"Finding the complete solution of special types of the second order partial differential equations with variable coefficients"

((إيجاد الحل الكامل لأنواع خاصة من المعادلات التفاضلية الجزئية من الرتبة الثانية ذات المعاملات المتغيرة))

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بحـــــث مســــــتل

Abstract

Our aim in this paper is to solve some kinds of partial differential equations of the second order with variable coefficients, which have the general form

$$F_1(x,y)Z_{xx} + F_2(x,y)Z_{xy} + F_3(x,y)Z_{yy} + F_4(x,y)Z_x + F_5(x,y)Z_y + F_6(x,y)Z = 0$$

such that $F_1(x,y)$,$F_2(x,y)$, $F_3(x,y)$, $F_4(x,y)$, $F_5(x,y)$ and $F_6(x,y)$ are functions of $x$and y.By using the assumptions$Z(x,y) = e^{\int \frac{U(x)}{x}dx + \int y^n V(y)dy}$ and $Z(x,y) = e^{\int U(x)dx + \int V(y)dy}$ these assumptions will transform the second order partial differential equations to linear first order ordinary differential equation with two independent functions $U(x)$ and $V(y)$.

المستخلص

الهدف من هذا البحث هو حل بعض أنواع المعادلات التفاضلية الجزئية من الرتبة الثانية ذات المعاملات المتغيرة التي صيغتها العامة :

$$F_1(x,y)Z_{xx} + F_2(x,y)Z_{xy} + F_3(x,y)Z_{yy} + F_4(x,y)Z_x + F_5(x,y)Z_y + F_6(x,y)Z = 0$$

حيث إن

$F_1(x,y)$, $F_2(x,y)$, $F_3(x,y)$$F_4(x,y)$,$F_5(x,y)$ و $F_6(x,y)$دوال بالمتغيرين x و y.

باستخدام التعويضات $Z(x,y) = e^{\int \frac{U(x)}{x}dx + \int y^n V(y)dy}$ و $Z(x,y) = e^{\int U(x)dx + \int V(y)dy}$ التي تحول المعادلات التفاضليةالجزئيةمن الرتبة الثانية الى معادلات تفاضلية اعتيادية خطية من الرتبة الأولى بالدالتين المستقلتين $U(x)$ و $V(y)$.

1.Introduction

Differential equations play an important role in plenty of the fields of the sciences as Physics, Chemistry and other sciences, and therefore the plenty of the scientists were studying this subject and they are trying to find modern methods for getting rid up the difficulties that facing them in the solving these equations.

Kudaer, in 2006 studied the linear second order ordinary differential equations, of **[1]** the form:

$y'' + P(x)y' + Q(x)y = 0$,

and she used the assumption the $y = e^{\int Z(x)dx}$ to find the general solution of it, in which the solution depends on the forms of $P(x)$ and $Q(x)$.

Abd Al-Sada, in 2006 studied the linear second order (P.D.Es) with constant coefficients, of **[2]**the form:

$A_1Z_{xx} + A_2Z_{xy} + A_3Z_{yy} + A_4Z_x + A_5Z_y + A_6Z = 0$,

where A_i are constants. (i=1,2,...,6) . She used the assumption

$Z(x,y)=e^{\int U(x)dx+\int V(y)dy}$ to find the complete solution of it, and the solution depended on the values of A_i .

Hani, in 2008studied the linear second order (P.D.Es),which have three independent variables, of **[3]** the form:
$AZ_{xx}+BZ_{xy}+CZ_{xt}+DZ_{yy}+EZ_{yt}+FZ_{tt}+GZ_x+HZ_y+IZ_t+JZ=0$,
where A, B, C, ... , I and J are arbitrary constants , and she used the assumption
$Z(x,y)=e^{\int U(x)dx+\int V(y)dy+\int W(t)dt}$
to find the complete solution of it . The solution depended on the values of A, B, C, ... , I and J.
Hanon, in 2009 studied the linear second order (P.D.Es), with variable coefficients, of **[4]** the form:
$A(x,y)Z_{xx}+B(x,y)Z_{xy}+C(x,y)Z_{yy}+D(x,y)Z_x+E(x,y)Z_y+F(x,y)Z=0$,
where some of $A(x,y), B(x,y), C(x,y), D(x,y), E(x,y)$ and $F(x,y)$ are functions of x only or y only or both x and y .
To solve this kind of equations , she used the assumptions
$Z(x,y)=e^{\int \frac{U(x)}{x}dx+\int V(y)\,dy}$, $Z(x,y)=e^{\int U(x)dx+\int \frac{V(y)}{y}dy}$ and $Z(x,y)=e^{\int \frac{U(x)}{x}dx+\int \frac{V(y)}{y}dy}$
These assumptions represent the complete solution of the above equation and the solution depends on the forms of $A(x,y), B(x,y), C(x,y), D(x,y), E(x,y)$ and $F(x,y)$.
Mohsin, in 2010studied the nonlinear second order (P.D.Es) ,of homogeneous degree, of **[5]** the form:
$F_1(x,y,Z,Z_x,Z_y,Z_{xx},Z_{xy},Z_{yy})Z_{xx}+F_2(x,y,Z,Z_x,Z_y,Z_{xx},Z_{xy},Z_{yy})Z_{xy}+$
$F_3(x,y,Z,Z_x,Z_y,Z_{xx},Z_{xy},Z_{yy})Z_{yy}+F_4(x,y,Z,Z_x,Z_y,Z_{xx},Z_{xy},Z_{yy})Z_x+$
$F_5(x,y,Z,Z_x,Z_y,Z_{xx},Z_{xy},Z_{yy})Z_y+F_6(x,y,Z,Z_x,Z_y,Z_{xx},Z_{xy},Z_{yy})Z=0$
where A, B, C, D, E and F are linear functions of dependent variable Z and partial derivatives of dependent variable with respect to the independent variables x and y, by the following assumptions
$Z(x,y)=e^{\int U(x)dx+\int V(y)dy}$, $Z(x,y)=e^{\int \frac{U(x)}{x}dx+\int V(y)\,dy}$
$Z(x,y)=e^{\int U(x)dx+\int \frac{V(y)}{y}dy}$ and $Z(x,y)=e^{\int \frac{U(x)}{x}dx+\int \frac{V(y)}{y}dy}$,
she found the complete solutions of the above kind of equations.
Finally, **ketap**, in 2011 studied the linear third order partial differential equations, of **[6]** the form
$AZ_{xxx}+BZ_{yyy}+CZ_{xxy}+DZ_{xyy}+EZ_{xx}+FZ_{yy}+GZ_{xy}+HZ_x+IZ_y+JZ=0$,
and it depend on the values of A,…,I and J. He used the assumption $Z(x,y)=e^{\int U(x)dx+\int V(y)dy}$ to find the complete solution of it.
These ideas made us to search functions $U(x)$ and $V(y)$, that give the complete solution of the linear second order partial differential equations with variable coefficients,
$F_1(x,y)Z_{xx}+F_2(x,y)Z_{xy}+F_3(x,y)Z_{yy}+F_4(x,y)Z_x+F_5(x,y)Z_y+F_6(x,y)Z=0$ and this solution depends on the forms of the functions $F_1(x,y)$, $F_2(x,y)$, $F_3(x,y)$, $F_4(x,y)$, $F_5(x,y)$ and $F_6(x,y)$.

2. Basic definitions

In this section we present the fundamental and necessary definitions related to this work.
2.1 Definition[7]: *A partial differential equation* is an equation involving one or more partial derivatives of an unknown function of several variables.
2.2 Definition[8]: Any relation between dependent variable and independent variables which satisfies(P. D. E.) and is free from partial derivatives is said to be a *solution of the partial differential equation.*
2.3 Definition[8]: The solution of the partial differential equation , which contains only arbitrary constants, is called *complete solution.*
2.4 Definition[7]: A linear partial differential equation of second order in the independent variables x and y is an equation of the form
$F_1(x,y)Z_{xx}+F_2(x,y)Z_{xy}+F_3(x,y)Z_{yy}+F_4(x,y)Z_x+$

$F_5(x,y)Z_y + F_6(x,y)Z = 0$ ...(1.1)

Such that $F_1(x,y)$,$F_2(x,y)$, $F_3(x,y)$, $F_4(x,y)$, $F_5(x,y)$ and $F_6(x,y)$ are functions of $x$and y.

3. Finding the complete solution

In order to solve the equation (1.1), we will choose some kinds of linear second order partial differential equations with variable coefficients and divide them into two kinds, according to the assumptions that we will suggested it help us to solve these equations.

Kind(1)

1) $\mathrm{A}xZ_x + \mathrm{B}\frac{Z_y}{y^n} + \mathrm{D}\frac{x}{y^n}Z_{xy} = 0$; ($y \neq 0$if n $\in \mathbb{R}^+$) and n $\in \mathbb{R}$,
$\ni$ A, B and D are not identically zero.

2) $\mathrm{A}\frac{Z_y}{y^n} + \mathrm{B}\frac{x}{y^n}Z_{xy} + \mathrm{D}Z = 0$;($y \neq 0$if n $\in \mathbb{R}^+$) and n $\in \mathbb{R}$,
$\ni$ A, B and Dare not identically zero.

3) $\mathrm{A}xZ_x + \mathrm{B}\frac{x}{y^n}Z_{xy} + \mathrm{D}Z = 0$;($y \neq 0$if n $\in \mathbb{R}^+$) and n $\in \mathbb{R}$,
$\ni$ A, B and Dare not identically zero.

4) $\mathrm{A}xZ_x + \mathrm{B}\frac{x}{y^n}Z_{xy} + \mathrm{D}\frac{Z_y}{y^n} + \mathrm{E}Z = 0$;($y \neq 0$if n $\in \mathbb{R}^+$) and n $\in \mathbb{R}$, $\ni$ A, B, D and Eare not identically zero.

5) $\mathrm{A}xZ_x + \mathrm{B}\frac{Z_y}{y^n} + \mathrm{C}x^2Z_{xx} + \mathrm{E}\frac{x}{y^n}Z_{xy} = 0$; ($y \neq 0$if n $\in \mathbb{R}^+$) and n $\in \mathbb{R}$, $\ni$ A, B, C and Eare not identically zero.

6) $\mathrm{A}xZ_x + \mathrm{B}\frac{Z_y}{y^n} + \mathrm{C}x^2Z_{xx} + \mathrm{E}\frac{x}{y^n}Z_{xy} + \mathrm{F}Z = 0$; ($y \neq 0$if n $\in \mathbb{R}^+$) and n $\in \mathbb{R}$, $\ni$ A, B, C, EandFare not identically zero.

Kind(2)

1) $\mathrm{A}\tan y\, Z_{xx} + \mathrm{B}Z_{xy} = 0$

2) $\mathrm{A}Z_{xx} + \mathrm{B}\, Z_x + \mathrm{D}\frac{Z_y}{f(y)} = 0$, such that $f(y) \neq 0$

3) $\mathrm{A}Z_{yy} + \mathrm{B}\, Z_y + \frac{\mathrm{D}Z_x+\mathrm{F}Z}{f(x)} = 0$, such that $f(x) \neq 0$

such that A, B, D and F are arbitrary constants and not identically zero.

In kind (1), we search about functions $U(x) \neq 0$ and $V(y) \neq 0$ such that the assumption

$$Z(x,y) = e^{\int \frac{U(x)}{x}dx + \int y^n V(y)dy} \quad , \quad x \neq 0 \qquad \text{... (1.2)}$$

gives the complete solution of the equations in kind (1). By finding Z_x,Z_y, Z_{xx}, Z_{yy} and Z_{xy} from (1.2), we get

$$Z_x = \frac{U(x)}{x}e^{\int \frac{U(x)}{x}dx + \int y^n V(y)\,dy}$$

$$Z_y = y^n V(y) e^{\int \frac{U(x)}{x}dx + \int y^n V(y)\,dy}$$

$$Z_{xx} = \frac{xU'(x)+U^2(x)-U(x)}{x^2}e^{\int \frac{U(x)}{x}dx + \int y^n V(y)\,dy}$$

$$Z_{xy} = \frac{y^n}{x}U(x)V(y)e^{\int \frac{U(x)}{x}dx + \int y^n V(y)\,dy}$$

By substituting Z_x, Z_y, Z_{xx}, $Z_{yy}$and Z_{xy} in the equations of kind(1), we get

1) $[\mathrm{A}U(x) + \mathrm{B}V(y) + \mathrm{D}U(x)V(y)]e^{\int \frac{U(x)}{x}dx + \int y^n V(y)\,dy} = 0$

2) $[\mathrm{A}V(y) + \mathrm{B}U(x)V(y) + \mathrm{D}]e^{\int \frac{U(x)}{x}dx + \int y^n V(y)\,dy} = 0$

3) $[\mathrm{A}U(x) + \mathrm{B}U(x)V(y) + \mathrm{D}]e^{\int \frac{U(x)}{x}dx + \int y^n V(y)\,dy} = 0$

4) $[\mathrm{A}U(x) + \mathrm{B}U(x)V(y) + \mathrm{D}V(y) + \mathrm{E}]e^{\int \frac{U(x)}{x}dx + \int y^n V(y)\,dy} = 0$

5)$\left[\mathrm{A}U(x) + \mathrm{B}V(y) + \mathrm{C}\left(xU'(x) + U^2(x) - U(x)\right) + \mathrm{E}U(x)V(y)\right]e^{\int \frac{U(x)}{x}dx + \int y^n V(y)\,dy} = 0$

6)$\left[\mathrm{A}U(x)+\mathrm{B}V(y)+\mathrm{C}\left(xU'+U^2(x)-U(x)\right)+\mathrm{E}U(x)V(y)+\mathrm{F}\right]e^{\int\frac{U(x)}{x}dx+\int y^nV(y)\,dy}=0$

since $e^{\int\frac{U(x)}{x}dx+\int y^nV(y)\,dy}\neq 0$. So,
1) $\mathrm{A}U(x)+\mathrm{B}V(y)+\mathrm{D}U(x)V(y)=0$
2) $\mathrm{A}V(y)+\mathrm{B}U(x)V(y)+\mathrm{D}=0$
3) $\mathrm{A}U(x)+\mathrm{B}U(x)V(y)+\mathrm{D}=0$
4) $\mathrm{A}U(x)+\mathrm{B}U(x)V(y)+\mathrm{D}V(y)+\mathrm{E}=0$
5) $\mathrm{A}U(x)+\mathrm{B}V(y)+\mathrm{C}\left(xU'(x)+U^2(x)-U(x)\right)+\mathrm{E}U(x)V(y)=0$
6) $\mathrm{A}U(x)+\mathrm{B}V(y)+\mathrm{C}\left(xU'+U^2(x)-U(x)\right)+\mathrm{E}U(x)V(y)+\mathrm{F}=0$
The last equations are of the first order (O. D. Es.) and contain two independent functions $U(x)$ and $V(y)$.

Solution(1):
Since $\mathrm{A}U(x)+\mathrm{B}V(y)+\mathrm{D}U(x)V(y)=0$, so, we can separate the variables in this equation[9].So:
$\Rightarrow -U(x)=\frac{\mathrm{B}V(y)}{\mathrm{A}+\mathrm{D}V(y)}=-\lambda$
Therefore$U(x)=\lambda$ and $V(y)=\frac{-\lambda\mathrm{A}}{\mathrm{B}+\lambda\mathrm{D}}$.
Then the complete solution is given by:
$Z(x,y)=\mathrm{k}x^{\lambda}e^{-\frac{\lambda\mathrm{A}}{(\mathrm{B}+\lambda\mathrm{D})(n+1)}y^{n+1}}$ when $n\neq -1$
and $Z(x,y)=\mathrm{k}\frac{x^{\lambda}}{y^{\frac{\lambda\mathrm{A}}{\mathrm{B}+\lambda\mathrm{D}}}}$ when $n=-1$
where $\mathrm{k}=e^{g}$ and λ are arbitrary constants.

Solution(2):
Since $\mathrm{A}V(y)+\mathrm{B}U(x)V(y)+\mathrm{D}=0$, this equation is with separate variables. So:
$\Rightarrow -V(y)=\frac{\mathrm{D}}{\mathrm{A}+\mathrm{B}U}=-\lambda$
Therefore$V(y)=\lambda$and $U(x)=-\frac{\lambda\mathrm{A}+\mathrm{D}}{\lambda\mathrm{B}}$.
Hence the complete solution is given by:
$Z(x,y)=\mathrm{k}\frac{e^{\frac{\lambda}{n+1}y^{n+1}}}{x^{\frac{\lambda\mathrm{A}+\mathrm{D}}{\lambda\mathrm{B}}}}$ when $n\neq -1$
and $Z(x,y)=\mathrm{k}\frac{y^{\lambda}}{x^{\frac{\lambda\mathrm{A}+\mathrm{D}}{\lambda\mathrm{B}}}}$ when $n=-1$
where $\mathrm{k}=e^{g}$ and λ are arbitrary constants.

Solution(3):
Since $\mathrm{A}U(x)+\mathrm{B}U(x)V(y)+\mathrm{D}=0$, this equation is with separate variables. Then by the same method in case (2), we get the complete solution which is given by:
$Z(x,y)=\mathrm{k}\frac{e^{\frac{\mathrm{D}-\lambda\mathrm{A}}{\mathrm{B}\lambda(n+1)}y^{n+1}}}{x^{\lambda}}$ when $n\neq -1$
and $Z(x,y)=\mathrm{k}\frac{y^{\frac{\mathrm{D}-\lambda\mathrm{A}}{\mathrm{B}\lambda}}}{x^{\lambda}}$ when $n=-1$
where $\mathrm{k}=e^{g}$ and λ are arbitrary constants.

Solution(4):
Since $\mathrm{A}U(x)+\mathrm{B}U(x)V(y)+\mathrm{D}V(y)+\mathrm{E}=0$, this equation is with separate variables . So:
$\Rightarrow -U(x)=\frac{\mathrm{D}V+\mathrm{E}}{\mathrm{A}+\mathrm{B}V}=-\lambda$
Therefore$U(x)=\lambda$ and $V(y)=\frac{-\lambda\mathrm{A}-\mathrm{E}}{\mathrm{D}+\mathrm{B}\lambda}$
Hence the complete solution is given by:
$Z(x,y)=\mathrm{k}x^{\lambda}e^{\left(\frac{-\lambda\mathrm{A}-\mathrm{E}}{\mathrm{D}+\mathrm{B}\lambda}\right)\frac{y^{n+1}}{n+1}}$ when $n\neq -1$

and $Z(x,y) = \mathrm{k}x^{\lambda} y^{\left(\frac{-\lambda \mathrm{A}-\mathrm{E}}{\mathrm{D}+\mathrm{B}\lambda}\right)}$ when $n = -1$
where $\mathrm{k} = e^{g}$ and λ are arbitrary constants.

Solution(5):
Since $\mathrm{A}U(x) + \mathrm{B}V(y) + \mathrm{C}\big(xU'(x) + U^2(x) - U(x)\big) + \mathrm{E}U(x)V(y) = 0$, this equation is with separate variables. So:
$\Rightarrow \mathrm{A}_1 U + V(\mathrm{B}_1 + \mathrm{E}_1 U) + (xU' + U^2 - U) = 0$,
such that $\mathrm{A}_1 = \frac{\mathrm{A}}{\mathrm{C}}$, $\mathrm{B}_1 = \frac{\mathrm{B}}{\mathrm{C}}$ and $\mathrm{E}_1 = \frac{\mathrm{E}}{\mathrm{C}}$.
$\Rightarrow -V(y) = \frac{\mathrm{A}_1 U + (xU' + U^2 - U)}{\mathrm{B}_1 + \mathrm{E}_1 U} = -\lambda$
$\Rightarrow V(y) = \lambda$ and $xU'(x) + U^2(x) + \mathrm{H}_1 U(x) + \mathrm{H}_2 = 0$
such that $\mathrm{H}_1 = \mathrm{A}_1 + \lambda \mathrm{E}_1 - 1$ and $\mathrm{H}_2 = \lambda \mathrm{B}_1$.
i) If $\mathrm{H}_2 \neq \frac{{\mathrm{H}_1}^2}{4}$, we get $\frac{dU}{U^2(x) + \mathrm{H}_1 U(x) + \mathrm{H}_2} + \frac{dx}{x} = 0$
$\Rightarrow \frac{dU}{(U + \frac{\mathrm{H}_1}{2})^2 + \mathrm{b}^2} + \frac{dx}{x} = 0$; $\mathrm{b}^2 = \mathrm{H}_2 - \frac{{\mathrm{H}_1}^2}{4}$
Therefore $U(x) = -\mathrm{b}\tan(\mathrm{b}\ln(cx)) - \frac{\mathrm{H}_1}{2}$; $cx > 0$
Then the complete solution is given by:

$$Z(x,y) = e^{\int \left(\frac{-\mathrm{b}\tan(\mathrm{b}\ln(cx)) - \frac{\mathrm{H}_1}{2}}{x}\right) dx + \int \lambda y^n \, dy}$$

$$= e^{\ln(\cos(\mathrm{b}\ln(cx))) - \frac{\mathrm{H}_1}{2}\ln(x) + \lambda \frac{y^{n+1}}{n+1} + g} \quad ; \cos(\mathrm{b}\ln(cx)) > 0 \text{ and } cx > 0$$

$$= \mathrm{k}\frac{e^{\lambda \frac{y^{n+1}}{n+1}} \cos(\mathrm{b}\ln(cx))}{x^{\frac{\frac{\mathrm{A}}{\mathrm{C}} + \lambda\frac{\mathrm{E}}{\mathrm{C}} - 1}{2}}} \quad \text{when } n \neq -1 \text{ ; } \mathrm{k} = e^{g} \text{ and } cx > 0$$

and $Z(x,y) = \mathrm{k}\frac{y^{\lambda}\cos(\mathrm{b}\ln(cx))}{x^{\frac{\frac{\mathrm{A}}{\mathrm{C}} + \lambda\frac{\mathrm{E}}{\mathrm{C}} - 1}{2}}}$ when $n = -1$; $\mathrm{k} = e^{g}$ and $cx > 0$
where k, c and λ are arbitrary constants.
ii) If $\mathrm{H}_2 = \frac{{\mathrm{H}_1}^2}{4}$, we get $\frac{dU}{(U + \frac{\mathrm{H}_1}{2})^2} + \frac{dx}{x} = 0$
$\Rightarrow \frac{-1}{U + \frac{\mathrm{H}_1}{2}} + \ln(x) = -\ln c_1$ Therefore $U(x) = \frac{1}{\ln(c_1 x)} - \frac{\mathrm{H}_1}{2}$; $c_1 x > 0$
Then the complete solution is given by:

$$Z(x,y) = e^{\int \left(\frac{\frac{1}{\ln(c_1 x)} - \frac{\mathrm{H}_1}{2}}{x}\right) dx + \int \lambda y^n \, dy} = e^{\int \left(\frac{1}{x\ln(c_1 x)} - \frac{\mathrm{H}_1}{2x}\right) dx + \int \lambda y^n \, dy}$$

$$= \mathrm{k}\frac{\ln(c_1 x) e^{\lambda \frac{y^{n+1}}{n+1}}}{x^{\frac{\frac{\mathrm{A}}{\mathrm{C}} + \lambda\frac{\mathrm{E}}{\mathrm{C}} - 1}{2}}} \quad \text{when } n \neq -1 \text{ ; } \mathrm{k} = e^{g} \text{ and } c_1 x > 0$$

and $Z(x,y) = \mathrm{k}\frac{y^{\lambda}\ln(c_1 x)}{x^{\frac{\frac{\mathrm{A}}{\mathrm{C}} + \lambda\frac{\mathrm{E}}{\mathrm{C}} - 1}{2}}}$ when $n = -1$; $\mathrm{k} = e^{g}$ and $c_1 x > 0$
where k, c_1 and λ are arbitrary constants.
Solution(6):
Since $\mathrm{A}U(x) + \mathrm{B}V(y) + \mathrm{C}\big(xU' + U^2(x) - U(x)\big) + \mathrm{E}U(x)V(y) + \mathrm{F} = 0$
this equation is with separate variables. So:
$\Rightarrow \mathrm{A}_1 U + V(\mathrm{B}_1 + \mathrm{E}_1 U) + (xU' + U^2 - U) + \mathrm{F}_1 = 0$,
such that $\mathrm{A}_1 = \frac{\mathrm{A}}{\mathrm{C}}$, $\mathrm{B}_1 = \frac{\mathrm{B}}{\mathrm{C}}$, $\mathrm{E}_1 = \frac{\mathrm{E}}{\mathrm{C}}$ and $\mathrm{F}_1 = \frac{\mathrm{F}}{\mathrm{C}}$.
$\Rightarrow -V(y) = \frac{\mathrm{A}_1 U + (xU' + U^2 - U) + \mathrm{F}_1}{\mathrm{B}_1 + \mathrm{E}_1 U} = -\lambda$
$\Rightarrow V(y) = \lambda$ and $xU'(x) + U^2(x) + \mathrm{H}_1 U(x) + \mathrm{H}_2 = 0$,

such that $H_1 = A_1 + \lambda E_1 + 1$ and $H_2 = \lambda B_1 + F_1$.

i) If $H_2 \neq \frac{{H_1}^2}{4}$, we get $U(x) = -b\tan(b\ln(c_2 x)) - \frac{H_1}{2}$; $c_2 > 0$ and $x > 0$

Then the complete solution is given by:

$$Z(x,y) = e^{\int\left(\frac{-b\tan(b\ln(c_2x)) - \frac{H_1}{2}}{x}\right)dx + \int \lambda y^n\,dy}$$

$$= e^{\ln(\cos(b\ln(c_2x))) - \frac{H_1}{2}\ln(x) + \lambda\frac{y^{n+1}}{n+1} + g} \quad ; \cos(b\ln(c_2x)) > 0 \text{ and } c_2x > 0$$

$$= k\frac{e^{\lambda\frac{y^{n+1}}{n+1}}\cos(b\ln(c_2x))}{x^{\frac{\frac{A}{C}+\lambda\frac{E}{C}-1}{2}}} \quad \text{when } n \neq -1 \text{ ; } k = e^g \text{ and } c_2x > 0$$

and $Z(x,y) = k\frac{y^{\lambda}\cos(b\ln(c_2x))}{x^{\frac{\frac{A}{C}+\lambda\frac{E}{C}-1}{2}}}$ when $n = -1$; $k = e^g$ and $c_2x > 0$

where k, c_2 and λ are arbitrary constants.

ii) If $H_2 = \frac{{H_1}^2}{4}$, we get $U(x) = \frac{1}{\ln(c_3x)} - \frac{H_1}{2}$; $c_3x > 0$

Hence the complete solution is given by:

$$Z(x,y) = e^{\int\left(\frac{\frac{1}{\ln(c_3x)} - \frac{H_1}{2}}{x}\right)dx + \int \lambda y^n\,dy} = e^{\int\left(\frac{1}{x\ln(c_3x)} - \frac{H_1}{2x}\right)dx + \int \lambda y^n\,dy}$$

$$= k\frac{\ln(c_3x)e^{\lambda\frac{y^{n+1}}{n+1}}}{x^{\frac{\frac{A}{C}+\lambda\frac{E}{C}-1}{2}}} \quad \text{when } n \neq -1 \text{ ; } k = e^g \text{ and } c_3x > 0$$

and $Z(x,y) = k\frac{y^{\lambda}\ln(c_3x)}{x^{\frac{\frac{A}{C}+\lambda\frac{E}{C}-1}{2}}}$ when $n \neq -1$; $k = e^g$ and $c_3x > 0$

where k, c_3 and λ are arbitrary constants.

In kind (2), we search about functions $U(x) \neq 0$ and $V(y) \neq 0$ such that the assumption

$$Z(x,y) = e^{\int U(x)dx + \int V(y)dy} \quad \text{... (1.3)}$$

gives the complete solution of the equations in kind (2). By finding Z_x, Z_y, Z_{xx}, Z_{yy} and Z_{xy} from (1.3), we get

$$Z_x = U(x)e^{\int U(x)dx + \int V(y)dy}$$

$$Z_{xx} = \left(U'(x) + U^2(x)\right)e^{\int U(x)dx + \int V(y)dy}$$

$$Z_y = V(y)e^{\int U(x)dx + \int V(y)dy}$$

$$Z_{yy} = \left(V'(y) + V^2(y)\right)e^{\int U(x)dx + \int V(y)dy}$$

$$Z_{xy} = U(x)V(y)e^{\int U(x)dx + \int V(y)dy}$$

by substituting Z_x, Z_y, Z_{xx}, Z_{yy} and Z_{xy} in the equations of kind(2), we get

1) $A\tan y\left(U'(x) + U^2(x)\right) + B\,U(x)V(y) = 0$

2) $A\left(U'(x) + U^2(x)\right) + B\,U(x) + \frac{DV(y)}{f(y)} = 0$

3) $A\left(V'(y) + V^2(y)\right) + B\,V(y) + \frac{DU(x)+FZ}{f(x)} = 0$

The last equations are of the first order (O. D. Es.) and contain two independent functions $U(x)$ and $V(y)$.

<u>Solution(1):</u>

Since $A\tan y\left(U'(x) + U^2(x)\right) + B\,U(x)V(y) = 0$, this equation is separate variables. So:

$$\Rightarrow \frac{U'(x)+U^2(x)}{U(x)} = \frac{-B}{A\tan y}V(y) = -\lambda$$

$$\Rightarrow V(y) = \frac{\lambda A}{B} \text{ and } U'(x) + U^2(x) + \lambda U(x) = 0 \quad \text{...(1.4)}$$

The equation(1.4) is similar to Bernoulli equation[2].

Then $U(x)=\frac{e^{-\lambda x}}{\int e^{-\lambda x}dx}$, [2]

Hence the complete solution is given by:

$Z(x,y)=(\cos(y))^{-\frac{A\lambda}{B}}(d_1e^{-\lambda x}+d_2)$

where $d_1=\frac{-k}{\lambda}$, $d_2=\frac{-km}{\lambda}$, λ, k and m are arbitrary constants.

Solution(2):

Since $A(U'(x)+U^2(x))+B\,U(x)+\frac{DV(y)}{f(y)}=0$, this equation is with separate variables. So:

$\Rightarrow(U'(x)+U^2(x)+\lambda U(x))=-\frac{DV(y)}{f(y)}=-\lambda$

Therefore $V(y)=\frac{A\lambda}{D}f(y)$ and

$U'(x)+U^2(x)+B_1U(x)+B_2=0$ ...(1.5)

such that $B_1=\frac{B}{A}$ and $B_2=\lambda$.

The equation (1.5) is similar to Riccati's equation [9][10].

i) If $B_2\neq\frac{{B_1}^2}{4}$, We get $U(x)=b\tan(f-bx)-\frac{B_1}{2}$; $b^2=B_2-\frac{{B_1}^2}{4}$ and f is constant.

Then the complete solution is given by:

$Z(x,y)=F(y)e^{\frac{A\lambda}{D}-\frac{B}{2A}x}\left[d_1\cos\sqrt{\lambda-\frac{B^2}{4A^2}}x+d_2\sin\sqrt{\lambda-\frac{B^2}{4A^2}}x\right]$

where k , $d_1=k\cos f$, $d_2=k\sin f$ and λ are arbitrary constants and $F(y)=e^{\int f(y)dy}$ is an arbitrary function of y.

ii) If $B_2=\frac{{B_1}^2}{4}$, we get $U(x)=\frac{1}{x-c_1}-\frac{B_1}{2}$. Then the complete solution is given by:

$Z(x,y)=k\,F(y)e^{\frac{A\lambda}{D}-\frac{B}{2A}x}(x-c_1)$

where k , c_1 and λ are arbitrary constants, and $F(y)=e^{\int f(y)dy}$ is an arbitrary function of y.

Solution(3):

Since $A(V'(y)+V^2(y))+B\,V(y)+\frac{DU(x)+F}{f(x)}=0$, this equation is with separate variables[10].we can solve it as follows:

$\Rightarrow(V'(y)+V^2(y))+B\,V(y)=-\frac{DU(x)+F}{f(x)}=-\lambda$

Therefore $U(x)=\frac{\lambda f(x)-F}{D}$ and

$V'(y)+V^2(y)+B_1V(y)+B_2=0$ ...(1.6)

such that $B_1=\frac{B}{A}$ and $B_2=\frac{\lambda}{A}$ are arbitrary constants.

i) If $B_2\neq\frac{{B_1}^2}{4}$, we get $V(y)=b\tan(f-by)-\frac{B_1}{2}$; $b^2=B_2-\frac{{B_1}^2}{4}$ and f is constant.

Then the complete solution is given by:

$Z(x,y)=G(x)e^{\frac{\lambda}{D}-Fx-\frac{B}{2A}y}\left[d_1\cos\sqrt{\frac{\lambda}{A}-\frac{B^2}{4A^2}}y+d_2\sin\sqrt{\frac{\lambda}{A}-\frac{B^2}{4A^2}}y\right]$

where $d_1=k\cos f$, $d_2=k\sin f$ and λ are arbitrary constants, and $G(x)=e^{\int f(x)dx}$ is an arbitrary function of x.

ii) If $B_2=\frac{{B_1}^2}{4}$, we get $U(x)=\frac{1}{y-c_2}-\frac{B_1}{2}$. Then the complete solution is given by:

$Z(x,y)=k\,G(x)e^{\frac{\lambda}{D}-Fx-\frac{B}{2A}y}(y-c_2)$

where k , c_2 and λ are arbitrary constants, and $G(x)=e^{\int f(x)dx}$ is an arbitrary function of x.

4.Examples

In this section, some illustrative examples are given for completeness purposes .

Example (3.1):To solve the (P. D. E):-

$3xZ_x + 5\frac{Z_y}{y^n} + 2\frac{x}{y^n}Z_{xy} = 0$ by using the (kind(1), 1).Then the complete solution is given by:

$Z(x,y) = \mathrm{k}x^{\lambda}e^{-\frac{3\lambda}{(5+\lambda 2)(n+1)}y^{n+1}}$ when $n \neq -1$

and $Z(x,y) = \mathrm{k}x^{\lambda}\mathrm{e}^{-\frac{\lambda A}{(3+\lambda 2)}\ln(y)} = \mathrm{k}\frac{x^{\lambda}}{y^{\frac{3\lambda}{5+2\lambda}}}$ when $n = -1$

where $\mathrm{k} = \mathrm{e}^{g}$ and λ are arbitrary constants.

Example (3.2):To solve the (P. D. E):-

$2xZ_x + 5\frac{x}{y^n}Z_{xy} + 7\frac{Z_y}{y^n} + Z = 0$by using the (kind(1), 4). Then the complete solution is given by:

$Z(x,y) = \mathrm{k}x^{\lambda}e^{\left(\frac{-2\lambda-1}{7+5\lambda}\right)\frac{y^{n+1}}{n+1}}$ when $n \neq -1$

and $Z(x,y) = \mathrm{k}x^{\lambda}y^{\left(\frac{-2\lambda-1}{7+5\lambda}\right)}$ when $n = -1$

where $\mathrm{k} = e^{g}$ and λ are arbitrary constants.

Example (3.3):To solve the (P. D. E):-

$6\tan y\, Z_{xx} + 5Z_{xy} = 0$ by using the (kind(2), 1).Then the complete solution is given by:

$Z(x,y) = (\cos(y))^{-\frac{6\lambda}{5}}\left(\mathrm{d}_1\mathrm{e}^{-\lambda x} + \mathrm{d}_2\right)$

where $\mathrm{d}_1 = \frac{-\mathrm{k}}{\lambda}$, $\mathrm{d}_2 = \frac{-\mathrm{km}}{\lambda}$, λ, k and m are arbitrary constants.

Example(3.4): To solve the P.D.E. :- $3xZ_x + \sqrt{5}\frac{x}{y^n}Z_{xy} + Z = 0$, ($y \neq 0$ if $n \in \mathbb{R}^+$) $\wedge$ $n \in \mathbb{R}$.

$A = 3$, $B = \sqrt{5}$ and $D = 1$, we use the formula (kind(1),3), then the complete solution is given by:

$Z(x,y) = \mathrm{k}\,\mathrm{e}^{\frac{1-3\lambda}{\sqrt{5}\,\lambda(n+1)}y^{n+1}}x^{-\lambda}$ when $n \neq -1$

and $Z(x,y) = \mathrm{k}y^{\frac{1-3\lambda}{\sqrt{5}\,\lambda}}x^{-\lambda}$ when $n = -1$

where $\mathrm{k} = e^{g}$ and $\lambda \neq 0$ are arbitrary constants.

Example(3.5): To solve the P.D.E. :-

$2xZ_x + 5\frac{Z_y}{y^n} + x^2Z_{xx} + 3\frac{x}{y^n}Z_{xy} + 2Z = 0$, $A = 2$, $B = 5$, $C = 1$, $E = 3$ and $F = 2$, ($y \neq 0$ if $n \in \mathbb{R}^+$) andn $\in \mathbb{R}$, we use the formula (kind(1),6), Then the complete solution is given by:

$Z(x,y) = \mathrm{k}\,\mathrm{e}^{\lambda\frac{y^{n+1}}{n+1}}\cos\left(\sqrt{(5\lambda+2) - \frac{(3+\lambda 3)^2}{4}}\ln(c_2 x)\right)x^{-\frac{3+3\lambda}{2}}$ when $n \neq -1$and $c_2 x > 0$

and $Z(x,y) = \mathrm{k}\,y^{\lambda}\cos\left(\sqrt{(5\lambda+2) - \frac{(3+\lambda 3)^2}{4}}\ln(c_2 x)\right)x^{-\frac{3+3\lambda}{2}}$ when $n = -1$and $c_2 x > 0$

where $\mathrm{k} = \mathrm{e}^{g}$, c_2 and λ are arbitrary constants.

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