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The cyclic decomposition of the group $D_{2^n} \times C_3$

التجزئة الدائرية للزمرة $D_{2^n} \times C_3$

Intissar Abd AL- Hur Kadum
AL-Kufa university \ College of Education for Girls\ Department of Mathematics
Email: intesara.kadhim@uokufa.edu.iq

Abstract

The group of all Z-valued characters of a finite group G over the group of induced unit characters from all cyclic subgroups of G forms a finite a belian group, called ***Artin Cokernel of G*** ,denoted by AC(G).The problem of finding the cyclic decomposition of Artin cokernel $D_{2^n} \times C_3$ has been considered in this paper ,the cyclic decomposition of $D_{2^n} \times C_3$ is :

$$AC(D_{2^n} \times C_3) = \mathop{\oplus}_{i=1}^{2n} C_2 .$$

Also we give the general form of rational character and Artin's characters tables of $D_{2^n} \times C_3$ group**.**

المستخلص :

إن زمرة كل الشواخص العمومية ذات القيم الصحيحة للزمرة المنتهية G على زمرة الشواخص المحتثة من الشواخص الأحادية للزمر الجزئية الدائرية من الزمرة G تكون زمرة ابيلية منتهية و تسمى النواة المشارك – آرتن للزمرة G ويرمز لها بالرمز (AC(G)).

إن مسألة إيجاد التجزئة الدائرية لزمرة القسمة AC(G) قد اعتبرت في هذه الرسالة للزمرة $D_{2^n} \times C_3$،حيث وجدنا إن التجزئة الدائرية للزمرة $D_{2^n} \times C_3$:

$$AC\,(D_{2^n} \times C_3) = \mathop{\oplus}_{i=1}^{2n} C_2$$

للزمرة. $D_{2^n} \times C_3$ وكذلك وجدنا الصيغة العامة لجدول الشواخص النسبية وجدول شواخص ارتن

Introduction:

The problem of determining the cyclic decomposition of AC(G) seem to be untouched. we use the concepts of invariant matrix in linear algebra to find the cyclic decomposition of AC(G), G is considered to be the group $D_{2^n} \times C_3$.In 1968 T.Y Lam [8] defined AC(G) and he studied AC(G),when G is a cyclic group.

In 2000 H.R .Yassin [4] studied the cyclic decomposition of AC(G) when G is an elementary abelian group . In 2006 A.S. Abed [2] found Ar(C_n) when C_n is the cyclic group of order n .

In this paper ,we find the rational valued character table and the artin's characters table of the direct product group $D_{2^n} \times C_3$,where D_{2^n} is the dihedral group of order 2^{n+1} and C_3 is the cyclic group of order 3 ,we also find the cyclic decomposition of the factor group $AC(D_{2^n} \times C_3)$.

1. Some Basic Concepts:

In this section, we give basic concepts, notations and theorems about matrix representation, characters and Artin characters.

Definition (1.1) [3]

The general linear group GL(n,F) is a multiplicative group of all non-singular $n \times n$ matrices over the field F.

Definition (1.2) [3]

A matrix representation of the group G is a homomorphism of G into GL(n,F) ,n is the degree of matrix representation T .In particular ,T is called a unit representation (principal) if T(g) =1 ,for all $g \in G$.

Definition (1.3) [3]

The trace of an $n \times n$ matrices A is the sum of the main diagonal elements, denoted by tr(A).

Definition (1.4) [3]

Let T be a matrix representation of degree n of a finite group G over the field F. The character χ of degree n of T is the mapping χ:G$\to$F defined by $\chi(g)=tr(T(g))$ for all $g \in G$. In particular ,the character of the principal representation if ($\chi(g)=1$,for all $g \in G$) is called the principal character.

Definition (1.5) [3]

Two elements g and h in a group G are said to be conjugate if $h = xgx^{-1}$ for some $x \in G$.The relation of conjugacy is an equivalence relation on G . The equivalence classes determined by this relation are referred to be as the conjugate classes, denoted by CL_g ,
$g \in G$ is the conjugate class of the element g.

Definition (1.6) [3]

The centralizer of x in G is the subgroup $C_G(x)=\{a \in G: a\, x\, a^{-1} = x\}$.

Definition (1.7) [3]

Let H be a subgroup of G and ϕ be a character of H ,the induced character on G is given by

$\phi\uparrow^G (g) = \frac{1}{|H|}\sum_{x \in G} \phi^\circ(xgx^{-1})$ where $g \in G$ and ϕ° is defined by $\phi^\circ(h) = \begin{cases} \phi(h)\, if & h \in H \\ 0 \quad if & h \notin H \end{cases}$.

Theorem (1.8) [4]

Let H be a cyclic subgroup of G and $h_1, h_2, h_3, \ldots, h_m$ are chosen representatives for the m-conjugate classes of H in CL_g ,$g \in G$,then

$$\phi\uparrow^G (g) = \frac{|C_G(g)|}{|C_H(g)|}\sum_{i=1}^{m} \varphi(h_i) \quad if \quad h_i \in H \cap CL(g)$$

$$\phi\uparrow^G (g) = 0 \text{ if } H \cap CL(g) = \phi$$

Definition (1.9) [4]

Let G be a finite group ,any character induced from the principal character of a cyclic subgroup of G is called Artin character of G.

Definition (1.10) [5]

Two elements of G are said to be Γ-conjugate if the cyclic subgroups they generate are conjugate in G, this defines an equivalence relation on G .Its classes are called Γ-classes.

Proposition (1.11): *[8]*

The number of all distinct Artin valued characters of a finite group G equal to the number of all distinct Γ-classes on G .

Definition (1.12): *[2]*

The complete information about Artin valued characters of a finite group G is displayed in a table called the Artin characters table of G.denoted by Ar(G) which is $l \times l$ matrix whose columns are Γ-classes and rows are the values of all Artin characters of G, where l is the number of Γ-classes.

Definition (1.13): *[3]*

A rational valued character θ of G is a character whose values are in Z, that is θ(g) ∈ Z, for all g ∈ G .

Proposition (1.14): [6]

The number of all distinct rational valued characters of a finite group G is equal to the number of all distinct Γ-classes on G .

Definition (1.15): [6]

The complete information about rational valued characters of a finite group G is displayed in a table called the rational valued characters table of G.denoted it by $\equiv^*(G)$ which is $l \times l$ matrix whose columns are Γ-classes and rows are the values of all rational valued characters of G, where l is the number of Γ-classes.

Theorem [Artin] *(1.16):* *[5]*

Every rational valued character of G can be written as a linear combination of Artin characters with coefficient rational numbers .

Theorem (1.17):[5]

Let $T_1: G_1 \rightarrow GL(n, K)$ and $T_2: G_2 \rightarrow GL(m, K)$ are two irreducible representations of the group G_1 and G_2 with characters χ_1 and χ_2 respectively , then $T_1 \otimes T_2$ is irreducible representation of the group $G_1 \times G_2$ with the character $\chi_1 \chi_2$.

Proposition (1.18):[6]

The rational valued characters $\theta_i = \sum_{\sigma \in Gal(Q(\chi_i)/Q)} \sigma(\chi_i)$ form basis for $\bar{R}(G)$, where χ_i are the irreducible characters of G and their numbers are equal to the number of all distinct Γ- classes of G.

2.The factor Group AC(G):-

The definition of the group AC(G) was introduced by T.Y Lam [8] in 1967. The applications of the factor group AC(G) not only in the mathematics but also in physics and chemistry .In this section we shall study AC(G), dihedral group D_n and $\equiv^*(D_n)$.

Definition (2.1): [8]

Let $\overline{R(G)}$ be the group of Z-valued generalized characters of G under the operation pointwise addition and T(G) is the normal subgroup of $\overline{R(G)}$ generated by Artin characters .The abelian group $\overline{R}(G)/T(G)$ is called ***Artin cokernel of G*** **,** denoted by AC(G) **.**

Definition (2.2): [6]

Let M be a matrix with entries in a principal ideal domain R . ***A k – minor of M*** is the determinant of k×k sub matrix preserving row and column order .

Definition (2.3): [6]

A k-th determinant divisor of M is the greatest common divisor (g.c.d) of all the k–minors of M , this is denoted by $D_k(M)$.

Theorem (2.4): [6]

Let M be an k×k matrix with entries in a principal ideal domain R , then there exits matrices P and W such that :

1 - P and W are invertible .
2 - P M W = D .
3 - D is a diagonal matrix .
4 -If we denote D_{jj} by d_j then there exists a natural number m ; 0 ≤ m ≤ k
such that j > m implies d_j =0 and j ≤ m implies $d_j \neq 0$ and 1 ≤ j ≤ m implies d_j / d_{j-1}

Definition (2.5): [6]

Let M be matrix with entries in a principal ideal domain R, equivalent to matrix D = diag $\{d_1, d_2, \ldots, d_m, 0, 0, \ldots, 0\}$ such that d_j / d_{j-1} for 1 ≤ j < m ,we call D ***the invariant factor matrix of*** M and $d_1, d_2, \ldots, d_m$ the invariant factors of M.

Remark(2.6) :

According to the Artin theorem (1.16) there exists an invertible matrix $M^{-1}(G)$ with entries in the set of rational numbers such that : $\overset{*}{\equiv}(G) = M^{-1}(G).Ar(G)$

and this implies, $M(G) = Ar(G).(\overset{*}{\equiv}(G))$

by theorem (2.4) there exist two matrices P(G), W(G) such that P(G).M(G).W(G)=

diag $\{d_1,d_2,\ldots,d_l\}$ =D(G) ,where $d_j = \overset{+}{-} D_j(M(G)) \,|\, D_{j-1}(M(G))$ and l is the number of Γ- classes.

Theorem (2.7): [4]

$$AC(G) = \bigoplus_{j=1}^{l} C_{d_j} \text{ where } d_j = \pm D_j(M(G)) \setminus D_{j+1}(M(G))$$

Definition(2.8)**[9]**

The group of all symmetries of the regular polygon with n sides , including both rotations and reflections , is called ***dihedral group*** and denoted by D_n .

The set of rotations generated by r - counterclockwise rotation with angle $2\pi/n$ of order n, and the set of reflections are of order 2 and every element s^j generates $\{1, s^j\}$, where 1 is the identity element in D_n.

In general we can write D_n as:

$D_n=\{s^j r^k : 0 \le k \le n-1 , 0 \le j \le 1\}$

Where $r^n = 1, s^2 = 1, sr^k s = r^{-k}$.

The element r generates the group C_n which is a cyclic subgroup of D_n .

Theorem(2.9) :[16]

The cyclic decomposition of $AC(D_{2^n})$ is: $AC(D_{2^n}) = \bigoplus_{i=1}^{n-1} C_2$

Remark (2.10):[5]

In this work we consider the direct product group $D_{2^n} \times C_3$,where C_3 is a cyclic group of the order 3 consisting of elements $\{1, r', r'^2\}$with $(r')^3=1$.

The order of the group $|D_{2^n} \times C_3| = |D_{2^n}| \cdot |C_3| = 2.2^n.3 = 3.2^{n+1}$

Proposition (2.11): [6]

The rational valued characters table of the cyclic group $(\equiv^* C_p)$, where p is a prime number can be given as follows :

$(\equiv^* C_p) =$

Γ-classes	[1]	[r]
θ_1	p-1	-1
θ_2	1	1

Proposition (2.12): [6]

The ***rational valued characters table of the cyclic group*** C_{2^n} of the rank n +1 which is denoted by $(\equiv^*(C_{2^n}))$, is given as follows:

Γ-classes	[1]	$[r^{2^{n-1}}]$	$[r^{2^{n-2}}]$	…	$[r^2]$	[r]
θ_1	2^{n-1}	-2^{n-1}	0	…	0	0
θ_2	2^{n-2}	2^{n-2}	-2^{n-2}	…	0	0
θ_3	2^{n-3}	2^{n-3}	2^{n-3}	…	0	0
⋮	⋮	⋮	⋮	⋱	⋮	⋮
θ_{n-1}	1	1	1	…	-2	0
θ_n	1	1	1	…	1	-1
θ_{n+1}	1	1	1	…	1	1

Theorem(2.13) :[4]

The rational valued character table of the dihedral group D_{2^n} is equal to $\equiv^*(D_{2^n}) =$

	Γ-classes of C_{2^n}	$[s]$	$[sr]$
θ_1	$\equiv^*(C_{2^n})$	0	0
θ_2		0	0
⋮		⋮	⋮
θ_n		-1	1
θ_{n+1}	1 1 ⋯ 1	1	1
θ_{n+2}	1 1 ⋯ 1	-1	-1
θ_{n+3}		1	-1

where n is the number of Γ-classes of the group C_{2^n}, $\theta_{n+3}(r^k)=1$ if k is an even number and $\theta_{n+3}(r^k)= -1$ if k is an odd number .

Theorem (2.14): [2] [3]

The general form of Artin character of C_2^n is given by table:

$Ar(C_2^n)=$

Γ-classes	$[1]$	$[r^{2^{n-1}}]$	$[r^{2^{n-2}}]$	…	$[r]$
φ_1	2^n	0	0	…	0
φ_2	2^{n-1}	2^{n-1}	0	…	0
φ_3	2^{n-2}	2^{n-2}	2^{n-2}	…	0
⋮	⋮	⋮	⋮	⋱	⋮
φ_n	n	n	n	…	0
φ_{n+1}	1	1	1	…	1

And the general form of Artin characters table of C_p when p is a prime number is given by:

$Ar(C_p)=$

Γ- classes	$[1]$	$[r]$
$\lvert CL_\alpha \rvert$	**1**	**1**
$\lvert C_{C_5}(CL_\alpha) \rvert$	**P**	**P**
φ'_1	P	0
φ'_2	1	1

Proposition (2.15): [7]

$$P(C_{2^n})=\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ & & & & & \ddots & & \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$ which is (n+1)×(n +1) square matrix .

$W\left(C_{2^n}\right)=I_{n+1}$ where I_{n+1} is an identity matrix and $D(C_{2^n})=\{1,1,\dots,1\}$

Remark :(2.16) :

We can write $M(C_{2^n})$ as the following :

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

which is (n+1)×(n+1) square matrix .

Theorem(217.):[1]

The Artin's character table of the dihedral group D_{2^n} when n is an even number is given as follows :

$Ar(D_{2^n})$ =

	$[1]$	$\left[r^{\frac{n}{2}}\right]$	**Γ – Classes of C_n**		$[s]$	$[sr]$
$\lvert CL_\alpha \rvert$	1	1	2 2 …	2	$2^n/2$	$2^n/2$
$\lvert C_{D_n}(CL_\alpha) \rvert$	2^{n+1}	2^{n+1}	2^n 2^n …	2^n	2^2	2^2
Φ_1					0	0
$\vdots$			$2.Ar(C_{2^n})$		$\vdots$	$\vdots$
Φ_l					0	0
Φ_{l+1}	2^n	0	…	0	0	2
Φ_{l+2}	2^n	0	…	0	2	0

Where l is the number of Γ-classes of C_{2^n} and $\Phi_j, 1 \le j \le l+2$ are the Artin's characters of the group D_{2^n} .

Proposition (2.18):[1]

$$M(D_{2^n}) = \left[\begin{array}{cccccccc} & & & & 1 & 1 & 1 & 1 \\ & 2M(C_{2^n}) & & & \vdots & \vdots & \vdots & \vdots \\ & & & & 1 & 1 & 1 & 1 \\ 0 & 0 & \cdots & 0 & & & & \\ & & & & & & & \\ 0 & 0 & \cdots & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & \cdots & 1 & 0 & 1 & 0 & 1 \\ \multicolumn{4}{c}{\underbrace{1 \quad 1 \quad \cdots \quad 1}_{n-1\times n-1}} & 1 & 1 & 0 & 0 \end{array}\right] \quad \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} n\ times$$

Which is $(n+3)\times(n+3)$ square matrix .

Proposition (2.19):[1]

The matrices $P(D_{2^n})$ and $W\left(D_{2^n}\right)$ are taking the forms :

$$P(D_{2^n}) = \left[\begin{array}{cccccc} & & & & 0 & 0 \\ & & & & 0 & 0 \\ & & & & \vdots & \vdots \\ & & P(C_{2^n}) & & & \\ & & & & 0 & 0 \\ & & & & -1 & 1 \\ & & & & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & -1 \\ 0 & 0 & \dots & 0 & 0 & 1 \end{array}\right]$$

$$\text{and } W(D_{2^n}) = \left[\begin{array}{cccccccc} & & & & & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 \\ & & & & & \vdots & \vdots & \vdots \\ & & & I_n & & & & \\ & & & & & 0 & 0 & 0 \\ -1 & -1 & \dots & -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & \dots & 1 & 1 & 1 & 0 & -1 \\ 0 & 0 & \dots & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

Where I_n is an identity matrix of the order n, $P(D_{2^n})$and $W(D_{2^n})$ are $(n+3)\times(n+3)$ square matrices.

3.The Main Results

In this section we give the general forms of rational valued character table , Artin's characters table and the cyclic decomposition of the factor group of the group $D_{2^n} \times C_3$.

Theorem(1.3) :-

The rational valued character table of the group $D_{2^n} \times C_3$ is equal to the tensor product of the rational valued characters table of $D_2{}^n$ and the rational valued characters table of C_3 that is

$\equiv^*(D_{2^n} \times C_3) = \equiv^*(D_{2^n}) \otimes \equiv^*(C_3)$

Proof :-

We denote by χ_i to the irreducible characters of D_{2^n} and θ_i , $1 < i < n+3$ to the rational valued characters of D_{2^n} ,

Since the character table of C_3 is equal to

$\equiv C_3 =$

CL_α	$[g'_1]$	$[g'_2]$	$[g'_3]$
χ'_1	1	1	1
χ'_2	1	ω	ω^2
χ'_3	1	ω^2	ω

$\omega = e^{2\pi i \backslash n}; i = 1,2,3$ and by proposition(2.11) ,the rational valued character table of C_3 is equal to

$\equiv$*(C3)=

Γ- Clases	$[g'_1]$	$[g'_2]$
θ'_1	2	-1
θ'_2	1	1

From definition of $D_{2^n} \times C_3$ and by theorem (1.17) we have each element g_{hk} in $D_{2^n} \times C_3$ can be written as follows $g_{hk} = g_h . g'_k$ where $g_h \in D_{2^n}$, h= 1,2,3,….., n+1 and $g'_k \in C_3$, k =1,2 and each irreducible character χ_{ij} of $D_{2^n} \times C_3$ can be written as follows

$\chi_{ij} = \chi_i . \chi'_j$

where χ_i is an irreducible character of $D_2{}^n$, i = 1,2,….., n+3 and χ'_j is the irreducible character of C_3 , j=1,2 .

then
$$\chi_{ij}(g_{hk}) = \begin{cases} 2\chi_i(g_h) & if \quad j=1 \quad and \quad k=1 \\ -\chi_i(g_h) & if \quad j=2 \quad and \quad k=1 \\ \chi_i(g_h) & if \quad j=1,2 \quad and \quad k=2 \end{cases}$$

denote by θ_{ij} to the rational valued characters of $D_{2^n} \times C_3$.

From Proposition (2.18)

[I] $\theta_{i1} = \sum_{\sigma \in Gal(Q(\chi_{i2})/Q)} \sigma(\chi_{i2})$

then $\theta_{i1}(g_{hk}) = \sum_{\sigma \in Gal(Q(\chi_{i2}(g_{hk}))/Q)} \sigma(\chi_{i2}(g_{hk}))$

(a) If k= 1 .

$\theta_{i1}(g_{hk}) = \sum_{\sigma \in Gal(Q(\chi_i(g_h))/Q)} \sigma(2\chi_i(g_h)) = 2\theta_i(g_h) = \theta_i(g_h) \,.\, 2 = \theta_i(g_h) \,.\, \theta_1'(g_k')$

(b) k=2

$\theta_{i1}(g_{hk}) = \sum_{\sigma \in Gal(Q(\chi_i(g_h))/Q)} \sigma(\chi_i(g_h)) = \sum_{\sigma \in Gal(Q(\chi_i(g_h))/Q)} \sigma(\chi_i(g_h))$

$= \sum_{\sigma \in Gal(Q(\chi_i(g_h))/Q)} \sigma(\chi_i(g_h)) \,.\, (1) = \theta_i(g_h) \,.\, \theta_1'(g_k') \,.$

[II] $\theta_{i2} = \sum_{\sigma \in Gal(Q(\chi_{i1})/Q)} \sigma(\chi_{i1})$

then $\theta_{i2}(g_{hk}) = \sum_{\sigma \in Gal(Q(\chi_{i1}(g_{hk}))/Q)} \sigma(\chi_{i1}(g_{hk}))$

(a) If k= 1 .

$\theta_{i2}(g_{hk}) = \sum_{\sigma \in Gal(Q(\chi_i(g_h))/Q)} \sigma(\chi_i(g_h)) = \theta_i(g_h) = \theta_i(g_h) \,.\, 1 = \theta_i(g_h) \,.\, \theta_2(g_k')$

(b) k=2

$\theta_{i2}(g_{hk}) = \sum_{\sigma \in Gal(Q(\chi_i(g_h))/Q)} \sigma(-\chi_i(g_h)) = -\theta_i(g_h) = \theta_i(g_h) \,.\, -1 = \theta_i(g_h) \,.\, \theta_2(g_k')$

From [I] and [II] we have $\theta_{ij} = \theta_i \cdot \theta_j'$

Then $\equiv^*(D_{2^n} \times C_3) = \equiv^*(D_{2^n}) \otimes \equiv^*(C_3)$.

Example(3.2) :

To find the matrix $\equiv^*(D_{2^3} \times C_3)$

From theorem (2.13) and proposition (2.11), we have

$$\equiv^*(D_{2^3}) = \begin{bmatrix} 4 & -4 & 0 & 0 & 0 & 0 \\ 2 & 2 & -2 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & 1 & -1 \end{bmatrix} \text{and} \quad C_3 = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

By theorem (1.3) we get $\equiv^*(D_{2^3} \times C_3) = \equiv^*(D_{2^3}) \otimes \equiv^*(C_3)$

$$
= \begin{bmatrix}
8 & -8 & 0 & 0 & 0 & 0 & -4 & 4 & 0 & 0 & 0 & 0 \\
4 & 4 & -4 & 0 & 0 & 0 & -2 & -2 & 2 & 0 & 0 & 0 \\
2 & 2 & 2 & -2 & -2 & 2 & -1 & -1 & -1 & 1 & 1 & -1 \\
2 & 2 & 2 & 2 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 \\
2 & 2 & 2 & 2 & -2 & -2 & -1 & -1 & -1 & -1 & 1 & 1 \\
2 & 2 & 2 & -2 & 2 & -2 & -1 & -1 & -1 & 1 & -1 & 1 \\
4 & -4 & 0 & 0 & 0 & 0 & 4 & -4 & 0 & 0 & 0 & 0 \\
2 & 2 & -2 & 0 & 0 & 0 & 2 & 2 & -2 & 0 & 0 & 0 \\
1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\
1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & -1
\end{bmatrix}
$$

Theorem(3.3):

The artin character table of the group $D_{2^n} \times C_3$ is equal to the tensor product of the artin characters table of D_{2^n} and the artin characters table of C_3 that is

$Ar(D_{2^n} \times C_3)=Ar(D_{2^n}) \otimes Ar(C_3)$

Proof :-

We denote by χ_i to the irreducible characters of D_{2^n} and θ_i , $1< i < n+3$ to the rational valued characters of D_{2^n} , Since the character table of C_3 equal to

$\equiv C_3 =$

CL_α	$[g'_1]$	$[g'_2]$	$[g'_3]$
ϕ'_1	1	1	1
ϕ'_2	1	ω	ω^2
ϕ'_2	1	ω^2	ω

$\omega = e^{2\pi i \backslash n}; i = 1,2,3$ and by proposition(2.14) ,the artin character table of C_3 is equal to

$Ar(C_3)=$

Γ- Clases	$[g'_1]$	$[g'_2]$
θ'_1	3	0
θ'_2	1	1

From definition of $D_{2^n} \times C_3$ and by theorem (1.17) we have each element g_{hk} in $D_{2^n} \times C_3$ can be written as follows $g_{hk} = g_h . g'_k$ where $g_h \in D_{2^n}$, h = 1,2,3,….., $n+3$ and $g'_k \in C_3$, k =1,2 and each irreducible character ϕ_{ij} of $D_{2^n} \times C_3$ can be written as follows

$\phi_{ij} = \phi_i . \phi'_j$

where ϕ_i is an irreducible character of $D_2{}^n$, i = 1,2,….., $n+3$ and ϕ'_j is the irreducible character of C_3 , j=1,2 .

then $$\phi_{ij}(g_{hk}) = \begin{cases} 3\phi_i(g_h) & if \quad j=1 \quad and \quad k=1 \\ 0 & if \quad j=2 \quad and \quad k=1 \\ \phi_i(g_h) & if \quad j=1,2 \quad and \quad k=2 \end{cases}$$

denote by $\Phi_{(i,j)}$ to the artin characters of $D_{2^n} \times C_3$.

If H is a cyclic subgroup of the group $D_{2^n} \times C_3$, we use theorem(1.8) to find each

$$\Phi_{(i,j)}(g_{hk}) = \frac{\left| C_{D_{2^n} \times C_3}(g_{hk}) \right|}{\left| C_H(g_{hk}) \right|} \sum_{t=1}^{m} \phi(x_t) \text{ if } H \cap CL(g) = \varnothing$$

and $\Phi_{(i,j)}(g) = 0$ if $H \cap CL(g) = \phi$ where $1 \le i \le l, 1 \le j \le 2$

$$\text{[I] } \Phi_{(i,2)}(g_{hk}) = \frac{\left| C_{D_{2^n} \times C_3}(g_{hk}) \right|}{\left| C_H(g_{hk}) \right|} \sum_{t=1}^{m} \phi_{ij}(x_t) \text{ then}$$

$$\Phi_{(i,2)}(g_{hk}) = \frac{\left| C_{D_{2^n}} \right| . \left| C_3 \right|}{\left| C_H(g_h) \right| . \left| C_H(g'_k) \right|} \sum_{t=1}^{m_1} \phi_i(x_t) \sum_{t=1}^{m_2} \phi'_2(x_t)$$

(a) If k= 1 then $\Phi_{(i,j)}(g) = 0$ (since $H \cap CL(g) = \phi$)

$$\Phi_{(i,2)}(g_{hk}) = \frac{\left| C_{D_{2^n}} \right| . \left| C_3 \right|}{\left| C_H(g_h) \right| . \left| C_H(g'_1) \right|} \sum_{t=1}^{m_1} \phi_i(x_t) \sum_{t=1}^{m_2} \phi'_2(x_t) = \frac{\left| C_{D_{2^n}} \right|}{\left| C_H(g_h) \right|} \sum_{t=1}^{m_1} \phi_i(x_t) \left(\frac{3}{3} \quad 0 \right)$$

$$= \Phi_i(g_h).0 = \Phi_i(g_h).\Phi_2(g'_1)$$

(b) If k= 2 then $\Phi_{(i,j)}(g) = 3$ (since $H \cap CL(g)$= {(g,1′),(g,r′),(g,r'_2)})

$$\Phi_{(i,2)}(g_{hk}) = \frac{\left| C_{D_{2^n}} \right| . \left| C_3 \right|}{\left| C_H(g_h) \right| . \left| C_H(g'_2) \right|} \sum_{t=1}^{m_1} \phi_i(x_t) \sum_{t=1}^{m_2} \phi'_2(x_t) = \frac{\left| C_{D_{2^n}} \right|}{\left| C_H(g_h) \right|} \sum_{t=1}^{m_1} \phi_i(x_t) \left(\frac{3}{3} \quad (3) \right)$$

$$= \Phi_i(g_h).0 = \Phi_i(g_h).\Phi_2(g'_2)$$

$$\text{[II] } \Phi_{(i,1)}(g_{hk}) = \frac{\left| C_{D_{2^n}} \right| . \left| C_3 \right|}{\left| C_H(g_h) \right| . \left| C_H(g'_k) \right|} \sum_{t=1}^{m_1} \phi_i(x_t) \sum_{t=1}^{m_2} \phi'_1(x_t)$$

(a) If k= 1then $\Phi_{(i,j)}(g) = 1$ (since $H \cap CL(g)$= {(g,1′)})

$$\Phi_{(i,1)}(g_{hk}) = \frac{\left| C_{D_{2^n}} \right| . \left| C_3 \right|}{\left| C_H(g_h) \right| . \left| C_H(g'_1) \right|} \sum_{t=1}^{m_1} \phi_i(x_t) \sum_{t=1}^{m_2} \phi'_1(x_t) = \frac{\left| C_{D_{2^n}} \right|}{\left| C_H(g_h) \right|} \sum_{t=1}^{m_1} \phi_i(x_t) \left(\frac{3}{3} \quad 1 \right)$$

$$= \Phi_i(g_h).1 = \Phi_i(g_h).\Phi_1(g'_1)$$

(b) If k= 2 then $\Phi_{(i,j)}(g) = 1$ (since $H \cap CL(g)$= {(g,r′)}

$$\Phi_{(i,1)}(g_{hk}) = \frac{\left|C_{D_{2^n}}\right| \; . \; |C_3|}{|C_H(g_h)| \; . \; |C_H(g_2')|} \sum_{t=1}^{m_1} \phi_i(x_t) \sum_{t=1}^{m_2} \phi_1'(x_t) = \frac{\left|C_{D_{2^n}}\right|}{|C_H(g_h)|} \sum_{t=1}^{m_1} \phi_i(x_t) \left(\frac{3}{3} \quad (1)\right)$$

$= \Phi_i(g_h).1 = \Phi_i(g_h).\Phi_1(g_2')$

From [I] and [II] we have

$\Phi_{ij} = \Phi_i . \Phi_j$

Then $\mathrm{Ar}(D_{2^n} \times \mathrm{C}_3) = \mathrm{Ar}(D_{2^n}) \otimes \mathrm{Ar}(\mathrm{C}_3)$.

Example(3.4):

To find the matrix $\mathrm{Ar}(\mathrm{D}_{2^3} \times \mathrm{C}_3)$ from theorems (2.17) and (2.14)

$$\mathrm{Ar}(\mathrm{D}_{2^3}) = \begin{bmatrix} 16 & 0 & 0 & 0 & 0 & 0 \\ 8 & 8 & 0 & 0 & 0 & 0 \\ 4 & 4 & 4 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 & 0 & 0 \\ 8 & 0 & 0 & 0 & 2 & 0 \\ 8 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \text{ and } \mathrm{Ar}(\mathrm{C}_3) = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}$$

Which is $(3+3)\times(3+3)$ square matrix .

Using theorem (3.3) , we have

$$\mathrm{Ar}(D_{2^3} \times \mathrm{C}_3) = \mathrm{Ar}(D_{2^3}) \otimes \mathrm{Ar}(\mathrm{C}_3) = \begin{bmatrix} 48 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 24 & 24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 12 & 12 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 6 & 6 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 24 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 24 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 16 & 0 & 0 & 0 & 0 & 0 & 16 & 0 & 0 & 0 & 0 & 0 \\ 8 & 8 & 0 & 0 & 0 & 0 & 8 & 8 & 0 & 0 & 0 & 0 \\ 4 & 4 & 4 & 0 & 0 & 0 & 4 & 4 & 4 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 & 0 & 0 & 2 & 2 & 2 & 2 & 0 & 0 \\ 8 & 0 & 0 & 0 & 2 & 0 & 8 & 0 & 0 & 0 & 2 & 0 \\ 8 & 0 & 0 & 0 & 0 & 2 & 8 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Proposition (3.5):

$$\mathrm{M}(\mathrm{D}_{2^3} \times \mathrm{C}_3) = \left[\begin{array}{c|c} \mathrm{M}(\mathrm{D}_{2^n}) & \mathrm{M}(\mathrm{D}_{2^n}) \\ \hline 0 & \mathrm{M}(\mathrm{D}_{2^n} \end{array}\right]$$

which is $[2(n+3)] \times [2(n+3)]$ square matrix .

Proof : By theorem(3.1) we obtain the Artin characters table $\mathrm{Ar}(\mathrm{D}_{2^n} \times \mathrm{C}_3)$ and from theorem(3.1) we find the rational valued characters table $\equiv^* (\mathrm{D}_{2^n} \otimes \mathrm{C}_3)$

Thus by definition of $\mathrm{M}(\mathrm{D}_{2^3} \times \mathrm{C}_3)$ we find the matrix $\mathrm{M}(\mathrm{D}_{2^3} \times \mathrm{C}_3)$:

$$M(D_{2^n} \times C_3) = Ar(D_{2^n} \times C_3).(\equiv^* (D_{2^n} \times C_3))^{-1} = \left[\begin{array}{c|c} M(D_{2^n}) & M(D_{2^n}) \\ \hline 0 & M(D_{2^n} \end{array}\right]$$

Example (3.6):

To find ($M(D_{2^n} \times C_3)$ we must

$$M(D_{2^3}) = \begin{bmatrix} 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Which is $2(3+1)\times 2(3+1)$ square matrix .

Then by proposition (3.5) we have

$$M(D_{2^n} \times C_3) = \left[\begin{array}{c|c} M(D_{2^3}) & M(D_{2^3})) \\ \hline 0 & M(D_{2^3})) \end{array}\right] = \begin{bmatrix} 2 & 2 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 1 & 0 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Theorem (3.7):

$$D(D_{2^n} \otimes C_3) = diag \; \{\overbrace{2, \; 2, \; \cdots \; 2}^{2n-times}, 1, 1, 1, 1, 1\}$$

Proof :the matrices $P(D_{2^n} \otimes C_3)$ and $W(D_{2^n} \otimes C_3)$ are taking the forms :

$$P(D_{2^n} \otimes C_3) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \otimes P(D_{2^n})$$

which is $[2(n+3)] \times [2(n+3)]$ square matrix.

and the form of $W(D_{2^n} \otimes C_3)$ is : $W(D_{2^n} \otimes C_3) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes W(D_{2^n})$

which is $[2(n+3)] \times [2(n+3)]$ square matrix.

By using theorem(2.4) we have $M(D_{2^n} \otimes C_3)$.

From proposition(3.3) and the above forms of $P(D_{2^n} \otimes C_3)$ and $W(D_{2^n} \otimes C_3)$) then :

$$P(D_{2^n}\otimes C_3)\ .M(D_{2^n}\otimes C_3)\ .W(D_{2^n}\otimes C_3)\ =\ diag\ \{\overbrace{2,\ \ 2,\ \ \cdots\ \ 2}^{2n-times},1,1,1,1,1\} = D(D_{2^n}\otimes C_3)$$

which is $\left[2(n+3)\right]\times\left[2(n+3)\right]$ square matrix.

Now we can find the invariant factors matrix of the group $D_{2^n}\otimes C_3$, $D(D_{2^n}\otimes C_3)$ and the cyclic decomposition of the factor group $AC(D_{2^n}\otimes C_3)$ by using theorem(3.7).

Corollary (3.8):

The cyclic decomposition of $AC(D_{2^n}\otimes C_3)$ is : $AC(D_{2^n}\times C_3)=\bigoplus_{i=1}^{2n} C_2$

Proof :

By theorem (3.7) , we have

$$D(D_{2^n}\otimes C_3) = \text{diag}\{\overbrace{2,2,2,\ldots,2}^{2n},1,1,1,1,1,1\}$$

Then by theorem (2.7) we have

$$AC(D_{2^n}\otimes C_3) = \bigoplus_{i=1}^{2n} C_2\ .$$

Example(3.9)

$$1-AC(D_{2^5}\otimes C_3) = \bigoplus_{i=1}^{2(5)} C_2 = \bigoplus_{i=1}^{10} C_2$$

$$2-AC(D_{2^3}\otimes C_3) = \bigoplus_{i=1}^{2(3)} C_2 = \bigoplus_{i=1}^{6} C_2$$

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