On the existence and the nonexistence of some $(\mathbf{k},\mathbf{n})$ −arcs in $\mathbf{PG}(\mathbf{2},\mathbf{41})$

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Abstract:

A $(k, n) -$ arc is a set of k points of a projective plane such that some n, but no n+1 of them, are collinear. The maximum size of a $(k, n) -$ arc in $PG(2\ q)$ is denoted by $m_n(2, q)$. In this paper we found $m_n(2{,}41)$ for $n = (22, 23, \dots, 40)$.

Keyword: Projective spaces, (k, n)-arcs, {l, t}-blocking set.

1. Introduction

Let $GF(q)$ be the Galois field of q elements and $V(3,q)$ be the vector space of row vectors in length three whose entries in $GF(q)$. Let $PG(2,q)$ be the corresponding projective plane. The points of $PG(2,q)$ are the non-zero vectors of $V(3,q)$ with the rule that $X = (x_1, x_2, x_3)$ and $Y = (\lambda x_1, \lambda x_2, \lambda x_3)$ are the same point, where $\lambda \in GF(q)\backslash\{0\}$. Since any non-zero vector has precisely $q-1$ non-zero scalar multiples, the number of points in the form $(1, x_1, x_2)$ is q^2 and the form $(1, x_1, 0)$ is q while the form (0,0,1) is 1, so the number of points of $PG(2,q)$ is $\frac{q^3-1}{q-1} = q^2 + q + 1$. If the point P(X) is the equivalence class of the vector X, then we will say that X is a vector representing P(X). A subspace of dimension one is a set of points all of them representing vectors that form a subspace of dimension two of $V(3,q)$, such subspaces are called lines. The number of lines in $PG(2,q)$ is $q^2 + q + 1$. There are $q+1$ points on every line and any two distinct points lie exactly on one line. There are $q+1$ lines throughout every point and any two distinct lines have exactly one common point.

(Bose, in 1947) proved that $m_2(2,q) = q+1$ for q odd, and $m_2(2,q) = q+2$ for q even. (Segre, in the mid of 1950s) proved that for q odd every $q+1$ −arc is a conic, for $q=2, q=4$ and $q=8$ every $q+2$ −arc is a conic plus its nucleus, and for $q=16, q=32, q=2^h$ $(h \geq 7)$, there exists a $q+2$ −arc other than the conic plus its nucleus. (Barlotti, in 1956) proved that the first of many results in the attempt to determine the value of $m_n(2,q)$, and this has been proved to be far from simple. The existence and the nonexistence of some (k, n)-arcs in PG(2,17) have been proved by (Daskalov, 2004). The existence and the nonexistence of some (k, n)-arcs in PG(2,31) have been proved by (Najem, 2010). Also The existence and the nonexistence of some (k, n)-arcs in PG(2,37) have been proved by (Khalid, 2013).

Definition 1.1 (Hirschfeld, 1998) A $(k, n)-$arc K is a set of k points, such that there is some n but no $(n+1)$ are collinear.

Definition 1.2 (Hirschfeld, 1998) An $\{l, t\}-$blocking set S in $PG(2, q)$ is a set of l points such that every line of $PG(2, q)$ intersects S in at least t points, and there is a line intersecting S in exactly t points.

Note that A $(k, n)-$ arc is the complement of a $\{q^2 + q + 1 - k, q + 1 - n\}$-blocking set in a projective plane $PG(2, q)$ and conversely.

Definition 1.3 Let M be a set of points in any plane. An $i-$secant is a line meeting M in exactly i points. Define T_i as the number of $i-$secants to a set M.

The T_i satisfy the next three diophantine equations in any projective plane, which are known as the standard equations (Tallini Scafati, 1966).

Lemma 1.1 (Daskalov, 2004) For any set of k points in $PG(2, q)$ the following hold:

1. $\sum_{i=0}^{q+1} T_i = q^2 + q + 1$

2. $\sum_{i=1}^{q+1} iT_i = k(q+1)$

3. $\sum_{i=2}^{q+1} i(i-1)T_i = k(k-1)$

In (Ball, 1994) the next theorem is proved:

Theorem 1.1

Let K be a $(k, n)-$arc in $PG(2, q)$ where q is prime, then

1. If $n \leq (q + 1)/2$, then $m_n(2, q) \leq (n-1)q + 1$.

2. If $n \geq (q + 3)/2$, then $m_n(2, q) \leq (n-1)q + n - (q+1)/2$.

From Theorem 1.1 the next corollary holds:

Corollary 1.1

$m_{22}(2,41) \leq 862$, $m_{23}(2,41) \leq 904$, $m_{24}(2,41) \leq 946$, $m_{25}(2,41) \leq 988$, $m_{26}(2,41) \leq 1030$, $m_{27}(2,41) \leq 1072$, $m_{28}(2,41) \leq 1114$, $m_{29}(2,41) \leq 1156$,

$m_{30}(2,41) \leq 1198$, $m_{31}(2,41) \leq 1240$, $m_{32}(2,41) \leq 1282$, $m_{33}(2,41) \leq 1324$, $m_{34}(2,41) \leq 1366$, $m_{35}(2,41) \leq 1408$, $m_{36}(2,41) \leq 1450$, $m_{37}(2,41) \leq 1492$, $m_{38}(2,41) \leq 1534$, $m_{39}(2,41) \leq 1576$, $m_{40}(2,41) \leq 1618$.

1.4 The Projective Plane $\mathbf{PG(2,41)}$

1.4.1 The cyclic projectivity of $\mathbf{PG(2,41)}$

The plane PG(2,41) contains 1723 points and 1723 lines, every line contains 42 points and every point passes through it 42 lines. It is convenience to use the numbers 0,1,2,3, …, 40 to be the elements of GF(41).

Let $f(x) = x^3 - 7x^2 - x - 1$ be a monic irreducible polynomial over GF(41) then the matrix $T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 7 \end{bmatrix}$ is a cyclic projectivity which is given by the right multiplication of the points of PG(2,41) and the order of T is 1723 .

1.4.2 The points of $\mathbf{PG(2,41)}$

Let P_0 be the initial point that is represented by the vector (1 0 0) then $P_i = P_{i-1}T$, $i = 1, \ldots, 1722$ are the 1722 points of PG(2,41).

1.4.3 The lines of $\mathbf{PG(2,41)}$

Let L_0 be the initial line which contains the points { 0 1 8 68 103 142 239 282 286 296 397 409 415 439 455 468 488 491 493 519 600 619 656 718 739 808 954 963 995 1029 1040 1084 1101 1189 1282 1297 1345 1367 1432 1459 1557 1607} and has the equation $X_3 = 0$ then $L_i = L_{i-1}T$, $i = 1, \ldots, 1722$ are the 1722 lines of PG(2,41).

2. The new arcs in $\mathbf{PG(2,41)}$

Theorem 2.1 There exist a $(810,23)-\text{arc}$, $(853,24)-\text{arc}$, $(893,25)-\text{arc}$, and $(936,26)-\text{arc}$,

Proof: We will construct the above arcs by using the equation $x_1x_2 + 39x_2x_3 + x_1x_3 = 0$ which forms the conic in PG(2,41).

i. The set of points β_1

0 1 2 70 1408 775 856 1657 827 1047 414 1537 266 193 840 203 1450 1038 418 498 258 1375 209 231 432 466 503 558 634 750 780 874 957 1092 1166 1318 1379 1482 1498 1666 1668 1682 4 13 470 1713 341 749 46 378 1413 687 89 207 109 523 1076 149 1405 1412 618 202 1167 587 1157 143 1073 449 1407 1140 1503 208 1299 531 1336 1459 1487 661 1168 1491 1593 962 1688 649 185 568 227 954 1331 1362 7 1684 611 1595 416 1385 1429 1571 229 69 44 831 532 217 1105 1671 619 140 1479 1386 781 1198 1242 316 1393 1153 1426 1507 1023 241 1133 804 1146 1575 1128 796 243 1067 299 1514 1342 253 803 1044 1578 1711 1297 672 1332 732 1036 1383 9 760 711 1594 412 837 420 973 1573 319 1259 515 1043 1052 318 537 947 1655 273 1000 1419 848 367 1469 955 1101 126 384 288 472 1708 183 1001 73 1388 895 958 1324 755 1021 966 18 1395 1509 382 712 112 931 59 1266 100 1468 1251 913 1606 1621 297 1257 670 1096 1404 1014 1515 151 136 1305 959 1254 933 1378 111 1285 373 355 389 754 519 206 938 1152 1637 347 1122 142 201 1103 123 1263 626 1241 295 1369 1080 1561 17 1699 736 181 197 1238 1464 104 562 1697 1513 75 1079 158 35 38 521 1436 909 1319 1587 313 1454 487 118 1551 1586 114 396 348 164 1119 419 444 809 1288 791 910 799 1323 1519 1290 1091 1360 580 1136 1527 998 565 45 415 1452 1258 581 1114 979 1004 47 62 1217 1060 162 368 132 1661 471 990 538 773 1526 673 1252 499 334 1215 982 222 1151 1197 407 212 1042 178 1028 961 324 1030 739 213 237 1409 677 1293 974 457 497 293 66 836 1663 877 459 948 1160 236 1317 1365 785 1026 376 858 1640 1560 305 1439 372 841 233 599 353 770 1234 925 946 881 814 1549 338 983 1139 1443 1649 214 872 1691 1134 49 400 483 76 1427 28 547 713 41 1530 1156 850 79 1117 1535 708 125 374 199 1533 871 1660 553 1083 1279 735 1172 731 287 458 818 409 1230 698 725 1244 298 539 1567 656 509 1346 752 722 1240 211 1221 1633 1612 1401 590 1540 1548 220 1715 891 1685 864 127 889 1291 1097 1430 370 1194 314 139 184 448 852 80 1625 490 101 1131 145 250 1280 1007 1246 77 668 344 1211 357 1425 331 508 554 1034 1590 1191 279 625 61 269 1118 1307 1599 1011 1411 1423 176 737 1223 179 386 1087 337 1123 1113 255 284 779 20 398 210 1641 1512 1051 312 718

1434 152 1239 1158 784 240 296 1277 249 36 42 491 1721 1652 1421 746 1132 551 1704 1142 1380 120 275 446 1502 235 1615 870 511 1602 1120 1206 1328 513 1027 1579 671 1177 577 1619 1673 992 180 1098 653 1283 1442 839 172 1399 873 67 1710 1428 442 1220 715 1147 300 346 859 226 633 339 1248 141 857 194 1185 512 321 1039 730 1273 1108 594 1505 1547 1086 1182 886 533 1187 1321 461 620 1389 1320 81 596 971 1364 1562 58 1189 1522 102 399 729 405 146 310 257 1371 717 1531 29 1374 481 924 358 666 1477 39 422 1202 335 1372 410 1610 989 918 349 697 501 1497 795 1687 812 134 85 1205 534 646 631 650 778 1552 495 408 1141 797 659 527 1714 6 854 1394 1680 960 196 667 43 90 360 336 1315 1574 463 1045 734 844 23 1237 326 1008 40 1475 1403 1173 320 1613 30 163 64 327 1508 55 758 1145 15 950 1646 765 1435 116 824 1124 985 1653 1709 177 244 148 753 1308 1209 1058 1563 1309 1675 904 723 756 364 941 1347 278 1361 550 1024 388 252 981 788 1233 1422 63 115 375 1472 91 1609 1626 919 484 1314 1059 593 987 1414 1055 359 566 1204 230 354 452 936 1218 1348 119 424 605 254 1183 1210 914 323 1622 963 1029 652 161 445 575 1473 1017 838 192 1629 1312 1169 901 592 404 1581 110 930 427 1651 793 830 769 259 21 1149 847 674 108 583 863 1707 294 1446 476 759 8 1226 1351 191 1624 365 16 377 1700 1256 1228 549 900 329 1214 1337 816 688 342 976 875 1272 911 1270 1164 766 894 417 362.

forms a $(810, 23) -$ arc in PG$(2,41)$ with secant distribution

T_i (i=0,...,23) = 0 0 0 0 0 2 1 3 5 6 12 17 27 28 51 51 48 89 103 152 213 260 393 262.

ii. A new $(853, 24) -$ arc set β_2 has been constructed by extension of the previous set β_1, by adding the next 43 points.

363 743 1542 1683 380 223 995 486 1247 724 1506 1311 33 1525 828 1306 965 1701 1373 514 800 591 1566 1441 528 245 504 951 274 27 205 685 522 942 456 200 328 1287 10 277 1712 665 5.

with secant distribution

T_i (i=0,...,24) = 0 0 0 0 0 1 0 3 1 5 9 11 21 17 40 36 51 60 73 106 147 209 258 406 269.

iii. A new $(893, 25)-$ arc set β_3 has been constructed by extension of the previous set β_2, by adding the next 40 points

477 600 1381 1410 564 390 608 474 1659 242 1496 1267 1432 401 975 3 1201 1456 726 1046 1138 478 1354 853 1219 1499 423 1524 678 54 1057 1148 1181 1390 282 1664 232 1150 716 949.

with secant distribution

T_i (i=0,...,25) = 0 0 0 0 0 1 0 0 2 3 5 9 14 15 24 24 48 57 44 94 96 152 195 271 425 244.

iv. A new $(936, 26)-$ arc set β_4 has been constructed by extension of the previous set β_3, by adding the next 43 points

88 135 1070 1170 1630 908 1645 187 1018 1193 595 1538 1635 182 1316 411 569 147 1471 1605 502 1075 710 1275 425 939 524 1019 1564 271 436 1261 1481 1199 1010 131 1327 1174 693 1490 171 263 615.

with secant distribution T_i (i=0,...,26) = 0 0 0 0 0 0 1 0 0 2 6 6 7 6 15 28 27 48 33 62 101 92 133 206 288 390 272.

3. The nonexistence of some arcs in $PG(2, 41)$

Theorem 3.1 (Daskalov, 2004) Let B be an $\{l, t\}-$blocking set in $PG(2, q)$ (q-prime).

1. If $t < \frac{q}{2}$ and $q > 3$, then $l \geq t(q + 1) + \frac{q+1}{2}$.
2. If $l = t(q + 1) + \frac{q+1}{2}$ then each point of B has exactly $\frac{q+3}{2}$ lines through it that are not $t-$secants and exactly $\frac{q-1}{2}$ lines that are $t-$secants. So the total number of $t-$secants is $\frac{l(q-1)}{2t}$.

Applying this theorem, the next theorem holds.

Theorem 3.2

1. There exists no $(904, 23)-$arc and hence $m_{23}(2,41) \leq 903$.
2. There exists no $(946, 24)-$arc and hence $m_{24}(2,41) \leq 945$.
3. There exists no $(988, 25)-$arc and hence $m_{25}(2,41) \leq 987$.
4. There exists no $(1030, 26)-$arc and hence $m_{26}(2,41) \leq 1029$.

5. There exists no $(1156, 29)$ −arc and hence $m_{29}(2,41) \leq 1155$.
6. There exists no $(1240, 31)$ −arc and hence $m_{31}(2,41) \leq 1239$.
7. There exists no $(1324, 33)$ −arc and hence $m_{33}(2,41) \leq 1323$.
8. There exists no $(1366, 34)$ −arc and hence $m_{34}(2,41) \leq 1365$.

Proof: 1. Finding a maximum $(904, 23)$ −arc is equivalent to find a $\{819,19\}$ −blocking set. Theorem 3.1 implies, that the total number of 19 −secants is $\frac{819.20}{19}$ which is not an integer (a contradiction).

The proof of the remaining cases is similar to the proof of the previous one.

Theorem 3.3 (Ball, 1996) Let B be an $\{l, t\}$ −blocking set in $PG(2, q)$ that contains a line. Then: If $(t - 1, q) = 1$ then $|B| = l \geq q(t + 1)$.

Theorem 3.4

1. There exists no $(862, 22)$ −arc and hence $m_{22}(2,41) \leq 861$.
2. There exists no $(1072, 27)$ −arc and hence $m_{27}(2,41) \leq 1071$.
3. There exists no $(1114, 28)$ −arc and hence $m_{28}(2,41) \leq 1113$.
4. There exists no $(1198, 30)$ −arc and hence $m_{30}(2,41) \leq 1197$.
5. There exists no $(1282, 32)$ −arc and hence $m_{32}(2,41) \leq 1281$.
6. There exists no $(1408, 35)$ −arc and hence $m_{35}(2,41) \leq 1407$.
7. There exists no $(1450, 36)$ −arc and hence $m_{36}(2,41) \leq 1449$.
8. There exists no $(1492, 37)$ −arc and hence $m_{37}(2,41) \leq 1491$.
9. There exists no $(1534, 38)$ −arc and hence $m_{38}(2,41) \leq 1533$.
10. There exists no $(1576, 39)$ −arc and hence $m_{39}(2,41) \leq 1575$.
11. There exists no $(1618, 40)$ −arc and hence $m_{40}(2,41) \leq 1617$.

Proof: 1. Finding a maximum $(1072, 27)$ −arc is equivalent to find a $\{651,15\}$ −blocking set. Theorem 3.1 implies, that the total number of 15 −secants is 868. Let r be the length of the longest secant. If r = 42, then B contains a line and Theorem 3.3 can be applied. It follows from Theorem 3.3

that $|B| = l \geq 41.16 = 656$, a contradiction.

If $37 \leq r \leq 41$ then considering lines through a point on the longest secant but not in B, B must have at least 15.41+r points, a contradiction.

Now consider the intersection of the 15 −secants with the longest secant.

If $r = 36$ then $T_{15} = 36.20 + 5.41 = 925 > 868$, a contradiction.

If $r = 35$ then $T_{15} = 35.20 + 6.40 = 940 > 868$, a contradiction.

If $r = 34$ then $T_{15} = 34.20 + 7.39 = 953 > 868$, a contradiction.

If $r = 33$ then $T_{15} = 33.20 + 8.38 = 964 > 868$, a contradiction.

If $r = 32$ then $T_{15} = 32.20 + 9.37 = 973 > 868$, a contradiction.

If $r = 31$ then $T_{15} = 31.20 + 10.36 = 980 > 868$, a contradiction.

If $r = 30$ then $T_{15} = 30.20 + 11.35 = 985 > 868$, a contradiction.

If $r = 29$ then $T_{15} = 29.20 + 12.34 = 988 > 868$, a contradiction.

If $r = 28$ then $T_{15} = 28.20 + 13.33 = 989 > 868$, a contradiction.

If $r = 27$ then $T_{15} = 27.20 + 14.32 = 988 > 868$, a contradiction.

If $r = 26$ then $T_{15} = 26.20 + 15.31 = 985 > 868$, a contradiction.

If $r = 25$ then $T_{15} = 25.20 + 16.30 = 980 > 868$, a contradiction.

If $r = 24$ then $T_{15} = 24.20 + 17.29 = 973 > 868$, a contradiction.

If $r = 23$ then $T_{15} = 23.20 + 18.28 = 964 > 868$, a contradiction.

If $r = 22$ then $T_{15} = 22.20 + 19.27 = 953 > 868$, a contradiction.

If $r = 21$ then $T_{15} = 21.20 + 20.26 = 940 > 868$, a contradiction.

If $r = 20$ then $T_{15} = 20.20 + 21.25 = 925 > 868$, a contradiction.

If $r = 19$ then $T_{15} = 19.20 + 22.24 = 908 > 868$, a contradiction.

If $r = 18$ then $T_{15} = 18.20 + 23.23 = 889 > 868$, a contradiction.

If □ = 17 then the standard equations for the set B are:

$$T\_15 + T\_16 + T\_17 = 1723$$

$$15T\_15 + 16T\_16 + 17T\_17 = 27342$$

$$〚210T〛\_15 + 240T\_16 + 272T\_17 = 423150$$

The unique solution of this system is $T\_15 = 8431,\ T\_16 = -14913,\ \ T\_17 = 8205$, a contradiction.

If $r = 16$ then the standard equations for the set B are:

$$T_{15} + T_{16} = 1723$$

$$15T_{15} + 16T_{16} = 27342$$

$$210T_{15} + 240T_{16} = 423150$$

From the first two equations we obtain $T_{15} = 226$ *and* $T_{16} =1497$. But $210.226 + 240.1497 = 406740$ and we have a contradiction again. This completes the proof.

The proof of the remaining cases is similar to the proof of the previous one.

Now we can summarize the results from this paper in the next Table I.

n	22	23	24	25	26	27	28	29
k	…861	810…903	853…945	893…987	936…1029	…1071	…1113	…1155

n	30	31	32	33	34	35	36	37	38	39	40
k	…1197	…1239	…1281	…1323	…1365	…1407	…1449	…1491	…1533	…1575	…1617

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حول الوجود وعدم الوجود لبعض الأقواس $-(k, n)$ في $PG(2, 41)$

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الخلاصة:-

القوس $-(k, n)$ هو مجموعة k من النقاط في المستوي الاسقاطي بحيث يوجد n ولا يوجد n+1 من هذه النقاط تقع على إستقامة واحدة. الحجم الأعظم للأقواس $-(k, n)$ في المستوي الإسقاطي $PG(2, q)$ يرمز له بالرمز $m\_n\ (2, q)$ في هذا البحث وجدنا $m_n(2,41)$ عندما $n = 22, 23, \dots, 40$.