C}\|v_h\|^4\leq\frac{1}{2C}\|g\|^2$$

Multiplying by the integrating factor $e^{2Ct}$and integrating from $t = 0\ to\ t = T$ gives,

$$e^{2CT}\|u_h(T)\|^2+\frac{\beta^2}{2C}\int_0^T e^{2Ct}\|u_h\|^4dt\leq\frac{1}{2C}\int_0^T e^{2Ct}\|f\|^2dt+\|u_h^0\|^2$$

$$e^{2CT}\|v_h(T)\|^2+\frac{\beta^2}{2C}\int_0^T e^{2Ct}\|v_h\|^4dt\leq\frac{1}{2C}\int_0^T e^{2Ct}\|g\|^2dt+\|v_h^0\|^2$$

Since ,the second terms are non-negative , we get

$$\|u_h(T)\|^2\leq\frac{1}{2C}\int_0^T e^{2C(t-T)}\|f\|^2dt+e^{-2CT}\|u_h^0\|^2$$

$$\|v_h(T)\|^2\leq\frac{1}{2C}\int_0^T e^{2C(t-T)}\|g\|^2dt+e^{-2CT}\|v_h^0\|^2$$

$$\|u_h(T)\|^2\leq e^{-2CT}\|u_h^0\|^2+\frac{1}{2C}\|f\|^2_{(C_1:(0,T))}$$

$$\|v_h(T)\|^2\leq e^{-2CT}\|v_h^0\|^2+\frac{1}{2C}\|g\|^2_{(C_1:(0,T))}$$ ■

Lemma(7.2): Let u_h, v_h are the solutions of equations (4.2), there exist a constant C >0 such that,

$$\|u_h(T)\|^2\leq e^{-2CT}\|u_h^0\|^2+\frac{1}{2C}\|f\|^2_{(C_1:(0,T))}.$$

$$\|v_h(T)\|^2\leq e^{-2CT}\|v_h^0\|^2+\frac{1}{2C}\|g\|^2_{(C_1:(0,T))}.$$

Proof : Similarly as the proof of lemma (7.1)

8- The error estimate

Theorem (8.1): Let u, v,u_h and $v_h$be the solutions of (3.2) and (4.1) respectively , then there exists constants C_1 ,C_2 independent of h such as,

$$\|u-u_h\|_{L^\infty(L^2)}\leq\|u_h^0-u^0\|+C_1h\left\{\|u^0\|+\|u\|_{L^\infty(H_0^1)}+\|u_t\|_{L^2(H_1^0)}\right\},$$

$\|v - v_h\|_{L^\infty(L^2)} \leq \|v_h^0 - v^0\| + C_2 h\{\|v^0\| + \|v\|_{L^\infty(H_0^1)} + \|v_t\|_{L^2(H_1^0)}\}$.

Proof . We write the errors in terms of elliptic projection pu and pv which satisfy

$a(pu, \varphi_h) = a(u, \varphi_h)$ and $a(pv, \varphi_h) = a(v, \varphi_h)$.

(8.1)

$u - u_h = (u - pu) - (u_h - pu) = \rho_1 - \theta_1$

$v - v_h = (v - pv) - (v_h - pv) = \rho_2 - \theta_2$

then ,

$\|u - u_h\|_{L^\infty(L^2)} \leq \|\rho_1\|_{L^\infty(L^2)} + \|\theta_1\|_{L^\infty(L^2)}$

$\|v - v_h\|_{L^\infty(L^2)} \leq \|\rho_2\|_{L^\infty(L^2)} + \|\theta_2\|_{L^\infty(L^2)}$

From [Johnson1981] , we have,

$\|\rho_1\|_{L^\infty(L^2)} \leq \mathrm{Ch}\|u\|_{L^\infty(H_0^1)}$ and $\|\rho_2\|_{L^\infty(L^2)} \leq \mathrm{Ch}\|v\|_{L^\infty(H_0^1)}$

(8.2)

To estimate θ_1^n and θ_2^n , note that,

$(\rho_{1,t} - \theta_{1,t}, \varphi_h) + a(\rho_1 - \theta_1, \varphi_h) + (uu_x - u_h u_{h,x}, \varphi_h) + (vu_y - v_h u_{h,y}, \varphi_h) = 0$

$(\rho_{2,t} - \theta_{2,t}, \varphi_h) + a(\rho_2 - \theta_2, \varphi_h) + (uv_x - u_h v_{h,x}, \varphi_h) + (v v_y - v_h v_{h,y}, \varphi_h) = 0$

From property of elliptic projection(8.1), we have

$(\theta_{1,t}, \varphi_h) + a(\theta_1, \varphi_h) - (uu_x - u_h u_{h,x}, \varphi_h) - (vu_y - v_h u_{h,y}, \varphi_h) = (\rho_{1,t}, \varphi_h)$

(8.3a)

$(\theta_{2,t}, \varphi_h) + a(\theta_2, \varphi_h) - (uv_x - u_h v_{h,x}, \varphi_h) - (vv_y - v_h v_{h,y}, \varphi_h) = (\rho_{2,t}, \varphi_h)$

(8.3b)

choosing $\varphi_h = \theta_1$, and $\varphi_h = \theta_2$ in (8.3a) and (8.3b) respectively gives ,

$(\theta_{1,t}, \theta_1) + a(\theta_1, \theta_1) - (uu_x - u_h u_{h,x}, \theta_1) - (vu_y - v_h u_{h,y}, \theta_1) = (\rho_{1,t}, \theta_1)$,

$(\theta_{2,t}, \theta_2) + a(\theta_2, \theta_2) - (uv_x - u_h v_{h,x}, \theta_1) - (vv_y - v_h v_{h,y}, \theta_1) = (\rho_{2,t}, \theta_2)$.

$(\theta_{1,t}, \theta_1) = \int_\Omega \theta_{1,t}\theta_1 dxd = \frac{1}{2}\frac{d}{dt}\int_\Omega \theta_1^2 dxdy = \frac{1}{2}\frac{d}{dt}\|\theta_1\|^2$,

$(\theta_{2,t}, \theta_2) = \int_\Omega \theta_{2,t}\theta_2 dxdy = \frac{1}{2}\frac{d}{dt}\int_\Omega \theta_2^2 dxdy = \frac{1}{2}\frac{d}{dt}\|\theta_2\|^2$,

and by using Cauchy-Schwartz inequality, applying Young's inequality and from[Boules 1990] ,

$a(u, u) \geq \alpha\|u\|^2$, we have

$$\frac{1}{2}\frac{d}{dt}\|\theta_1\|^2 + \alpha_1\|\theta_1\|^2 + \frac{1}{2a_1}\|u\,u_x - u_h u_{h,x}\|^2 + \frac{a_1}{2}\|\theta_1\|^2 + \frac{1}{2a_2}\|v\,u_y - v_h u_{h,y}\|^2 + \frac{a_2}{2}\|\theta_1\|^2$$
$$\leq \frac{1}{8c_1}\|\rho_{1,t}\|^2 + 2c_1\|\theta_1\|^2$$

$$\frac{1}{2}\frac{d}{dt}\|\theta_2\|^2 + \alpha_2\|\theta_2\|^2 + \frac{1}{2a_3}\|u\,v_x - u_h v_{h,x}\|^2 + \frac{a_3}{2}\|\theta_2\|^2 + \frac{1}{2a_4}\|v\,v_y - v_h v_{h,y}\|^2 + \frac{a_4}{2}\|\theta_2\|^2$$
$$\leq \frac{1}{8c_2}\|\rho_{2,t}\|^2 + 2c_2\|\theta_2\|^2$$

since $\|u\,u_x - u_h u_{h,x}\|^2$, $\|v\,u_y - v_h u_{h,y}\|^2$, $\|u\,v_x - u_h v_{h,x}\|^2$ and $\|v\,v_y - v_h v_{h,y}\|^2$ are nonnegative terms and by Putting $\alpha_1 = a_1 = a_2 = c_1 = c_2$ and $\alpha_2 = a_3 = a_4 = c_3 = c_4$, we get ,

$$\frac{d}{dt}\|\theta_1\|^2 \leq \frac{1}{4c_1}\|\rho_{1,t}\|^2 \text{ and } \frac{d}{dt}\|\theta_2\|^2 \leq \frac{1}{4c_2}\|\rho_{2,t}\|^2,$$

(8.4)

note that, there exist $0 \leq t^* \leq T$ such that ,

$\|\theta_1(t^*)\| = max_{0\leq t\leq T}\|\theta_1\| = \|\theta_1\|_{L^\infty(L^2)}$,

$\|\theta_2(t^*)\| = max_{0\leq t\leq T}\|\theta_2\| = \|\theta_2\|_{L^\infty(L^2)}$, Integrating equation (8.4) from $t = 0$ to $t = t^*$ gives ,

$$\|\theta_1(t^*)\|^2 \leq \|\theta_1(0)\|^2 + \frac{1}{4c_1}\int_0^{t^*}\|\rho_{1,t}\|^2\,dt \leq \|\theta_1(0)\|^2 + \frac{1}{4c_1}\int_0^{T}\|\rho_{1,t}\|^2\,dt$$

$$\|\theta_2(t^*)\|^2 \leq \|\theta_2(0)\|^2 + \frac{1}{4c_2}\int_0^{t^*}\|\rho_{2,t}\|^2\,dt \leq \|\theta_2(0)\|^2 + \frac{1}{4c_2}\int_0^{T}\|\rho_{2,t}\|^2\,dt$$

then,

$\|\theta_1\|^2_{L^\infty(L^2)} \leq \|\theta_1(0)\|^2 + \frac{1}{4c_1}\int_0^{T}\|\rho_{1,t}\|^2\,dt$ and $\|\theta_2\|^2_{L^\infty(L^2)} \leq$
$\|\theta_2(0)\|^2 + \frac{1}{4c_2}\int_0^{T}\|\rho_{2,t}\|^2\,dt$

this implies that,

$$\|\theta_1\|_{L^\infty(L^2)} \leq \|\theta_1(0)\| + \left(\frac{1}{4c_1}\int_0^{T}\|\rho_{1,t}\|^2\,dt\right)^{\frac{1}{2}}$$

$$\|\theta_2\|_{L^\infty(L^2)} \leq \|\theta_2(0)\| + \left(\frac{1}{4c_2}\int_0^{T}\|\rho_{2,t}\|^2\,dt\right)^{\frac{1}{2}}$$

The first terms on right hand sides give,

$\|\theta_1(0)\| \leq \|u_h^0 - pu^0\| \leq \|u_h^0 - u^0\| + \|u^0 - pu^0\| \leq \|u_h^0 - u^0\| + Ch\|u^0\|$
(8.5a)

$\|\theta_2(0)\| \leq \|v_h^0 - pv^0\| \leq \|v_h^0 - v^0\| + \|v^0 - pv^0\| \leq \|v_h^0 - v^0\| + Ch\ \|v^0$
(8.5b)

for the second terms, we have

$$\left(\frac{1}{4c_1}\int_0^{T}\|\rho_{1,t}\|^2\,dt\right)^{\frac{1}{2}} \leq \left(\frac{1}{4c_1}\int_0^{T}\|u_t - pu_t\|^2\,dt\right)^{\frac{1}{2}} \leq \left(C_1h^2\int_0^{T}\|u_t\|^2\,dt\right)^{\frac{1}{2}}$$

$$\leq C_1 h\left(\int_0^T \|u_t\|^2\, dt\right)^{\frac{1}{2}} \leq C_1 h \|u_t\|_{L^2(H_1^0)}$$

(8.6a)

$$\left(\frac{1}{4c_1}\int_0^T \|\rho_{2,t}\|^2\, dt\right)^{\frac{1}{2}} \leq \left(\frac{1}{4c_2}\int_0^T \|v_t - pv_t\|^2\, dt\right)^{\frac{1}{2}} \leq \left(C_2 h^2 \int_0^T \|v_t\|^2\, dt\right)^{\frac{1}{2}}$$

$$\leq C_2 h\left(\int_0^T \|v_t\|^2\, dt\right)^{\frac{1}{2}} \leq C_2 h\ \|v_t\|_{L^2(H_1^0)}$$

(8.6b)

then ,

$$\|\theta_1\|_{L^\infty(L^2)} \leq \|u_h^0 - u^0\| + C_1 h\left\{\|u^0\| + \|u_t\|_{L^2(H_1^0)}\right\},$$

$$\|\theta_2\|_{L^\infty(L^2)} \leq \|v_h^0 - v^0\| + C_2 h\left\{\|v^0\| + \|v_t\|_{L^2(H_1^0)}\right\}.$$

with equation (8.2) and these results the proof is complete. ■

Theorem 8.2. Let u, v, u_h and v_h be the solutions of (3.3) and (4.2) respectively , then there exists constants C_1 ,C_2 independent of h such as,

$$\|u - u_h\|_{L^\infty(L^2)} \leq \|u_h^0 - u^0\| + C_1 h\left\{\|u^0\| + \|u\|_{L^\infty(H_0^1)} + \|u_t\|_{L^2(H_1^0)}\right\} ,$$

$$\|v - v_h\|_{L^\infty(L^2)} \leq \|v_h^0 - v^0\| + C_2 h\left\{\|v^0\| + \|v\|_{L^\infty(H_0^1)} + \|v_t\|_{L^2(H_1^0)}\right\}.$$

Proof: As the proof of theorem (8.1) we have ,

$$\frac{1}{2}\frac{d}{dt}\|\theta_1\|^2 + \alpha_1\|\theta_1\|^2 + \frac{1}{2a_1}\left\|\frac{1}{2}(u)_x^2 - \frac{1}{2}(u_h)_x^2\right\|^2 + \frac{a_1}{2}\|\theta_1\|^2 + \frac{1}{2a_2}\|v\, u_y - v_h u_{h,y}\|^2 + \frac{a_2}{2}\|\theta_1\|^2 \leq \frac{1}{8c_1}\|\rho_{1,t}\|^2 + 2c_1\|\theta_1\|^2$$

$$\frac{1}{2}\frac{d}{dt}\|\theta_2\|^2 + \alpha_2\|\theta_2\|^2 + \frac{1}{2a_3}\|u\, v_x - u_h v_{h,x}\|^2 + \frac{a_3}{2}\|\theta_2\|^2 + \frac{1}{2a_4}\left\|\frac{1}{2}(v)_y^2 - \frac{1}{2}(u_h)_y^2\right\|^2 + \frac{a_4}{2}\|\theta_2\|^2 \leq \frac{1}{8c_2}\|\rho_{2,t}\|^2 + 2c_2\|\theta_2\|^2$$

since

$\| \frac{1}{2}(u)_x^2 - \frac{1}{2}(u_h)_x^2 \|^2$, $\| v\,u_y - v_h u_{h,y} \|^2$, $\| u\,v_x - u_h v_{h,x} \|^2$ and $\| \frac{1}{2}(v)_y^2 - \frac{1}{2}(u_h)_y^2 \|^2$

are nonnegative terms and by Putting $\alpha_1 = a_1 = a_2 = c_1 = c_2$ and $\alpha_2 = a_3 = a_4 = c_3 = c_4$,we get,

$\frac{d}{dt}\|\theta_1\|^2 \le \frac{1}{4c_1}\|\rho_{1,t}\|^2$, and $\frac{d}{dt}\|\theta_2\|^2 \le \frac{1}{4c_2}\|\rho_{2,t}\|^2$, these imply

$$\|\theta_1\|_{L^\infty(L^2)} \le \|\theta_1(0)\| + \left(\frac{1}{4c_1}\int_0^T \|\rho_{1,t}\|^2\, dt\right)^{\frac{1}{2}}, \quad (8.7a)$$

$$\|\theta_2\|_{L^\infty(L^2)} \le \|\theta_2(0)\| + \left(\frac{1}{4c_2}\int_0^T \|\rho_{2,t}\|^2\, dt\right)^{\frac{1}{2}}, \quad (8.7b)$$

applying the bounds given by (8.5) and (8.6) to the first and second terms on the right hand sides respectively, to (8.7) gives,

$$\|\theta_1\|_{L^\infty(L^2)} \le \|u_h^0 - u^0\| + C_1 h\left\{\|u^0\| + \|u_t\|_{L^2(H_1^0)}\right\},$$

$$\|\theta_2\|_{L^\infty(L^2)} \le \|v_h^0 - v^0\| + C_2 h\left\{\|v^0\| + \|v_t\|_{L^2(H_1^0)}\right\}.$$

With these results the proof is complete. ■

9- Improvement of G and G- C finite element method

The G and G-C finite element method (4.1) and (4.2) may produce an oscillating solutions if $\epsilon < h$. To handle the difficulties connected with the G and G-C with $\epsilon < h$ is to avoid these situations completely. This can be done either by decreasing h until $\epsilon > h$, which may impractical if ϵ is very small or simply by solving, instead of the original problem (3.1) and (3.3) with diffusion terms $-\epsilon\Delta u$ and $-\epsilon\Delta v$ a modified problems with diffusions term $-h\Delta u$ and $-h\Delta v$ obtained by adding the terms $-\delta\Delta u$ and $-\delta\Delta v$ to the problems (3.1) and (3.3) respectively, where $\delta = h - \epsilon$ this the idea of the classical artificial diffusion method [Johnson (1987)], this method for solving Equations (3.1) and (3.3) reads: Find $u_h, v_h \in V_h$ such that,

$$(u_{h,t} ,\varphi_h) + h(\nabla u_h,\nabla\varphi_h) + (u\, u_x,\varphi) + (v\, u_y,\varphi) = (f,\varphi_h), \quad (9.1a)$$

$$(v ,\varphi_h) + h(\nabla v_h,\nabla\varphi_h) + (u\ v_x,\varphi) + (v\ v_y,\varphi) = (g, \varphi_h) \quad (9.1b)$$

$$(u_{h,t} ,\varphi_h) + h(\nabla u_h,\nabla\varphi_h) + (\tfrac{1}{2}(u_h)_x^2,\varphi_h) + (v_h\, u_{h,y},\varphi) = (f,\varphi_h) \quad (9.2a)$$

$$\left(v_{h,t} ,\varphi_h\right) + h(\nabla v_h,\nabla\varphi_h) + \left(u_h\, v_{h,x},\varphi\right) + (\tfrac{1}{2}(v_h)_y^2,\varphi_h) = (g,\varphi_h) \quad (9..2b)$$

10. The numerical solution.

In this section we introduce the two dimensional approximation schemes. The first is the G-finite element method to the coupled Burgers' problem (3.1) . The approximate solution is written as an expansion of the basis functions. In particular, we assume that,

$u_h = \sum_{j=1}^{N} d_j(t)\varphi_j(x,y)$ and $v_h = \sum_{j=1}^{N} h_j(t)\varphi_j(x,y)$

The second is the G-C finite element method to the coupled Burgers' problem (3.3), we introduce the following approximate solution for $u_h^2(x,y,t)$ and $v_h^2(x,y,t)$:

$u_h^2 = \sum_{j=1}^{N} d_j^2(t)\, \varphi_j(x,y)$,$v_h^2 = \sum_{j=1}^{N} h_j^2(t)\, \varphi_j(x,y)$,

where each $d_j(t)$, $h_j(t)$ are nodal unknowns and $\varphi_j(x,y)$ is the j^{th} linear basis function defined on Ω . Substitute the approximate solution u_h for u,v_h for v, u_h^2

for u^2, v_h^2 for $v^2$and replace φ by φ_i in (4.1a) ,(4.1b) , (4.2a) and (4.2b) respectively to get a system of ordinary differential equation,

$$\begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}\begin{bmatrix} \mathrm{D}' \\ \mathrm{H}' \end{bmatrix} + \left(\epsilon \begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix} + \begin{bmatrix} J_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & J_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ J_3 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & J_4 \end{bmatrix} \right) \begin{bmatrix} D \\ H \end{bmatrix} = \begin{bmatrix} \breve{F} \\ \breve{G} \end{bmatrix},$$

$$\begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}\begin{bmatrix} \mathrm{D}' \\ \mathrm{H}' \end{bmatrix} + \left(\epsilon \begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & J_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ J_3 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & B_2 \end{bmatrix} \right) \begin{bmatrix} D \\ H \end{bmatrix} = \begin{bmatrix} \breve{F} \\ \breve{G} \end{bmatrix},$$

where,

$\mathrm{D}=\begin{bmatrix} d_1(t) \\ \vdots \\ d_N(t) \end{bmatrix}$, $\mathrm{H}=\begin{bmatrix} h_1(t) \\ \vdots \\ h_N(t) \end{bmatrix}$, $D'=\begin{bmatrix} d'_1(t) \\ \vdots \\ d'_N(t) \end{bmatrix}$, $H'=\begin{bmatrix} h'_1(t) \\ \vdots \\ h'_N(t) \end{bmatrix}$, $M(m_{ij})=\int_\Omega \varphi_j \varphi_i \, dxdy$,

$G(g_{ij})=\int_\Omega \nabla\varphi_j \, \nabla\varphi_i dxdy$, $J_1=(J_{ikj})=\int_\Omega d_k(t)\varphi_j \frac{\partial \varphi_k}{\partial x} \varphi_i dx\, dy$,

$J_2=(J_{ikj})=\int_\Omega d_k(t)\varphi_j \frac{\partial \varphi_k}{\partial y} \varphi_i dx\; dy$, $J_3=(J_{ikj})= \int_\Omega h_k(t)\varphi_j \frac{\partial \varphi_k}{\partial x} \varphi_i dx\, dy$,

$J_4=(\mathrm{J}_{ikj})=\int_\Omega h_k(t)\varphi_j \frac{\partial \varphi_k}{\partial y} \varphi_i \, dx\; dy$, $B_1 = (b_{ij})=\frac{1}{2}\int_\Omega d_j(t) \frac{\partial \varphi_j}{\partial x} \varphi_i dxdy$,

$B_2=(\ b_{ij})= \frac{1}{2}\int_\Omega h_j(t) \frac{\partial \varphi_j}{\partial y} \varphi_i dxdy$, $\breve{F}=(f_k)= \int_\Omega f\varphi_i dxdy$, and $\breve{G}=(g_k)= \int_\Omega g\varphi_i dxdy$,

for i, j=1,2,…,N .

10.1 Test problem

In this subsection, we present the test problem to illustrate the different behaviors of Euler–Galerkin and Galerkin-Conservation methods time dependent coupled Burgers’.The exact solutions of Burgers’ equations (3.1.a) ,(3.1.b), (3.3.a) and (3.3.b) can be generated by using the Hopf–Cole transformation (*see* [Heitmann 2002] which are :

$$u(x,y,t)=\frac{3}{4}-\frac{1}{4\left[1+e^{\frac{(-4x+4y-t)}{32\epsilon}}\right]}, \quad v(x,y,t)=\frac{3}{4}+\frac{1}{4\left[1+e^{\frac{(-4x+4y-t)}{32\epsilon}}\right]}.$$

The initial conditions are obtained from the exact solution with t=0. In this problem ϵ, t, k can be take on various values and $f=g=0$. The domain Ω. We use a uniform mesh of $N \times N$ nodal points with $2(N-1)^2$ same size triangles, for some integer $N > 1$ with mesh width parameter $h=\frac{1}{N-1}$, we take N =18 and $\bar{\Omega}$ = [0,1] × [0,1].

10.2 Numerical Results.

Numerical example is tested to illustrate these schemes and the numerical results by using ODE 15s, ordinary differential equation, solvers matlab. We discuss two cases.

Case1($\epsilon > h$): t In this case the problem was run with ϵ= 1.14, $h = 0 \cdot 05$, we compared our results obtained from the two methods G and G-C methods on the bases of accuracy and speed (computational execution time) and found that the two methods both produced solutions that converged to the exact solution at better than the expected $\left(\frac{1}{N}\right)^{\frac{1}{2}}$rate [Fletcher 1984] {see Figure (3.1),(3.2) ,(3.3) }.

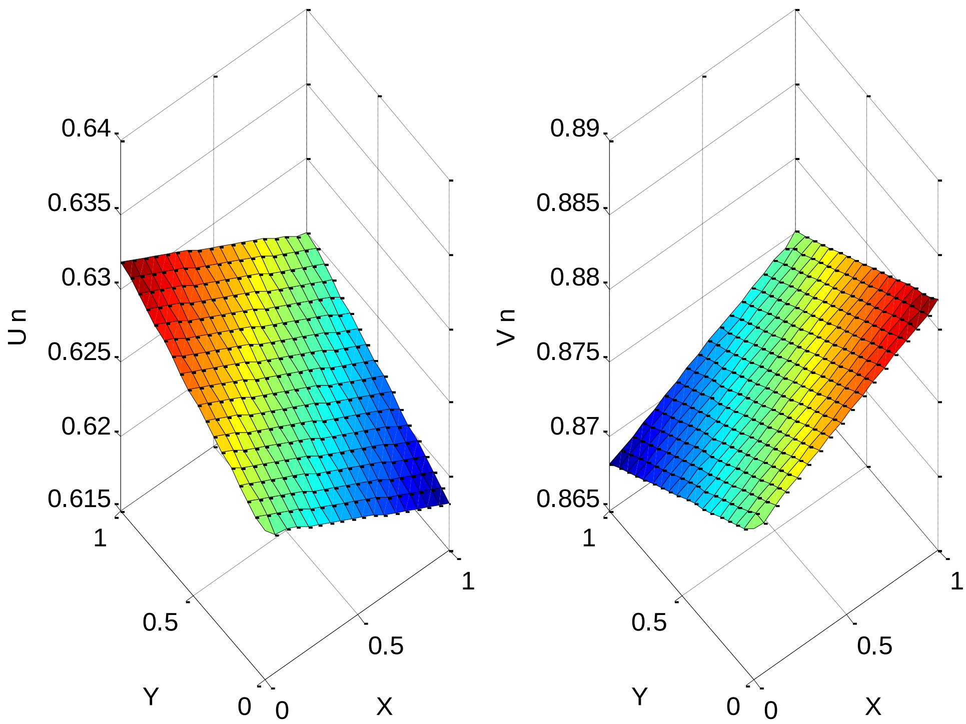

Figure 3.1 : Solution of G method at N =18, t=0.5 , ϵ = 1.14, and $h = 0 \cdot 05$

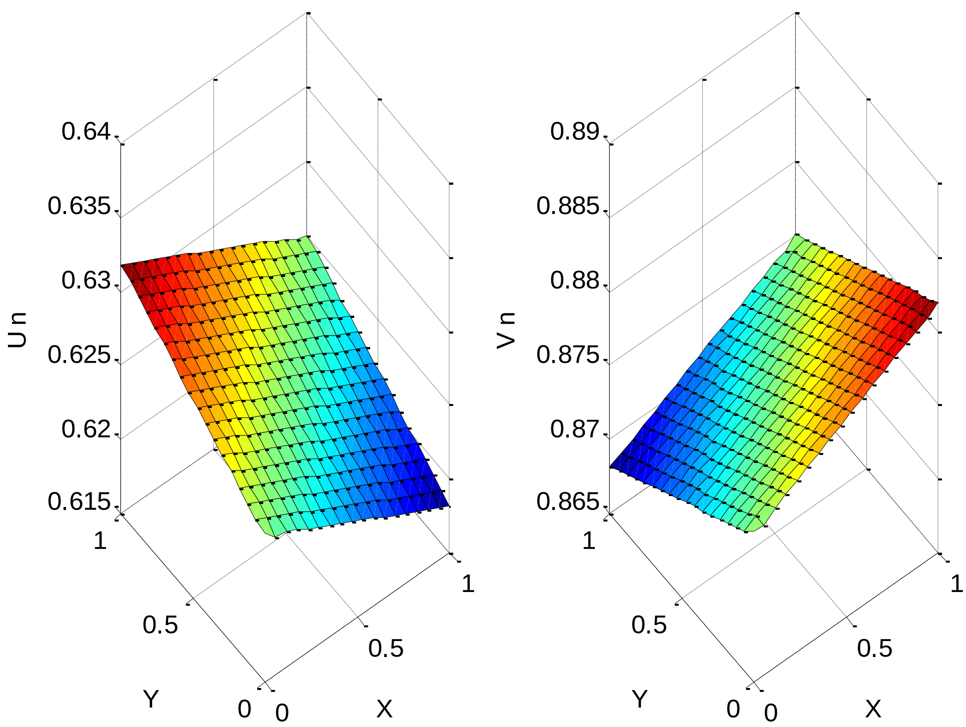

Figure3.2 : Solution of G -C method at N=18, t=0.5**,** ϵ= 1.14and $h = 0 \cdot 05$

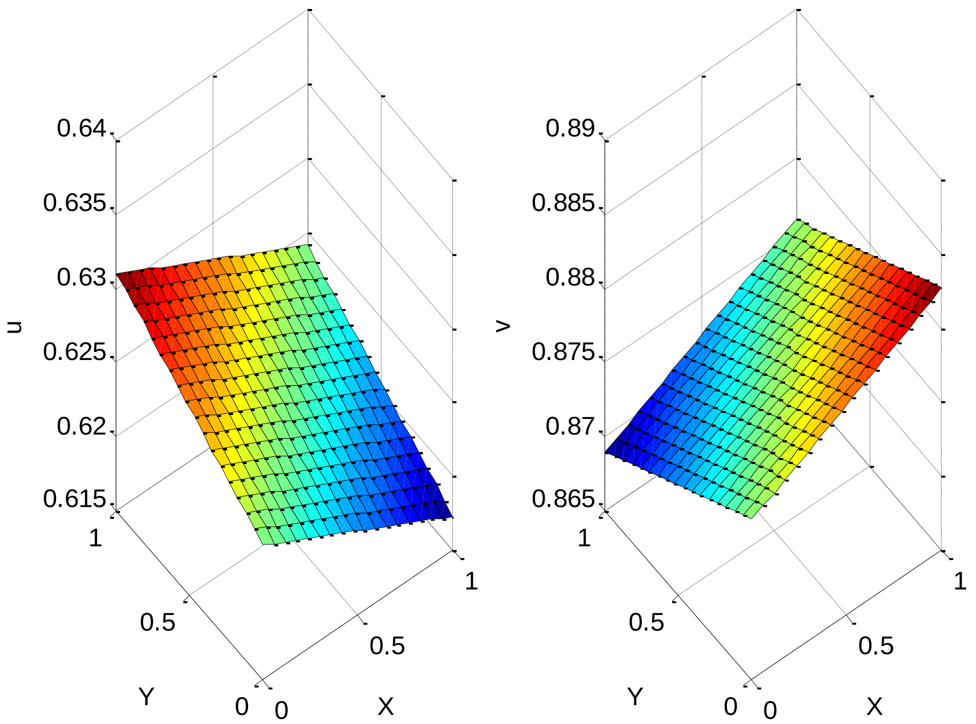

Figure3.3 : Exact solution at N=18, t=0.5, ϵ= 1.14, $h = 0 \cdot 05$

Case 2$(\epsilon < h)$**:** In this case we take ϵ =0.004, $h = 0.0588$ and $t = 0.5$. we see that the exact solution, G and G-C finite element method got oscillation as shown in figures (3.4), (3.5) and (3.6) but when we use the artificial diffusion method we remove the oscillation and improve the result that are shown in figures (3.7), (3.8) and (3.9).

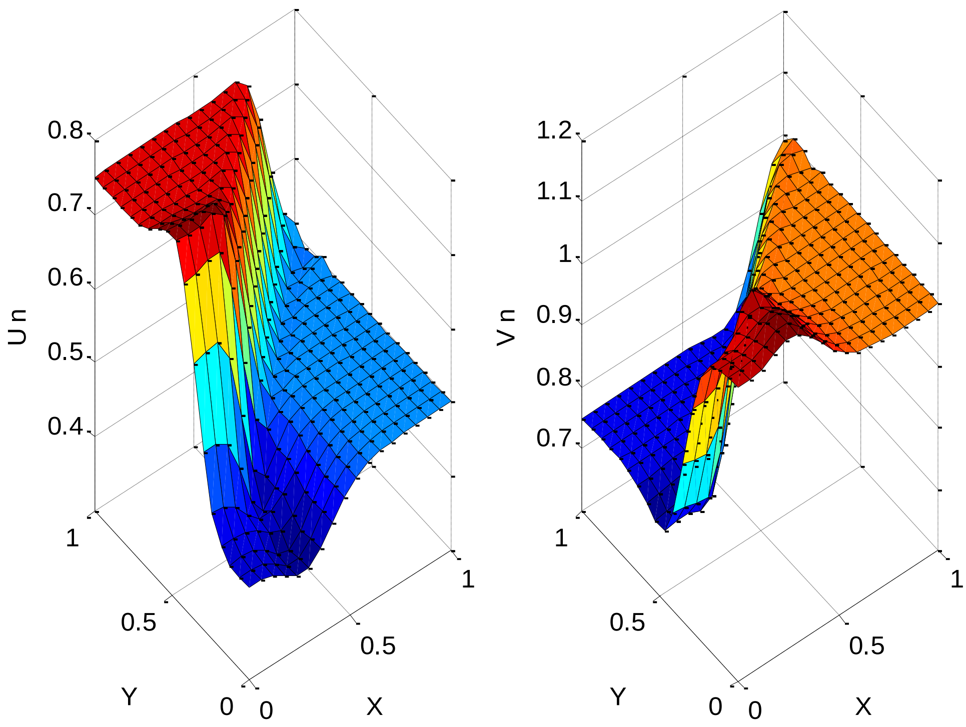

Figure 3.4: Solution of G method at $t = 0.5$, ϵ =0.004 and $h = 0.0588$

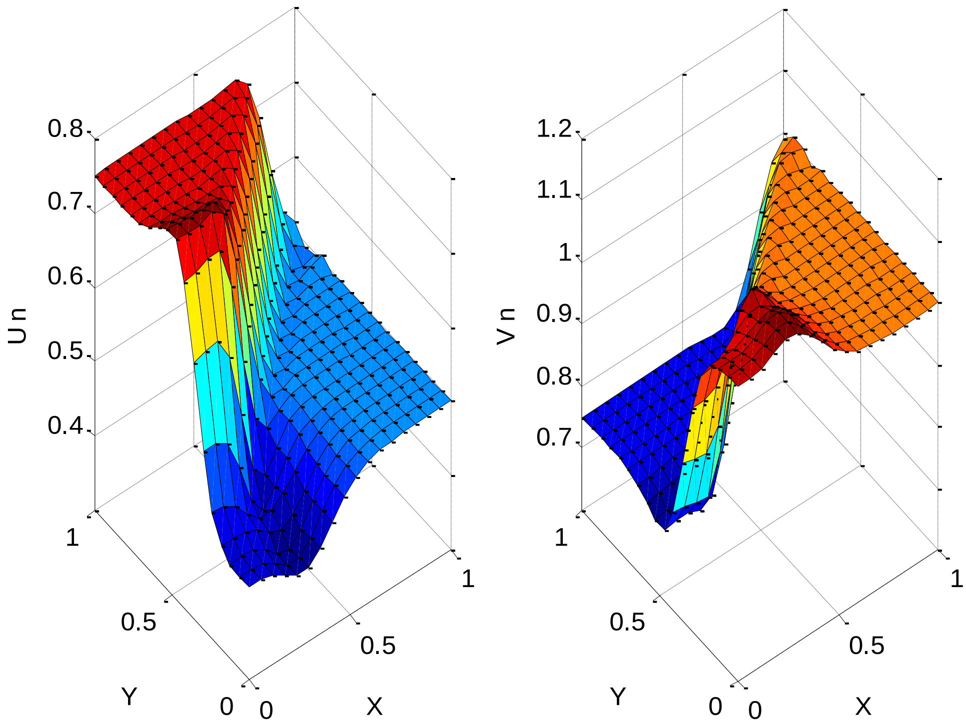

Figure 3.5 : Solution of G-C method at $t = 0.5$, ϵ =0.004 and $h = 0.0588$

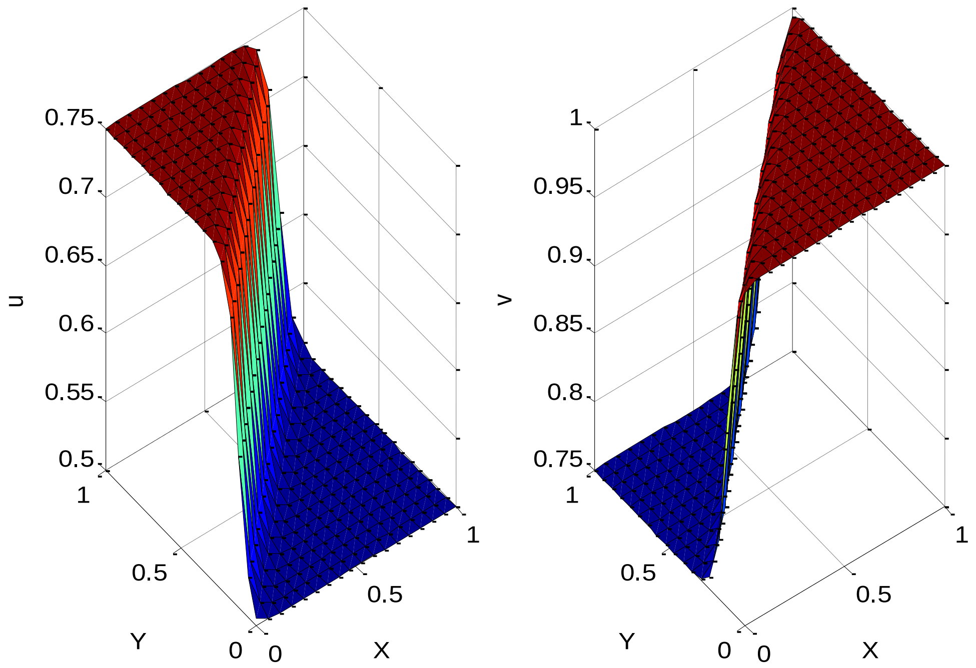

Figure 3.6 : Exact solution at t=.5, t =.5, ϵ =0.004 and $h = 0.0588$

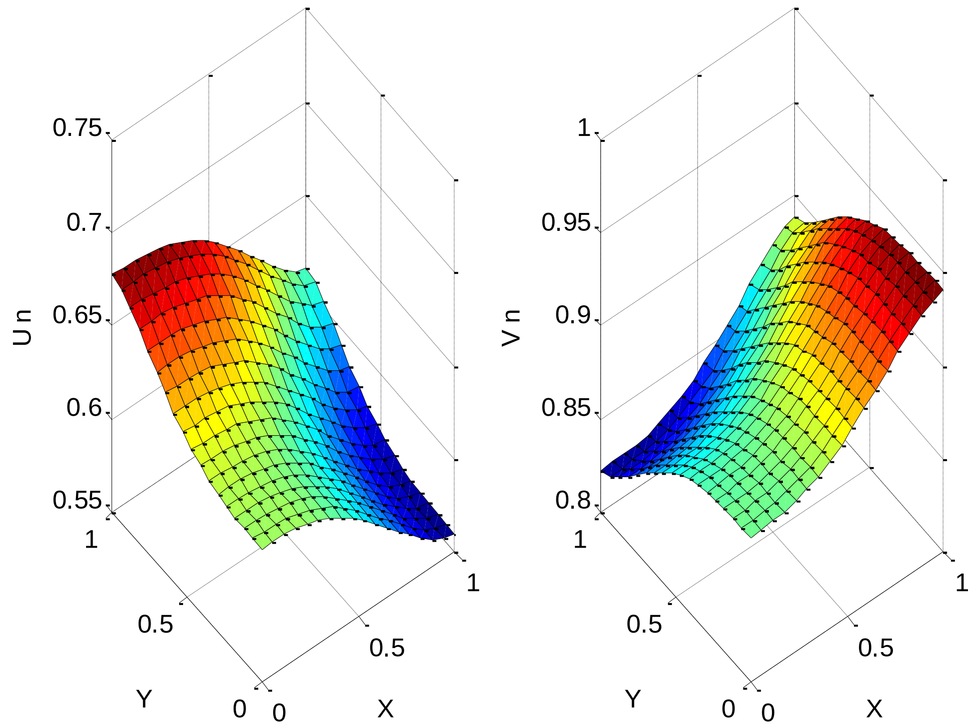

Figure 3.7 : Solution of G method of u and v with classical artificial viscosity at t=.5, t =.5, ϵ =0.004 and $h = 0.0588$

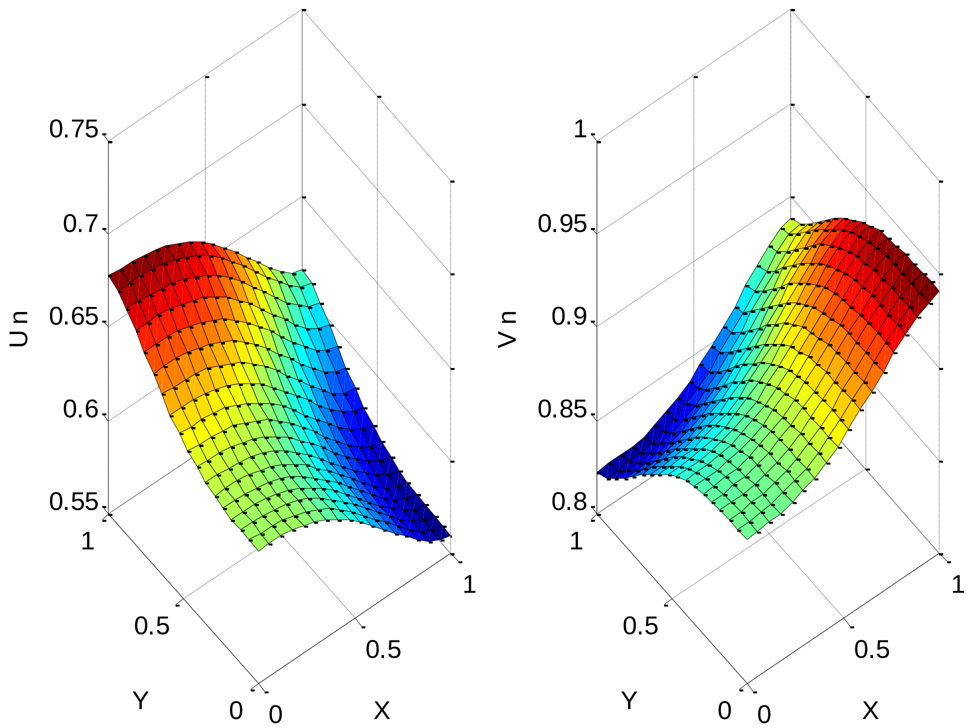

Figure 3.8: Solution of G-C method of u and v with classical artificial viscosity at t=.5, t =.5, ϵ =0.004 and $h = 0.0588$

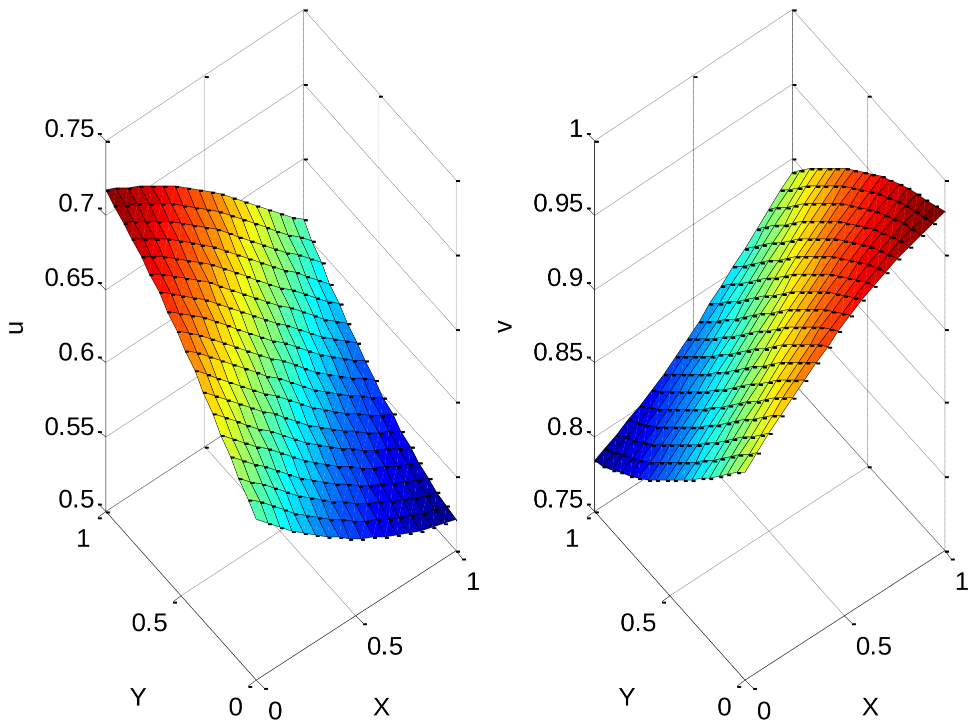

Figure 3.9 : Exact solution at t=.5, t =.5, ϵ =0.004 and $h = 0.0588$

11- Conclusions

From the theoretical analysis and the numerical results, we can conclude that the following.

1-Theoretical analysis shows that the G and G –C finite element methods are convergent with error $O(h)$

2-The numerical results for G and G –C finite element methods are convergent to the exact solutions when$(\epsilon > h)$, see figs (3.1, 3.2, 3.3), but the G.–C. finite elements method required less CPU time.

3-A special attention is paid particularly to problems with convection dominating over diffusion. The problem may arise from the weakness of the diffusion term. Such case makes the exact solution, G and G –C finite element methods lose stability and produce an oscillating solutions see figs (3-4, 3-5, 3-6). One way to overcome these difficulties is adding a classical artificial diffusion to the problem.

4-The classical artificial diffusion method removed all oscillations occur on the analytics solution, see fig (3-9) and removed all oscillations occur on G and G –C finite element methods, see figs (3-7, 3-8)

5- The classical artificial diffusion method produces non-oscillating but has the drawback of introducing a considerable amount of extra diffusion. Also we can modified these methods by using the streamline method [Johnson 1987] with less crosswind diffusion than the classical diffusion method but still corresponds to an $O(h)$-perturbation of the solution of the original problem . This is left to interested readers for future research.

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