Least Common Multiplier and Greatest Factorial Factorization for Gosper's Algorithm

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Abstract

The objective of this paper is to give an approach of Gosper's algorithm that frequently used in proving combinatorial identities. We give an approach for Gosper's algorithm by using greatest factorial factorization (GFF) and least common multiplier (lcm) concepts. This approach can be easily extends to the q–analogues. To illustrate the applicability of our approach, example is presented.

Key word: Gosper's algorithm, hypergeometric solution, greatest factorial factorization, least common multiplier, rational solution, universal denominator.

1. Introduction

Sister Mary Celine is the first one who began the subject of computerized proofs of identities with her Ph.D. thesis [Frasenmyer, Sister Mary Celine, 1945] at the university of Michigan in 1945. After that in [Frasenmyer, Sister Mary Celine, 1947, Frasenmyer, Sister Mary Celine, 1949], she developed a method to find recurrence relations for hypergeometric polynomials directly from the series expansions of the polynomials. In many parts of mathematics and computer science some expressions like $s_n = \sum_{k=0}^{n-1} t_k$ (called indefinite hypergeometric summation), arise in a natural way, for instance in combinatorics or complexity analysis. Usually one is interested in finding a solution for such an expression as an expression in n, Gosper's algorithm is an automatic procedure for evaluating these kinds of sums of hypergeometric terms in the form of the difference of a hypergeometric term and a constant, provided such an expression exists see for example [R.W. Jr. Gosper, 1978, R.L. Graham et al., 1994, W. Koepf, 1995, M. Petkovšek et al., 1996]. Since it often happens that during the analysis of a problem in combinatorial theory one encounters a large sum involving factorials and binomial coefficients, one would like to know whether or not that sum can be expressed in a simpler way. Gosper's algorithm is a procedure that discovers the answer systematically. Another aspect concerns the theoretical foundation of a q–analogue of Gosper's algorithm. Besides Karr's [M. Karr, 1981] approach which covers indefinite q–hypergeometric summation in the general frame of his theory of difference field extensions, up to now it had been a kind of a surprise that Gosper's algorithm can be carried over to the q–case almost word by word; Koornwinder [T.H. Koorrnwinder, 1993]. In [W. Koepf, 1995], Koepf considers the more general case and extended version of Gosper's algorithm for indefinite summation. In [H.L. Saad, 2006 2] Saad gave a q–analogue of Koepf's algorithm, and generalize it to find solutions of recurrence equations. It turns out that the new algebraic concept of "greatest factorial factorization" introduced by Paule [P. Paule, 1995] provides an algebraic explanation not only of Gosper's algorithm, but also of its analogue for q–hypergeometric telescoping. In [H.L. Saad, 2006 1], Saad extended the greatest factorial factorization to the m-greatest factorial factorization and presented an approach to the problem.

Let N be the set of natural numbers, k be the field of characteristic zero, $\mathrm{k}(n)$ be the field of rational functions of n over k, $\mathrm{k}[n]$ be the ring of polynomials of n over k. If $p(n)\in \mathrm{k}[n]$ is a nonzero polynomial we will denote its leading coefficient by $lc(p(n))$, $p(n)\in \mathrm{k}[n]$ is said to be monic if $lc(p(n))=1$, E be the shift operator on $K[n]$, i.e. $(Ep)(n)=p(n+1)$ for any $p\in K[n]$, $\deg(p)$ denotes the polynomial degree (in n) of any $p\in \mathrm{k}[n]$, $p\neq 0$. We define $\deg(0)=-1$, $\gcd(p,q)$ denotes the greatest common divisor for any polynomials $p,q\in \mathrm{k}[n]$. We assume that the gcd always takes a value as a monic polynomial, $lcm(p,q)$ denotes the least common multiplier for any polynomials $p,q\in \mathrm{k}[n]$. The pair $\langle f,g\rangle$ $f,g\in \mathrm{k}[n]$ is called the reduced form of a rational function if $r=\frac{f}{g}$, g monic and $\gcd(f,g)=1$, [H.L. Saad, 2005].

A nonzero sequence t_n is called a hypergeometric term (or shortly hypergeometric) over k if there exists a rational function $r(n)\in \mathrm{k}(n)$ such that $\frac{t_{n+1}}{t_n}=r(n)$.

For any monic polynomial $p(n)\in \mathrm{k}[n]$, and $m\in \mathbb{N}$, the m^{th} falling factorial $[p(n)]^{\underline{m}}$ of $p(n)$ is defined as [P. Paule, 1995]

$$[p(n)]^{\underline{m}}=\prod_{i=0}^{m-1}E^{-i}p(n)=p(n)p(n-1)\ldots p(n-m+1)$$
.

Let $p_1, p_2, \dots, p_k$, $p \in \mathrm{k}[n]$ then $\langle p_1, p_2, \dots, p_k \rangle$ is called a GFF-form of a monic polynomial $p(n) \in \mathrm{k}[n]$ if the following conditions hold [P. Paule, 1995]:

(GFF1) $p(n) = [p_1]^{\underline{1}} [p_2]^{\underline{2}} \dots [p_k]^{\underline{k}}$.

(GFF2) each $p_i(n)$ monic, and $k > 0$ implies $\deg(p_k) > 0$.

(GFF3) $i \le j \Rightarrow \gcd([p_i]^{\underline{i}}, Ep_j) = 1 = \gcd([p_i]^{\underline{i}}, E^{-j} p_j)$.

We can use the following lemma to compute the GFF.

Lemma 1.2.1. [P. Paule, 1995] Let $p(n) \in \mathrm{k}[n]$ be a monic polynomial with GFF-form $\langle p_1, p_2, p_3, \dots, p_k \rangle$. Then

$\mathrm{GFF}(\gcd(p, Ep)) = \langle p_2, p_3, \dots, p_k \rangle$, and $p_1(n) = \dfrac{p(n)}{[p_2]^{\underline{2}} \dots [p_k]^{\underline{k}}}$.

In 1978, Gosper [R.W. Jr. Gosper, 1978] developed algorithm for finding the sum $s_n = \sum_{k=0}^{n-1} t_k$ depends on finding at first the hypergeometric term z_n that satisfies

$$z_{n+1} - z_n = t_n. \qquad (1.1)$$

If we can find z_n, then we will express the above sum in the simple form of a single hypergeometric term plus a constant. Conversely any solution z_n of (1.1) will have the form

$$z_n = z_{n-1} + t_{n-1} = z_{n-2} + t_{n-2} + t_{n-1} = \dots = z_0 + \sum_{k=0}^{n-1} t_k = s_n + z_0,$$

where z_0 is a constant.

Gosper showed that any rational function can be written in the form

$$r(n) = \frac{a(n)c(n+1)}{b(n)c(n)}, \qquad (1.2)$$

where $a(n), b(n), c(n)$ are polynomials in n over k and

$$\gcd(a(n), b(n+h)) = 1, \text{ for all nonnegative integer } h \tag{1.3}$$

The representation in equation (1.2) such that equation (1.3) satisfied is called Gosper's representation. In [M. Petkovšek, 1992], Petkovšek proved that Gosper representation is unique which is called Gosper-Petkovšek representation, or shortly GP representation, if the polynomials $b(n), c(n)$ are monic such that

$\gcd(a(n), c(n)) = 1,$

$$\gcd(b(n), c(n+1)) = 1.$$

Petkovšek gave an algorithm to compute the GP representation. In [M. Petkovšek, 1994], Petkovšek generalized Gosper's algorithm for recurrences of arbitrary order. In [P. Paule, 1995], equipped with the Greatest Factorial Factorization, Paule presented a new approach to indefinite hypergeometric summation which leads to the same algorithm as Gosper's, but in a new setting. In [W.Y.C. Chen and H.L. Saad, 2005], Chen and Saad showed that the uniqueness of the Gosper-Petkovšek representation of rational functions can be utilized to give a simpler version of Gosper's algorithm. In [W.Y.C. Chen et al., 2008], Chen et al. found a convergence property of the gcd of the rising factorial and the falling factorial. Based on this property, they presented an approach to compute the universal denominators as given by Gosper's algorithm.

Many approaches, even ours, for Gosper's algorithm can be reduced to find rational solutions. So in this paper, we need to mention to the rational solutions $y(n)$ for the linear difference equation

$$p_0(n)y(n) + p_1(n)y(n+1) = p(n), \tag{1.4}$$

where $p_0(n), p_1(n), p(n) \in K[n]$ are given polynomials such that $p_0(n)$ and $p_1(n) \neq 0$.

A polynomial $g(n) \in K[n]$ is called universal denominator for (1.4) if for every solution $y(n) \in K(n)$ for (1.4) there exists $f(n) \in K[n]$ such that $y(n) = f(n)/g(n)$.

2. The Fundamental lcm Lemma

The "gcd–shift", i.e. the gcd of a polynomial p and its shift Ep ($\gcd(p, Ep)$) plays a basic role in hypergeometric summation. In [P. Paule, 1995], Paule gave the following fundamental lemma of "lcm –shift", which is fundamental in deriving Gosper's algorithm.

Lemma 2.1. (Fundamental lcm Lemma [P. Paule, 1995]) Let $p(n) \in k[n]$ be a monic polynomial with GFF-form $\langle p_1, p_2, \ldots, p_k \rangle$. Then the $lcm(p, Ep)$ has GFF-form $\langle 1, Ep_1, Ep_2, \ldots, Ep_k \rangle$.

The "lcm–shift", i.e. the lcm of a polynomial p and its shift Ep ($lcm(p, Ep)$) plays a basic role in hypergeometric summation. The following lemma is very important in our approach to derive Gosper's algorithm.

Lemma 2.2. Let $B \in k[n]$ be a monic polynomial with GFF-form $\langle p_1, p_2, \ldots, p_k \rangle$. Then

(1) $B_0 = \dfrac{lcm(B, EB)}{B} = E(p_1 . p_2 \ldots p_k)$.

(2) $B_1 = \dfrac{lcm(B, EB)}{EB} = p_1 . E^{-1} p_2 . E^{-2} p_3 \ldots E^{-k+1} p_k$.

(3) $\gcd(B_0, B_1) = 1$.

Proof .

From Lemma 2.1., we get

(1) $$B_0 = \frac{lcm(B, EB)}{B} = \frac{[Ep_1]^{2}[Ep_2]^{3}\dots[Ep_k]^{k+1}}{[p_1]^{1}[p_2]^{2}\dots[p_k]^{k}}$$

$$= \frac{p_1(n+1)p_1(n)p_2(n+1)p_2(n)p_2(n-1)\dots p_k(n+1)p_k(n)\dots p_k(n-k+1)}{p_1(n)p_2(n)p_2(n-1)\dots p_k(n)\dots p_k(n-k+1)}$$

$$= E(p_1(n)\cdot p_2(n)\dots p_k(n))$$

(2) $$B_1 = \frac{lcm(B, EB)}{EB} = \frac{[Ep_1]^{2}[Ep_2]^{3}\dots[Ep_k]^{k+1}}{E([p_1]^{1}[p_2]^{2}\dots[p_k]^{k})}$$

$$= \frac{p_1(n+1)p_1(n)p_2(n+1)p_2(n)p_2(n-1)\dots p_k(n+1)p_k(n)\dots p_k(n-k+1)}{p_1(n+1)p_2(n+1)p_2(n)\dots p_k(n+1)\dots p_k(n-k+2)}$$

$$= p_1(n)\cdot E^{-1}p_2(n)\cdot E^{-2}P_3(n)\dots E^{-k+1}p_k(n).$$

(3) From (1) and (2), we get

$$\gcd(B_0, B_1) = \gcd(E(p_1p_2\dots p_k), p_1E^{-1}p_2\dots E^{-k+1}p_k).$$

If $i \le j$ we get

$$\gcd(Ep_i, E^{-j+1}p_j) = E\gcd(p_i, E^{-j}p_j) \mid E\gcd([p_i]^{i}, E^{-j}p_j) = 1.$$

Then

$$\gcd(Ep_i, E^{-j+1}p_j) = 1.$$

If $i > j$ we get

$$\gcd(Ep_i, E^{-j+1}p_j) \mid \gcd(Ep_i, [p_j]^{j}) = 1.$$

Then

$$\gcd(Ep_i, E^{-j+1}p_j) = 1.$$

Hence $$\gcd(B_0, B_1) = 1.$$

3. An Approach for Gosper's Algorithm

In this section, we present an approach for Gosper's algorithm. Our approach depends on finding the universal denominator for the first-order linear recurrence relation (3.1) (below) by using the least common divisor and the greatest factorial factorization. Once a universal denominator is found, then it is easy to find the rational solutions of the linear recurrence relation (3.1) by finding the polynomial solutions of the resulting equation. Equipped with GFF and lcm concepts we present an algebraically motivated approach to the problem. Given a hypergeometric term t_n and suppose that there exists a hypergeometric term z_n satisfying equation (1.1). Let $r(n)=\frac{t_{n+1}}{t_n}$ and $y(n)=\frac{z_n}{t_n}$. Then equation (1.1) can be written as

$$r(n)y(n+1)-y(n)=1. \tag{3.1}$$

Thus, we have reduced the problem of finding hypergeometric solution of equation (1.1) to the problem of finding rational solutions of equation (3.1). Let $\langle a,b\rangle$, $\langle f,g\rangle$ be the reduced form of $r(n)$ and $y(n)$, respectively. Then equation (3.1) becomes

$$a(n)\frac{f(n+1)}{g(n+1)}-b(n)\frac{f(n)}{g(n)}=b(n). \tag{3.2}$$

Let $h(n)=lcm(g(n),g(n+1))$. Multiplying equation (3.2) by $h(n)$, we get

$$a(n)\frac{h(n)}{g(n+1)}f(n+1)-b(n)\frac{h(n)}{g(n)}f(n)=b(n)h(n). \tag{3.3}$$

Let $h_i(n)=\frac{h(n)}{E^i g(n)}$, $i\in\{0,1\}$. Since $g(n)\mid h(n)$ and $g(n+1)\mid h(n)$, then both $h_0(n)$ and $h_1(n)$ are polynomials. Thus equation (3.3) can be written as;

$$a(n)\cdot h_1(n)\cdot f(n+1)-b(n)\cdot h_0(n)\cdot f(n)=b(n)\cdot h(n). \tag{3.4}$$

Now, if $\langle p_1, p_2, \ldots, p_k \rangle$ is the GFF-form of $g(n)$, it follows from $\gcd(h_0, h_1) = 1 = \gcd(f, g)$ that

$h_0(n) \mid a(n)$ and $h_1(n) \mid b(n)$.

From Lemma 2.2., we get

$h_0(n) = E(p_1 \cdot p_2 \ldots p_k) \mid a(n)$,

and

$h_1(n) = p_1 \cdot E^{-1}p_2 \cdot E^{-2}p_3 \ldots E^{-k+1}p_k \mid b(n)$.

Hence

$p_1 \gcd(p_2 p_3 \ldots p_k, E^{-1}p_2 E^{-2}p_3 \ldots E^{-k+1}p_k) \mid \gcd(a(n-1), b(n))$.

and then, we get

$p_1 \mid \gcd(a(n-1), b(n))$.

by the same way we can get

$$p_i \mid \gcd(a(n-1), E^{i-1}b(n)), \text{ for } i = 1, 2, \ldots, k \qquad (3.5)$$

This observation gives rise to a simple and straightforward algorithm for computing a multiple $V = [P_1]^1 [P_2]^2 \ldots [P_m]^m$ of g. For instance, if $P_1 = \gcd(E^{-1}a, b)$ then obviously $p_1 \mid P_1$. Indeed, we shall see below that by exploiting GFF-properties one can extract iteratively p_i–multiples P_i such that $EP_i \mid a$ and $E^{-i+1}P_i \mid b$. Hence, we obtained the same algorithm obtained by Paule [P. Paule , 1995] which can be stated as follows:

Algorithm 3.1. VMULT. [P. Paule ,1995, p. 253]

INPUT : The reduced form $\langle a, b \rangle$ of $r \in \mathrm{k}(n)$

OUTPUT : Polynomials $\langle P_1, P_2, \ldots, P_m \rangle$ such that $V = [P_1]^1 [P_2]^2 \ldots [P_m]^m$ is a multiple of the reduced denominator g of $y \in \mathrm{k}(n)$.

(i) Compute $m = \min\{ j \in \mathbb{N} \mid \gcd(E^{-1}a, E^{k-1}b) = 1 \text{for all integers } k > j \}$.

(ii) Set $a_0 = a$, $b_0 = b$, and compute for i from 1 to m:

$$P_i = \gcd(E^{-1}a_{i-1}, E^{i-1}b_{i-1}),$$

$$a_i = a_{i-1}/EP_i ,$$

$$b_i = b_{i-1}/E^{-i+1}P_i . \qquad \square$$

From equation (3.2), we get

$$a(n)g(n)f(n+1) - b(n)g(n+1)f(n) = b(n)g(n)g(n+1) \tag{3.6}$$

The next step is to set

$g(n) = V(n)$

in equation (3.5). If equation (3.5) can be solved for $f(n) \in \mathrm{k}[n]$, then

$$z_n = \frac{f(n)}{g(n)} \cdot t_n , \tag{3.7}$$

is a hypergeometric solution of (1.1), otherwise no hypergeometric solution of (1.1) exists.

Example 3.1. Evaluate the following sum $\sum_{k=1}^{n-1} \frac{k^2 - 2k - 1}{k^2(k+1)^2} 2^k$

Solution. Let

$$t_n = \frac{(n^2 - 2n - 1)}{n^2(n+1)^2} 2^n ,$$

then

$$r(n) = \frac{t_{n+1}}{t_n} = \frac{2n^2(n^2 - 2)}{(n+2)^2(n^2 - 2n - 1)} .$$

Hence $a(n) = 2n^2(n^2 - 2)$, $b(n) = (n+2)^2(n^2 - 2n - 1)$, where $\langle a(n), b(n) \rangle$ is the reduced form of the rational function $r(n)$. Let $\langle p_1, p_2, \ldots, p_k \rangle$ be the GFF-form of $g(n)$. Then

$$h_0(n) = \frac{lcm(g, Eg)}{g(n)} = E(p_1 \cdot p_2 \ldots p_k) \mid 2n^2(n^2 - 2),$$

$$h_1(n) = \frac{lcm(g, Eg)}{Eg(n)} = p_1 \cdot E^{-1}p_2 \cdot E^{-2}p_3 \ldots E^{-k+1}p_k \mid (n+2)^2(n^2 - 2n - 1),$$

and from the algorithm VMULT, we get

$$P_1(n) = (n^2 - 2n - 1),$$

and

$$P_2(n) = P_3(n) = \ldots = P_k(n) = 1.$$

Hence

$$g(n) = V(n) = P_1(n) = n^2 - 2n - 1,$$

then from equation (3.6), we get

$$2n^2 \cdot f(n+1) - (n+2)^2 \cdot f(n) = (n+2)^2(n^2 - 2n - 1)$$

Case 1 [M. Petkovšek et al., 1996] yields

$$\deg(f(n)) = 4 - 2 = 2.$$

The polynomial $f(n) = n^2 + 2n + 1 = (n+1)^2$ is a solution to the above equation. By (3.7), we have

$$z_n = \frac{f(n)}{g(n)} t_n = \frac{2^n}{n^2}.$$

Hence

$$\sum_{k=1}^{n-1} \frac{k^2 - 2k - 1}{k^2(k+1)^2} 2^k = \frac{2^n}{n^2} - 2.$$

<u>Conclusions</u>

(1) The lcm is equivalent to the gcd in deriving Gosper's algorithm.

(2) Our approaches for Gosper's algorithm are easily extended to the q-case.

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المضاعف المشترك الأصغر والتحليل للعامل الأعظم لخوارزمية كَوسبر

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الخلاصة:

الهدف في هذا البحث هو إعطاء أسلوب خوارزمية كَوسبر التي تستخدم بكثرة في برهان المتطابقات التوافقية . نعطي أسلوب خوارزمية كَوسبر باستخدام مفهومي التحليل للعامل الأعظم (GFF) والمضاعف المشترك الأصغر(lcm) .هذا الأسلوب يمكن توسيعه للأسلوب المطابق -q .نقدم مثال لتوضيح تطبيق أسلوبنا.