Generating $3 \times \frac{n}{3}$ - Contingency Tables Using The Action of Dihedral Group D_n , n is Multiple of 6

توليد $3 \times \frac{n}{3}$ جداول طوارئ بأستخدام تاثير زمرة ثنائية السطوح D_n , n من مضاعفات 3

Husein Hadi Abbass
Department of Mathematics, Faculty of Education for Girls
University of Kufa, Iraq
E-mail: hussienh.abbas@uokufa.edu.iq
Hussein Salman Mohammed Hussein
Department of Mathematics, Faculty of Education for Girls
University of Kufa, Iraq
E-mail: husseinsalman88@yahoo.com

بحـــــث مستـــــل

Abstract

In this paper, we find six types of $\frac{n^2-3n}{3} \times 3 \times \frac{n}{3}$ - contingency tables with fixed two dimensional marginal using the action of largest subgroup H of dihedral group D_n , n is multiple of 6 and the Markov basis **B** such that **B** is H-invariant, where **B** is found by H. H. Abbass and H. S. Mohammed Hussein [7].

Keywords linear transformation, contingency table, action, dihedral group, Markov basis, toric ideal.

الخلاصة

في هذا البحث نحن نجد 6 أنواع من $\frac{n^2-3n}{3} \times 3 \times \frac{n}{3}$ جداول طوارئ مع ثبوت البعدين الجانبيين باستخدام تأثير اكبر زمرة جزئية H من زمرة ثنائية السطوح D_n ، n من مقسمات 6 و أساس ماركوف **B** بحيث **B** هي غير متغيره بالنسبة ل H عندما **B** وجدت بواسطة حسين هادي عباس وحسين سلمان محمد حسين[7]

1. Introduction

A Contingency table is a matrix of nonnegative integers with prescribed positive row and column sums [6].

Let I be a finite set $n = |I|$ elements, we call an element of I a cell and denoted by $\boldsymbol{i} \in I$. $\boldsymbol{i}$ is often multi-index $\boldsymbol{i} = i_1 \ldots i_l$. A non-negative integer $x_{\boldsymbol{i}} \in \mathbb{N} = \{1, 2, \ldots\}$ denotes the frequency of a cell $\boldsymbol{i}$. The set of frequencies is called ***a contingency table*** and denoted as $\boldsymbol{x} = \{x_{\boldsymbol{i}}\}_{\boldsymbol{i} \in I}$, with an appropriate ordering of the cell, we treat a contingency table $\boldsymbol{x} = \{x_{\boldsymbol{i}}\}_{\boldsymbol{i} \in I} \in \mathbb{N}^n$ as a n-dimensional column vector of non-negative integers. Not that a contingency table can also be considered as a function from I to $\mathbb{N}$ defined as $\boldsymbol{i} \mapsto x_{\boldsymbol{i}}$, The L_1-norm of $\boldsymbol{x} \in \mathbb{N}^n$ is called the ***sample size*** and denoted as $|\boldsymbol{x}| = \sum_{\boldsymbol{i} \in I} x_{\boldsymbol{i}}$. We will denote $\mathbb{Z}$ be the set of integer numbers, also we denote to the $a_j \in \mathbb{Z}^n, j = 1, \ldots, v$, as fixed column vectors consisting of integers. A v-dimantional column vector $\boldsymbol{t} = (t_1, \ldots, t_v)' \in \mathbb{Z}^v$ as $t_j = a'_j \boldsymbol{x}, j = 1, \ldots, v$. Here $'$ denotes the transpose of a vector or matrix. We also define a $v \times p$ matrix A, with its j-row being a'_j given by $A = \begin{bmatrix} a'_1 \\ \vdots \\ a'_v \end{bmatrix}$, and if

$\boldsymbol{t} = A\boldsymbol{x}$ is a v-dimensional column vector, we define the set $T = \{\boldsymbol{t}: \boldsymbol{t} = A\boldsymbol{x}, \boldsymbol{x} \in \mathbb{N}^n\} = A\mathbb{N}^n \subset \mathbb{Z}^v$, where $\mathbb{N}$ is the set of natural numbers. In typical situations of a statistical theory, $\boldsymbol{t}$ is *sufficient statistic* for the nuisance parameter. The set of $\boldsymbol{x}$'s for a given $\boldsymbol{t}$, $A^{-1}[\boldsymbol{t}] = \{\boldsymbol{x} \in \mathbb{N}^n: A\boldsymbol{x} = \boldsymbol{t}\}$ ($\boldsymbol{t}$-fibers), is considered for performing *similar tests*, for the case of the independence model of two–way contingency tables, for example, $\boldsymbol{t}$ is the row sums and column sums of $\boldsymbol{x}$, and $A^{-1}[\boldsymbol{t}]$ is the set of $\boldsymbol{x}'$s with the same row sums and column sums to $\boldsymbol{t}$. The set of $\boldsymbol{t}$-fibers gives a decomposition of $\mathbb{N}^n$. An important observation is that $\boldsymbol{t}$-fiber depends on given only through its kernel, ker (A). For different A's with the same kernel, the set of $\boldsymbol{t}$-fibers are the same. In fact, if we define $\boldsymbol{x}_1 \sim \boldsymbol{x}_2 \Leftrightarrow \boldsymbol{x}_1 - \boldsymbol{x}_2 \in \ker(A)$. This relation is an equivalence relation and $\mathbb{N}^n$ is partitioned into disjoint equivalence classes. The set of $\boldsymbol{t}$-fibers is simply the set of these equivalence classes. Furthermore, $\boldsymbol{t}$ may be considered as labels of these equivalence classes, A n-dimensional column vector of integers $\boldsymbol{z} = \{z_i\}_{i \in I} \in \mathbb{Z}^n$ is called a move if it is in the kernel of A, i.e. $A\boldsymbol{z} = 0$ [3]. A set of finite moves B is called **Markov basis** if for all $\boldsymbol{t}$, $A^{-1}[\boldsymbol{t}]$ constitutes one B equivalence class [1]. If a group G acts on $A^{-1}[\boldsymbol{t}]$ on the left, B is a Markov basis, and $G(B) = \{gz: z \in B, g \in G\}, B$ is called invariant under G (or G- invariant) if $G(B) = B$. We will denote to the polynomials in the p indeterminates (polynomial variables) $p_1, p_2, \dots, p_p$ over the complex field $₵$ by either $₵[p_1, p_2, \dots, p_p]$ or $₵[P]$, $P = (p_1, p_2, \dots, p_p)$. Let $A: \mathbb{Z}^n \to \mathbb{Z}^d$ be a linear transformation, the toric ideal $I_A$is the ideal $< P^u - P^v: u , v \in \mathbb{N}^n, A(u) = A(v) > \subseteq ₵[P_1, \dots, P_p]$ where $P^u = {P_1}^{u_1} {P_2}^{u_2} \dots {P_p}^{u_p}$[6].

In 2008, A. Takemura, and S. Aoki defined an invariant Markov basis for a connected Markov chain over the set of contingency tables with fixed marginals and derived some characterizations of minimality of the invariant basis, they give a necessary and sufficient condition for uniqueness of minimal invariant Markov bases. By considering the invariance, Markov bases can be presented very concisely. As an example, also present minimal invariant Markov bases for all 2×2×2×2 hierarchical models [3]. In the same year, A. Takemura, and S. Aoki defined the largest group of invariance for a given toric ideal and the associated Markov basis. Reduction by invariance leads to a concise description of an invariant Markov basis and a sampling scheme in terms of the group and a list of representative elements from the orbits of the Markov basis, they also give explicit forms of the largest group of invariance for several standard statistical problems [4]. In 2014 H. H. Abbass and H. S. Mohammed Hussein found a Markov basis and toric ideals for $\frac{n^2-3n}{3} \times 3 \times \frac{n}{3}$ - contingency tables with fixed two dimensional marginals, n is a multiple of 3 greater than or equal 6 [7].

Contingency tables are used in statistics to store date from sample surveys. One of related problems for a survey of contingency tables is how to generate tables from the set of all non-negative $K_1 \times K_2$ integer tables with given row and column sums. In this paper, we find the largest subgroup H of the dihedral group such that the Markov basis **B** is H-invariant to generate five subsets of $\boldsymbol{t}$-fibers each subset contains $\frac{n^2-3n}{3}$ - contingency tables.

2. Preliminaries

In this section, we review some basic definitions and notations of contingency tables, moves, Markov basis, dihedral group, and action of group on the set that we need in our work.

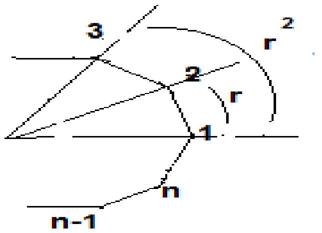
Definition (2.1) [2].

Let n be a positive integer greater than or equal 3. The group of all symmetries of the regular polygon with n sides, including both rotations and reflections, is called **dihedral group** and denoted by D_n . If we center the regular polygon at origin then the elements of the dihedral group acts as linear transformation of the plane. Lets us represent the elements of D_n as matrix, with composition multiplication. Dihedral groups are among simplest examples of finite groups and they play an important role in group theory, geometry, and chemistry. The set of rotations is generated by r -

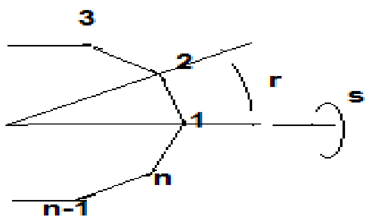
counterclockwise rotation with angle $2\pi / n$ of order n, and the set of reflections is of order 2 and every element sr^j generates $\{e, sr^{\,j}\}$, where e is the identity element in D_n . The 2n elements in D_n can be written as:$\{e\,, r\,, r^2\,,\ \ldots\,, r^{n-1}\,, s\,, sr\,, sr^2\,, \ldots\,, sr^{\,n-1}\}$.In general, we can write D_n as: D_n=$\{\ s^j\ r^k : 0 \le k \le n-1\ ,\ 0 \le \mathrm{j} \le 1\}$ which has the following properties: $r^{\,n} = 1$, $sr^k s = r^{-k}$, $(sr^k)^2 = 1\ ,\ for\ all\ 0\ \le\ k\ \le\ n-1$ .The composition of two elements of the D_n is given by $r^i r^j = r^{i+j}$, $r^i sr^j = sr^{j-i}$, $sr^i r^j = sr^{i+j}$, $sr^i s\, r^j = r^{j-i}$.

Remark (2.2) [2].

If we label the vertices (of the regular n-gon) 1 to n in a counterclockwise direction around n-gon then the elements of D_n can be written as permutations of vertices, let r be a counterclockwise rotation, and let s be the reflection of the n-gon about an axis through the center and vertex 1 , as indicated in below . The element r generates C_n the cyclic group of order n which is a normal cyclic subgroup of D_n . In all cases, addition and subtraction should be performed using modular arithmetic with modulus n.



Elements of C_n



Elements of D_n

Any symmetry will fix the origin and is determined by the image of two adjacent vertices , say 1 and 2 .The vertex 1 can be taken to any of n vertices and then the vertex 2 must be taken to one of the two vertices adjacent to the image of 1. Hence, D_n is a non abelian group of order $2n$ generated by r and s.

Now, we give some concepts about the action of a group on a set that we use later.

Definition (2.3) [3].

Let G be a group and W be a set. A left action of G in W is a function from $G \times W$ into W, usually denote by $(g, w) \to gw \in W$ such that $g(hw) = (gh)w$ and $ew = w$ for all $g, h \in G$ and $w \in W$where e is the identity element of G. We also say that G acts on W on the left.

Definition (2.4) [3].

Let a group G act on a set W, and $U \subseteq W$, $G_{(U)} = \{g\colon gu = u, \forall\, u \in U\}$ is called the pointwise stabilizer of U.

Definition (2.5) [3].

Let a group G acts on a set W, $U \subseteq W$, and $GU = \{gu\colon u \in U, g \in G\}$. We call U invariant under G (or G-invariant) if $GU = U$.

Remark (2.6) [1].

For a move $\boldsymbol{z}$, the positive part $\boldsymbol{z}^+ = \{z^+{}_i\}_{i\in I}$ and the negative part $\boldsymbol{z}^- = \{z^-{}_i\}_{i\in I}$ are defined by $z^+{}_i = \max(z_i, 0)$, $z^-{}_i = \max(-z_i, 0)$, respectively, Then $\boldsymbol{z} = \boldsymbol{z}^+ - \boldsymbol{z}^-$ and $\boldsymbol{z}^+, \boldsymbol{z}^- \in \mathbb{N}^n$. moreover, $\boldsymbol{z}^+$ and $\boldsymbol{z}^-$ are in the same $\boldsymbol{t}$-fiber, i.e., $\boldsymbol{z}^+, \boldsymbol{z}^- \in A^{-1}[\boldsymbol{t}]$ for $\boldsymbol{t} = A\boldsymbol{z}^+ = A\boldsymbol{z}^-$. We define the *degree* of $\boldsymbol{z}$ **as the sample size of** $\boldsymbol{z}^{+}$or $(\boldsymbol{z}^{-})$ and denote it by $\deg(\boldsymbol{z}) = |\boldsymbol{z}^{+}| = |z^{-}|$. In the following we denote the set of moves (for a given A) by $M = M_A = \mathbb{Z}^n \cap \ker(A)$.

Theorem (2.7) [1].

A collection of binomials $\{p^{z^+} - p^{z^-} : z \in B\} \subset I_A$ is generating set of toric ideal I_A if and only if $\pm B$ is a Markov basis for A.

In [7] the Authors gave a Markov basis $\boldsymbol{B}$ for $\frac{n^2-3n}{3} \times 3 \times \frac{n}{3}$ - contingency tables with fixed two dimensional marginal as follows:

Remark (2.8) [7].

Let n be a multiple of 3 such that $n \geq 6$, and let $\boldsymbol{x}_j \in A^{-1}[t]$, $j = 1, \dots, k$ be the representative elements of the set of $3 \times \frac{n}{3}$ −contingency tables and $\mathbf{B} = \{\boldsymbol{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_k\}$ such that each $\boldsymbol{z}_j$,$j = 1,2, \dots k$, is a matrix of dimension $3 \times \frac{n}{3}$ either has two columns $(1, -1, 0)', (-1, 1, 0)'$ ($(1, 0, -1)'$, $(-1, 0, 1)'$ or either $(0,1,-1)'$, $(0, -1, 1)'$) and the other columns are zero denoted by $+\boldsymbol{z}_j$, or it has two columns $(-1,1,0)', (1,-1,0)'$ ($(-1,0,1)'$, $(1,0,-1)'$ or $(0,-1,1)'$, $(0,1,-1)'$) and the other columns are zero denoted by $-\boldsymbol{z}_j$,like

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$$
$$, \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}.$$

Also, we can write all elements of $\mathbf{B}$ as one-dimensional column vector as follows:
$\boldsymbol{z_j} = (z_1, \dots, z_n)', j = 1, \dots, k$ and $z_t = 1 \; or - 1 \; or \; 0$ such that
$If \; t = 1, 2, \dots, \frac{n}{3}$

$$z_t = \begin{cases} 1 & if \; z_{t+\frac{n}{3}} + z_{t+\frac{2n}{3}} = -1 \; and \sum_{\substack{i=1 \\ i \neq t}}^{\frac{n}{3}} z_i = -1 \\ -1 & if \; z_{t+\frac{n}{3}} + z_{t+\frac{2n}{3}} = 1 \; and \sum_{\substack{i=1 \\ i \neq t}}^{\frac{n}{3}} z_i = 1 \\ 0 & if \; z_{t+\frac{n}{3}} + z_{t+\frac{2n}{3}} = 0 \; and \sum_{\substack{i=1 \\ i \neq t}}^{\frac{n}{3}} z_i = 0 \end{cases} \qquad (1)$$

If $t = \frac{n}{3} + 1, \frac{n}{3} + 2, \dots, \frac{2n}{3}$

$$z_t = \begin{cases} 1 & if \; z_{t-\frac{n}{3}} + z_{t+\frac{n}{3}} = -1 \; and \sum_{\substack{i=\frac{n}{3}+1 \\ i \neq t}}^{\frac{2n}{3}} z_i = -1 \\ -1 & if \; z_{t-\frac{n}{3}} + z_{t+\frac{n}{3}} = 1 \; and \sum_{\substack{i=\frac{n}{3}+1 \\ i \neq t}}^{\frac{2n}{3}} z_i = 1 \\ 0 & if \; z_{t-\frac{n}{3}} + z_{t+\frac{n}{3}} = 0 \; and \sum_{\substack{i=\frac{n}{3}+1 \\ i \neq t}}^{\frac{2n}{3}} z_i = 0 \end{cases} \qquad (2)$$

If $t = \frac{2n}{3} + 1, \frac{2n}{3} + 2, \dots, n$

$$z_t = \begin{cases} 1 & if\ z_{t-\frac{2n}{3}} + z_{t-\frac{n}{3}} = -1\ and \sum_{\substack{i=\frac{2n}{3}+1 \\ i\neq t}}^{n} z_i = -1 \\ -1 & if\ z_{t-\frac{2n}{3}} + z_{t-\frac{n}{3}} = 1\ and \sum_{\substack{i=\frac{n}{3}+1 \\ i\neq t}}^{n} z_i = 1 \\ 0 & if\ z_{t-\frac{2n}{3}} + z_{t-\frac{n}{3}} = 0\ and \sum_{\substack{i=\frac{n}{3}+1 \\ i\neq t}}^{n} z_i = 0 \end{cases} \qquad (3)$$

Theorem (2.9) [7].

The number of elements in **B** equal to $\frac{n^2-3n}{3}$.

Remark (2.10) [7].

Given a contingency table $\boldsymbol{x} = (x_1, x_2, \dots, x_n)'$ **,** the entry of the matrix A in the column indexed by $x_1, x_2, \dots, x_n$ respectively and its rows indexed by $\sum_{i=1}^{\frac{n}{3}} x_i, \sum_{i=\frac{n}{3}+1}^{\frac{2n}{3}} x_i, \sum_{i=\frac{2n}{3}+1}^{n} x_i, x_1 + x_{\frac{n}{3}+1} + x_{\frac{2n}{3}+1}, x_2 + x_{\frac{n}{3}+2} + x_{\frac{2n}{3}+2}, \dots, x_{\frac{n}{3}} + x_{\frac{2n}{3}} + x_n$ respectively. The entry in the column indexed by x_i in the matrix **A** will be equal to one, if x_i appears in the index of its row, and otherwise it will be zero. Then

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}_{\frac{n+9}{3} \times n}$$

Theorem (2.11) [7].

The set $\mathbf{B} = \{\boldsymbol{z}_1, \dots, \boldsymbol{z}_{\frac{n^2-3n}{3}}\}$ is a set of moves.

Remark (2.12) [7].

We can find $\frac{n^2-3n}{3}$ contingency table $x_0, x_1, \dots, x_{\frac{n^2-3n}{3}-1}$ in $A^{-1}[\boldsymbol{t}]$ (each of them is $3 \times \frac{n}{3}$ - contingency table) from the elements of the set $\mathbf{B} = \{\boldsymbol{z}_1, \boldsymbol{z}_2, \dots, \boldsymbol{z}_{\frac{n^2-3n}{3}}\}$, where $\boldsymbol{x}_i = \boldsymbol{x}_{i-1} + \boldsymbol{z}_i$, for all $i = 1, 2, \dots, \frac{n^2-3n}{3} - 1$ and $\boldsymbol{x}_0 = \boldsymbol{x}_{\frac{n^2-3n}{3}-1} + \boldsymbol{z}_{\frac{n^2-3n}{3}}$.

Corollary (2.13) [7].

The set **B** of moves in theorem (2.11) is a Markov basis**.**

Corollary (2.14) [7].

The toric ideal I_A for $\frac{n^2-3n}{3} \times 3 \times \frac{n}{3}$ - contingency tables are $I_A =< P_{i+l} P_{j+k} - P_{j+l} P_{i+k}$: $i, j = 1, 2, \dots, \frac{n}{3}$ and $l, k = 0, \frac{n}{3}, \frac{2n}{3}$, such that $i < j$ and $l < k >$ $\subset$ ₵$[P_1, P_2, \dots, P_n]$.

3. The Main Results

Let n be multiple of 6, $\boldsymbol{x}_j \in A^{-1}[t]$, $j = 0, \dots, \frac{n^2-3n}{3} - 1$ be representative elements of the set of $3 \times \frac{n}{3}$ −contingency tables , and H be the subgroup $\{e, r^{\frac{n}{3}}, r^{\frac{2n}{3}}, sr, sr^{\frac{n}{3}+1}, sr^{\frac{2n}{3}+1}\}$ of dihedral group D_n where $r = (1\ 2\ 3\ \dots n)$ and $s = (2\ \ n)(3\ \ n-1) \dots (\frac{n}{2}\ \ \frac{n}{2}+2)$. we will write, each $g \in D_n$ as a $n \times n$ permutation matrix $T_g = \{p_{ij}\} = \{\delta_i, g(i)\}$, where δ is the Kronecker's delta such that $T_{g_1 . g_2} = T_{g_1} T_{g_2}$ for $g_1, g_2 \in D_{n,}$ and$T_{g^{-1}} = T^t{}_g$. Then the identity matrix of order n is denoted by E_n for the unit element $e \in D_n$.

Now, we consider a left action of dihedral group D_n , $n = |I|$, on $A^{-1}[t]$ the set of $\frac{n^2-3n}{3} \times 3 \times \frac{n}{3}$ -contingency tables, and the action of dihedral group D_n on the set of Markov basis **B**.

Definition (3.1). Let $A^{-1}[t]$ be the set of $3 \times \frac{n}{3}$ −contingency tables A left action of D_n on $A^{-1}[t]$ is a function from $D_n \times A^{-1}[t]$ into $A^{-1}[t]$ such that $(g, \boldsymbol{x}) \to g\boldsymbol{x} = T_g\boldsymbol{x} \in A^{-1}[t]$.

Remark (3.2). A left action of $\boldsymbol{D_n}$ on the set of all n-diminsional column vectors of integers $\mathbb{Z}^{\boldsymbol{n}}$ is a function $(g, \boldsymbol{v}) \to g\boldsymbol{v} = T_g\boldsymbol{v} \in \mathbb{Z}^n$ of $D_n \times \mathbb{Z}^n$ into $\mathbb{Z}^{\boldsymbol{n}}$**,** where T is a permutation matrix representation of D_n and T_g is the permutation matrix of g, $\mathrm{A}^{-1}[\boldsymbol{t}]$, $\mathbf{B} \subseteq \mathbb{Z}^{\boldsymbol{n}}$ when the element of $\mathrm{A}^{-1}[\boldsymbol{t}]$ and **B** tread as all n-dimensional column vectors**.** If $\boldsymbol{x} \in \mathrm{A}^{-1}[\boldsymbol{t}]$**,** $\boldsymbol{z} \in \mathbf{B}$**,** and $g \in G$**,** then $T_g\boldsymbol{x}$**,** $T_g\boldsymbol{z} \in \mathbb{Z}^n$ but $T_g\boldsymbol{x}$ may not be in $\mathrm{A}^{-1}[\boldsymbol{t}]$ and $T_g\boldsymbol{z}$ may not be in **B** as in the following example.

Example (3.3). Consider 3 × 3- contingency table

$\boldsymbol{x} =$

1	4	5
2	1	3
4	2	6
7	7	14

$\boldsymbol{x}$ can be represented as 6-dimensional column vector of non negative integer $\boldsymbol{x} = (1,4,2,1,4,2)' \in \mathbb{N}^6$. Then $\boldsymbol{x} \in \mathrm{A}^{-1}[\boldsymbol{t}]$, where $\mathrm{A}^{-1}[\boldsymbol{t}] = \{\boldsymbol{x} \in \mathbb{N}^6 : A\boldsymbol{x} = \boldsymbol{t}\}$,

$\mathrm{A} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}_{5\times 6}$, and $\boldsymbol{t} = (5,3,6,7,7)'$. If $g = r = (1\,2\,3\,4\,5\,6)$, then $T_g\boldsymbol{x} =$

$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 2 \\ 1 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \\ 2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} x_6 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \notin \mathrm{A}^{-1}[\boldsymbol{t}]$**,** since

$$AT_g\boldsymbol{x}=\begin{bmatrix}1&1&0&0&0&0\\0&0&1&1&0&0\\0&0&0&0&1&1\\1&0&1&0&1&0\\0&1&0&1&0&1\end{bmatrix}_{5\times 6}\begin{bmatrix}2\\1\\4\\2\\1\\4\end{bmatrix}=\begin{bmatrix}3\\6\\5\\7\\7\end{bmatrix}\neq\begin{bmatrix}5\\3\\6\\7\\7\end{bmatrix}=\boldsymbol{t},\text{ and if } \boldsymbol{z}=(-1,1,\ 1,-1,0,0)'.$$

$$\text{Then } T_g\boldsymbol{z}=\begin{bmatrix}0&0&0&0&0&1\\1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\end{bmatrix}\begin{bmatrix}-1\\1\\1\\-1\\0\\0\end{bmatrix}=\begin{bmatrix}0\\-1\\1\\1\\-1\\0\end{bmatrix}\notin \mathbf{B}.$$

Now, we will prove H is the largest subgroups of the dihedral group D_n, such that the Markov basis **B** invariant under their actions.

Remark (3.4). The left action of H on **B** is a function $H\times\mathbf{B}\ \rightarrow g\boldsymbol{z}_j\in\ \mathbf{B}$.

Theorem (3.5). The Markov basis **B** is H-invariant.

Proof

To prove **B** is H-invariant

Let $\boldsymbol{z}_j\in\mathbf{B}$, $\boldsymbol{z}_j=(z_1,z_2,\ \dots\ ,z_{\frac{n}{3}},z_{\frac{n}{3}+1},\ \dots\ ,z_{\frac{2n}{3}},z_{\frac{2n}{3}+1},\ \dots\ ,z_{n-1},z_n)'$.

$\Longrightarrow e\boldsymbol{z}_j=T_e\boldsymbol{z}_j\in\mathbf{B}$.

Since $r=(1\ 2\ \dots\ n)$, then $r^{\frac{n}{3}}=\left(1\ \ \frac{n}{3}+1\ \ \frac{2n}{3}+1\right)\left(2\ \ \frac{n}{3}+2\ \ \frac{2n}{3}+2\right)\dots\left(\frac{n}{3}\ \ \frac{2n}{3}\ \ n\right)$. This implies

$r^{\frac{n}{3}}\boldsymbol{z}_j=T_{r^{\frac{n}{3}}}\boldsymbol{z}_j=(z_{\frac{2n}{3}+1},z_{\frac{2n}{3}+2},\dots\ ,\ z_n,z_1,\ \dots\ ,z_{\frac{n}{3}},z_{\frac{n}{3}+1},\dots\ ,z_{\frac{2n}{3}-1},z_{\frac{2n}{3}})'\in\mathbf{B}$.

And $r^{\frac{2n}{3}}=\left(1\ \ \frac{2n}{3}+1\ \ \frac{n}{3}+1\right)\left(2\ \ \frac{2n}{3}+2\ \ \frac{n}{3}+2\right)\dots\left(\frac{n}{3}\ \ n\ \ \frac{2n}{3}\right)$. This implies

$r^{\frac{2n}{3}}\boldsymbol{z}_j=T_{r^{\frac{2n}{3}}}\boldsymbol{z}_j=(z_{\frac{n}{3}+1},z_{\frac{n}{3}+2},\dots\ ,z_{\frac{2n}{3}},z_{\frac{2n}{3}+1},\dots\ ,\ z_n,z_1,\ \dots\ ,z_{\frac{n}{3}-1},z_{\frac{n}{3}})'\in\mathbf{B}$.

Since $r=(2\ \ n)(3\ \ n-1)\ \dots\left(\frac{n}{2}\ \ \frac{n}{2}+2\right)$, then

$sr=(1\ \ n)(2\ \ n-1)\dots\left(\frac{n}{3}\ \ \frac{2n}{3}+1\right)\left(\frac{n}{3}+1\ \ \frac{2n}{3}\right)\dots\ (\frac{n}{2}\ \ \frac{n}{2}+1)$. This implies

$sr\boldsymbol{z}_j=T_{sr}\boldsymbol{z}_j=(z_n,z_{n-1},\dots\ ,z_{\frac{2n}{3}+1},z_{\frac{2n}{3}},\dots\ ,z_{\frac{n}{3}+1},z_{\frac{n}{3}},\dots\ ,z_2,z_1)'\in\mathbf{B}$. Also

$sr^{\frac{n}{3}+1}=\left(1\ \ \frac{2n}{3}\right)\left(2\ \ \frac{2n}{3}-1\right)\dots\left(\frac{n}{3}\ \ \frac{n}{3}+1\right)\left(\frac{2n}{3}+1\ \ n\right)\dots\left(\frac{5n}{6}\ \ \frac{5n}{6}+1\right)$. Hence

$sr^{\frac{n}{3}+1}\boldsymbol{z}_j=T_{sr^{\frac{n}{3}+1}}\boldsymbol{z}_j=(\ z_{\frac{2n}{3}},z_{\frac{2n}{3}-1},\dots\ ,z_{\frac{n}{3}+1},z_{\frac{n}{3}},\dots\ ,z_1,z_n,\dots\ ,z_{\frac{2n}{3}+2},z_{\frac{2n}{3}+1})'\in\mathbf{B}$, and $sr^{\frac{2n}{3}+1}=$ $\left(1\ \ \frac{n}{3}\right)\left(2\ \ \frac{n}{3}-1\right)\cdots\left(\frac{n}{6}\ \ \frac{n}{6}+1\right)\left(\frac{n}{3}+1\ \ n\right)\left(\frac{n}{3}+2\ \ n-1\right)\cdots\left(\frac{2n}{3}\ \ \frac{2n}{3}+1\right)$. Therefore

$sr^{\frac{2n}{3}+1}\boldsymbol{z}_j=T_{sr^{\frac{2n}{3}+1}}\boldsymbol{z}_j=(z_{\frac{n}{3}},z_{\frac{n}{3}-1},\dots\ ,z_1,z_n,\dots\ ,z_{\frac{2n}{3}+1},z_{\frac{2n}{3}},\dots\ ,z_{\frac{n}{3}+2},z_{\frac{n}{3}+1})'\in\mathbf{B}$. Then $H(\mathbf{B})=\mathbf{B}$, and this follows **B** is H-invariant.

Corollary (3.6). The subgroup H is the Largest Subgroup of the group $\boldsymbol{D}_n$ such that the Markov basis B is H-invariant.

Proof

The Markov basis **B** is H-invariant (By Theorem (3.5))

Now, let $g\in\boldsymbol{D}_n\ and\ g\notin H$. Then, we have

If $g=r^j$ then $j\notin\{\frac{n}{3}\ ,\frac{2n}{3}\ ,n\}$

$$r^j\boldsymbol{z}_l = T_{r^j}\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_j \\ z_{j+1} \\ z_{j+2} \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} z_{n-j+1} \\ z_{n-j+2} \\ \vdots \\ z_n \\ z_1 \\ z_2 \\ \vdots \\ z_{n-j} \end{bmatrix} \notin \mathbf{B},$$

And if $g = sr^j$ then $j \notin \{1, \frac{n}{3}+1, \frac{2n}{3}+1\}$

$$sr^j\boldsymbol{z}_j = T_{sr^j}\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n-j+1} \\ z_{n-j+2} \\ z_{n-j+3} \\ \vdots \\ z_{n-1} \\ z_n \end{bmatrix} = \begin{bmatrix} z_{n-j+1} \\ z_{n-j} \\ \vdots \\ z_1 \\ z_n \\ z_{n-1} \\ \vdots \\ z_{n-j+3} \\ z_{n-j+2} \end{bmatrix} \notin \mathbf{B}.$$

Implies that $g \notin G_{(\boldsymbol{B})}$ *where* $G = \boldsymbol{D}_n$

Thus H is the largest subgroup of the group $\boldsymbol{D}_n$ such that **B** is H-invariant.

Now, we use $H(\mathbf{B})$ to generate $3 \times \frac{n}{3}$ contingincy tables with given row and colum sums, and Markov basis **B**.

Remark (3.7). Let $\boldsymbol{t} = \left(t_1, t_2, t_3, \dots, t_{\frac{n}{3}+3}\right)'$, $\boldsymbol{x}_i \in \mathrm{A}^{-1}[\boldsymbol{t}]$ and $g \in H$**.** Then $g\boldsymbol{x}_i \in \mathrm{A}^{-1}[g\boldsymbol{t}]$ where $g\boldsymbol{t} = \left(gt_1, gt_2, gt_3, \dots, gt_{\frac{n}{3}+3}\right)'$, $\mathrm{A}^{-1}[g\boldsymbol{t}] = \{\boldsymbol{x} \in \mathbb{N}^{\boldsymbol{n}} : \boldsymbol{Ax} = g\boldsymbol{t}\}$**.** Therefore, we have six types of $g\boldsymbol{t}$-fibers $\mathrm{A}^{-1}[\boldsymbol{t}]$, $\mathrm{A}^{-1}\left[r^{\frac{n}{3}}\boldsymbol{t}\right]$, $\mathrm{A}^{-1}\left[r^{\frac{2n}{3}}\boldsymbol{t}\right]$, $\mathrm{A}^{-1}[sr\boldsymbol{t}]$,
$\mathrm{A}^{-1}\left[sr^{\frac{n}{3}+1}\boldsymbol{t}\right]$ and $\mathrm{A}^{-1}\left[sr^{\frac{2n}{3}+1}\boldsymbol{t}\right]$.

Theorem (3.8). If $g \in H$, then **B** is a Markov basis for $\frac{n^2-3n}{3}$ contingency tables $g\boldsymbol{x}_0, g\boldsymbol{x}_1, \dots, g\boldsymbol{x}_{\frac{n^2-3n}{3}-1}$ in $\mathrm{A}^{-1}[g\boldsymbol{t}]$.

Proof

By remark (2.12) we have $\boldsymbol{x}_i = \boldsymbol{x}_{i-1} + \boldsymbol{z}_i$, for all $i = 1, 2, \dots, \frac{n^2-3n}{3} - 1$ and $\boldsymbol{x}_0 = \boldsymbol{x}_{\frac{n^2-3n}{3}-1} + \boldsymbol{z}_{\frac{n^2-3n}{3}}$.

Now, if $g \in H$

$g\boldsymbol{x}_i = T_g\boldsymbol{x}_i = T_g(\boldsymbol{x}_{i-1} + \boldsymbol{z}_i) = T_g\boldsymbol{x}_{i-1} + T_g\boldsymbol{z}_i$ for all $i = 1, 2, \dots, \frac{n^2-3n}{3} - 1$ and

$$g\boldsymbol{x}_0 = T_g\boldsymbol{x}_0 = T_g\left(\boldsymbol{x}_{\frac{n^2-3n}{3}-1} + \boldsymbol{z}_{\frac{n^2-3n}{3}}\right) = T_g\boldsymbol{x}_{\frac{n^2-3n}{3}-1} + T_g\boldsymbol{z}_{\frac{n^2-3n}{3}}.$$

Then $T_g\boldsymbol{x}_i = g\boldsymbol{x}_i \in \mathrm{A}^{-1}[g\boldsymbol{t}]$ for all $i = 0, 1, 2, \dots, \frac{n^2-3n}{3} - 1$, and **B** generate $\frac{n^2-3n}{3}$ contingency tables $g\boldsymbol{x}_0, g\boldsymbol{x}_1, \dots, g\boldsymbol{x}_{\frac{n^2-3n}{3}-1}$.

To prove **B** is a Markov basis for the contingency tables $g\boldsymbol{x}_0, g\boldsymbol{x}_1, \dots, g\boldsymbol{x}_{\frac{n^2-3n}{3}-1}$

Let $g\boldsymbol{x}_i, g\boldsymbol{x}_j \in A^{-1}[g\boldsymbol{t}]$, such that $j \geq i$.

By Remark (3.7), we get

$g\boldsymbol{x}_j = g\boldsymbol{x}_i + \sum_{k=i+1}^{j} g\boldsymbol{z}_k$ imples that $g\boldsymbol{x}_j - g\boldsymbol{x}_i = \sum_{k=i+1}^{j} g\boldsymbol{z}_k$, and

$g\boldsymbol{z}_k \in Ker(A)$. Hence $\sum_{k=i+1}^{j} g\boldsymbol{z}_k \in Ker(A)$ and so $g\boldsymbol{x}_j - g\boldsymbol{x}_i \in Ker(A)$.
Therefore $\boldsymbol{x}_i \sim \boldsymbol{x}_j$.
$A^{-1}[g\boldsymbol{t}]$ Constitutes one **B** equivalence class implies that **B** is a Markov basis.
Example (3.9). Consider the 3×3 −contingency table $\boldsymbol{x}$ in Example (3.3). $\boldsymbol{x}$ can be represented as 6-dimensional column vector as $\boldsymbol{x} = (1,4,2,1,4,2)' \in \mathbb{N}^6$. Then $\boldsymbol{x} \in A^{-1}[\boldsymbol{t}]$, where $A^{-1}[\boldsymbol{t}] = \{\boldsymbol{x} \in \mathbb{N}^6: A\boldsymbol{x} = \boldsymbol{t}\}$, and $\boldsymbol{t} = (5,3,6,7,7)'$. Then by remark (2.8) we can find the elements of **B**

$\mathbf{z}_1 =$

1	-1
-1	1
0	0

, $\mathbf{z}_2 =$

1	-1
0	0
-1	1

, $\mathbf{z}_3 =$

0	0
1	-1
-1	1

$\mathbf{z}_4 =$

-1	1
1	-1
0	0

, $\mathbf{z}_5 =$

-1	1
0	0
1	-1

, $\mathbf{z}_6 =$

0	0
-1	1
1	-1

We can find 6 elements in $A^{-1}[\boldsymbol{t}]$ of 3×2 −contingency table as

$\mathbf{x_0} =$

1	4	5
2	2	4
4	2	6
7	8	15

, $\mathbf{x_1} =$

2	3	5
1	3	4
4	2	6
7	8	15

, $\mathbf{x_2} =$

3	2	5
1	3	4
3	3	6
7	8	15

$\mathbf{x_3} =$

3	2	5
2	2	4
2	4	6
7	8	15

, $\mathbf{x_4} =$

2	3	5
3	1	4
2	4	6
7	8	15

, $\mathbf{x_5} =$

1	4	5
3	1	4
3	3	6
7	8	15

Now $r^2 = (1\ 3\ 5)(2\ 4\ 6)$ in D_6. To find $T_{r^2}\boldsymbol{z}_1$. $\boldsymbol{z_1}$ can be represented as 6-dimensional column vector of integer $\boldsymbol{z}_1 = (1,-1,-1,1,0,0)' \in \mathbb{Z}^6$

$$T_{r^2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \text{ then}$$

$$T_{r^2}\mathbf{z}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \mathbf{z}_2.$$

Similarly, we obtain
$T_{r^2}\mathbf{z}_2 = \boldsymbol{z}_6, T_{r^2}\mathbf{z}_3 = \boldsymbol{z}_4, T_{r^2}\mathbf{z}_4 = \boldsymbol{z}_5, T_{r^2}\mathbf{z}_5 = \boldsymbol{z}_3, T_{r^2}\mathbf{z}_6 = \boldsymbol{z}_1$**.** Thus $T_{r^2}\mathbf{B} = \{\mathbf{z}_2, \mathbf{z}_6, \mathbf{z}_4, \mathbf{z}_5, \mathbf{z}_3, \mathbf{z}_1\} = \mathbf{B}$. We also use r^2 to generate the set $\{T_{r^2}\boldsymbol{x}_0, T_{r^2}\boldsymbol{x}_1, T_{r^2}\boldsymbol{x}_2, T_{r^2}\boldsymbol{x}_3, T_{r^2}\boldsymbol{x}_4, T_{r^2}\boldsymbol{x}_5, T_{r^2}\boldsymbol{x}_6,\} \subseteq \mathrm{A}^{-1}[r^2\boldsymbol{t}]$ ($r^2\boldsymbol{t}$-fibers**),** where $r^2\boldsymbol{t} = (6, 5, 3, 7, 7)'$, and

$$T_{r^2}\mathbf{x_0} = \begin{array}{|c|c|c|} \hline 4 & 2 & 6 \\ \hline 1 & 4 & 5 \\ \hline 2 & 2 & 4 \\ \hline 7 & 8 & 15 \\ \hline \end{array}, T_{r^2}\mathbf{x_1} = \begin{array}{|c|c|c|} \hline 4 & 2 & 6 \\ \hline 2 & 3 & 5 \\ \hline 1 & 3 & 4 \\ \hline 7 & 8 & 15 \\ \hline \end{array}, T_{r^2}\mathbf{x_2} = \begin{array}{|c|c|c|} \hline 3 & 3 & 6 \\ \hline 3 & 2 & 5 \\ \hline 1 & 3 & 4 \\ \hline 7 & 8 & 15 \\ \hline \end{array}$$

$$T_{r^2}\mathbf{x_3} = \begin{array}{|c|c|c|} \hline 2 & 4 & 6 \\ \hline 3 & 2 & 5 \\ \hline 2 & 2 & 4 \\ \hline 7 & 8 & 15 \\ \hline \end{array}, T_{r^2}\mathbf{x_4} = \begin{array}{|c|c|c|} \hline 2 & 4 & 6 \\ \hline 2 & 3 & 5 \\ \hline 3 & 1 & 4 \\ \hline 7 & 8 & 15 \\ \hline \end{array}, T_{r^2}\mathbf{x_5} = \begin{array}{|c|c|c|} \hline 3 & 3 & 6 \\ \hline 1 & 4 & 5 \\ \hline 3 & 1 & 4 \\ \hline 7 & 8 & 15 \\ \hline \end{array}$$

And $r^4 = (1\ \ 5\ \ 3)(2\ \ 6\ \ 4)$ in D_6. To find $T_{r^4}\boldsymbol{z}_1$**.**

$$T_{r^4} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ then}$$

$$T_{r^4}\mathbf{z}_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \mathbf{z}_6.$$

Similarly, we find $T_{r^4}\mathbf{z}_2 = \boldsymbol{z}_1, T_{r^4}\mathbf{z}_3 = \boldsymbol{z}_5, T_{r^4}\mathbf{z}_4 = \boldsymbol{z}_3, T_{r^4}\mathbf{z}_5 = \boldsymbol{z}_4, T_{r^4}\mathbf{z}_6 = \boldsymbol{z}_2$**.** Thus $T_{r^4}\mathbf{B} = \{\mathbf{z}_6, \mathbf{z}_1, \mathbf{z}_5, \mathbf{z}_3, \mathbf{z}_4,\ \boldsymbol{z}_2\} = \mathbf{B}$. We also use r^4 to generate the set $\{T_{r^4}\boldsymbol{x}_0, T_{r^4}\boldsymbol{x}_1, T_{r^4}\boldsymbol{x}_2, T_{r^4}\boldsymbol{x}_3, T_{r^4}\boldsymbol{x}_4, T_{r^4}\boldsymbol{x}_5, T_{r^4}\boldsymbol{x}_6,\} \subseteq \mathrm{A}^{-1}[r^4\boldsymbol{t}]$ ($r^4\boldsymbol{t}$-fibers**),** where $r^4\boldsymbol{t} = (3, 6, 5, 7, 7)'$, and

$$T_{r^4}\mathbf{x_0} = \begin{array}{|c|c|c|}\hline 2 & 2 & 4\\\hline 4 & 2 & 6\\\hline 1 & 4 & 5\\\hline 7 & 8 & 15\\\hline\end{array}\ , T_{r^4}\mathbf{x_1} = \begin{array}{|c|c|c|}\hline 1 & 3 & 4\\\hline 4 & 2 & 6\\\hline 2 & 3 & 5\\\hline 7 & 8 & 15\\\hline\end{array}\ , T_{r^4}\mathbf{x_2} = \begin{array}{|c|c|c|}\hline 1 & 3 & 4\\\hline 3 & 3 & 6\\\hline 3 & 2 & 5\\\hline 7 & 8 & 15\\\hline\end{array}$$

$$T_{r^4}\mathbf{x_3} = \begin{array}{|c|c|c|}\hline 2 & 2 & 4\\\hline 2 & 4 & 6\\\hline 3 & 2 & 5\\\hline 7 & 8 & 15\\\hline\end{array}\ , T_{r^4}\mathbf{x_4} = \begin{array}{|c|c|c|}\hline 3 & 1 & 4\\\hline 2 & 4 & 6\\\hline 2 & 3 & 5\\\hline 7 & 8 & 15\\\hline\end{array}\ , T_{r^4}\mathbf{x_5} = \begin{array}{|c|c|c|}\hline 3 & 1 & 4\\\hline 3 & 3 & 6\\\hline 1 & 4 & 5\\\hline 7 & 8 & 15\\\hline\end{array}$$

Since $s = (2\ 6)(3\ 5)$ and $r = (1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6)$. Then $sr = (1\ 6)(2\ 5)(3\ 4)$. To find $T_{sr}\boldsymbol{z}_1$**,**

$$T_{sr} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ then}$$

$$T_{sr}\mathbf{z}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1\\ -1\\ -1\\ 1\\ 0\\ 0 \end{bmatrix} = \begin{bmatrix} 0\\ 0\\ 1\\ -1\\ -1\\ 1 \end{bmatrix} = \mathbf{z}_2$$

Similarly, we find $T_{sr}\mathbf{z}_2 = \boldsymbol{z}_1, T_{sr}\mathbf{z}_3 = \boldsymbol{z}_3, T_{sr}\mathbf{z}_4 = \boldsymbol{z}_5, T_{sr}\mathbf{z}_5 = \boldsymbol{z}_4, T_{sr}\mathbf{z}_6 = \boldsymbol{z}_6$**.** Thus $T_{sr}\mathbf{B} = \{\mathbf{z}_2, \mathbf{z}_1, \mathbf{z}_3, \mathbf{z}_5, \mathbf{z}_4, \mathbf{z}_6\} = \mathbf{B}$. We also use sr to generate the set $\{T_{sr}\,\boldsymbol{x}_0, T_{sr}\,\boldsymbol{x}_1, T_{sr}\,\boldsymbol{x}_2, T_{sr}\,\boldsymbol{x}_3, T_{sr}\,\boldsymbol{x}_4, T_{sr}\,\boldsymbol{x}_5, T_{sr}\,\boldsymbol{x}_6\} \subseteq \mathrm{A}^{-1}[sr\boldsymbol{t}]$ ($sr\boldsymbol{t}$-fibers**),** where $sr\boldsymbol{t} = (6, 3, 5, 7, 7)'$, and

$T_{sr}\mathbf{x_0} =$

2	4	6
2	2	4
4	1	5
8	7	15

$, T_{sr}\mathbf{x_1} =$

2	4	6
3	1	4
3	2	5
8	7	15

$, T_{sr}\mathbf{x_2} =$

3	3	6
3	1	4
2	3	5
8	7	15

$T_{sr}\mathbf{x_3} =$

4	2	6
2	2	4
2	3	5
8	7	15

$, T_{sr}\mathbf{x_4} =$

4	2	6
1	3	4
3	2	5
8	7	15

$, T_{sr}\mathbf{x_5} =$

3	3	6
1	3	4
4	1	5
8	7	15

And $sr^3 = (1\ \ 4)(2\ \ 3)(5\ \ 6)$. To find $T_{sr^3}\boldsymbol{z}_1$,

$$T_{sr^3} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \text{ then}$$

$$T_{sr^3}\mathbf{z}_1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \mathbf{z}_1$$

Similarly, we obtain $T_{sr^3}\mathbf{z}_2 = \boldsymbol{z}_6, T_{sr^3}\mathbf{z}_3 = \mathbf{z}_5, T_{sr^3}\mathbf{z}_4 = \boldsymbol{z}_4, T_{sr^3}\mathbf{z}_5 = \boldsymbol{z}_3, T_{sr^3}\mathbf{z}_6 = \boldsymbol{z}_2$ Thus $T_{sr^3}\mathbf{B} = \{\mathbf{z}_1, \mathbf{z}_6, \mathbf{z}_5, \mathbf{z}_4, \mathbf{z}_3, \boldsymbol{z}_2\} = \mathbf{B}$. We also use sr to generate the set $\{T_{sr^3}\boldsymbol{x}_0, T_{sr^3}\boldsymbol{x}_1, T_{sr^3}\boldsymbol{x}_2, T_{sr^3}\boldsymbol{x}_3, T_{sr^3}\boldsymbol{x}_4, T_{sr^3}\boldsymbol{x}_5, T_{sr^3}\boldsymbol{x}_6\} \subseteq A^{-1}[sr^3\boldsymbol{t}]$ ($sr^3\boldsymbol{t}$-fibers**),** where $sr^3\boldsymbol{t} = (3, 5, 6, 7, 7)'$, and

$T_{sr^3}\mathbf{x_0} =$

2	2	4
4	1	5
2	4	6
8	7	15

$, T_{sr^3}\mathbf{x_1} =$

3	1	4
3	2	5
2	4	6
8	7	15

$, T_{sr^3}\mathbf{x_2} =$

3	1	4
2	3	5
3	3	6
8	7	15

$T_{sr^3}\mathbf{x_3} =$

2	2	4
2	3	5
4	2	6
8	7	15

$, T_{sr^3}\mathbf{x_4} =$

1	3	4
3	2	5
4	2	6
8	7	15

$, T_{sr^3}\mathbf{x_5} =$

1	3	4
4	1	5
3	3	6
8	7	15

And $sr^5 = (1\ \ 2)(3\ \ 6)(4\ \ 5)$. To find $T_{sr^5}\boldsymbol{z}_1$**,**

$$T_{sr^5} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \text{ then}$$

$$T_{sr^5}\mathbf{z}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \mathbf{z}_6$$

Similarly, we obtain $T_{sr^5}\mathbf{z}_2 = \boldsymbol{z}_2, T_{sr^5}\mathbf{z}_3 = \boldsymbol{z}_4, T_{sr^5}\mathbf{z}_4 = \boldsymbol{z}_3, T_{sr^5}\mathbf{z}_5 = \boldsymbol{z}_5, T_{sr^5}\mathbf{z}_6 = \boldsymbol{z}_1$**.** Thus $T_{sr^5}\mathbf{B} = \{\mathbf{z}_6, \mathbf{z}_2, \mathbf{z}_4, \mathbf{z}_3, \mathbf{z}_5, \mathbf{z}_1\} = \mathbf{B}$. We also use sr to generate the set $\{T_{sr^5}\boldsymbol{x}_0, T_{sr^5}\boldsymbol{x}_1, T_{sr^5}\boldsymbol{x}_2, T_{sr^5}\boldsymbol{x}_3, T_{sr^5}\boldsymbol{x}_4, T_{sr^5}\boldsymbol{x}_5, T_{sr^5}\boldsymbol{x}_6\} \subseteq \mathrm{A}^{-1}[sr^5\boldsymbol{t}]$ ($sr^5\boldsymbol{t}$**-**fibers**),** where $sr^5\boldsymbol{t} = (5, 6, 3, 7, 7)'$, and

$T_{sr^5}\mathbf{x_0} =$

4	1	5
2	4	6
2	2	4
8	7	15

$, T_{sr^5}\mathbf{x_1} =$

3	2	5
2	4	6
3	1	4
8	7	15

$, T_{sr^5}\mathbf{x_2} =$

2	3	5
3	3	6
3	1	4
8	7	15

$T_{sr^5}\mathbf{x_3} =$

2	3	5
4	2	6
2	2	4
8	7	15

$, T_{sr^5}\mathbf{x_4} =$

3	2	5
4	2	6
1	3	4
8	7	15

$, T_{sr^5}\mathbf{x_5} =$

4	1	5
3	3	6
1	3	4
8	7	15

Corollary (3.10). The toric ideal for $\frac{n^2-3n}{3} \times 3 \times \frac{n}{3}$ -contingency table in $A^{-1}[g\boldsymbol{t}]$ is $I_A =< P_{g(i+l)}\, P_{g(j+k)} - P_{g(j+l)}\, P_{g(i+k)}$: $i, j = 1, 2, \dots, \frac{n}{3}$ and $l, k = 0, \frac{n}{3}, \frac{2n}{3}$, such that $i < j$ and $l < k >$ $\subset \mathbb{C}[P_1, P_2, \dots, P_n]$**,** for all $g \in H$.

Proof

Let $g \in H$. By theorem (2.7), we have $g\mathbf{Z} = T_g\mathbf{Z} = T_g\mathbf{Z}^+ - T_g\mathbf{Z}^- = g\mathbf{Z}^+ - g\mathbf{Z}^- \in \mathbf{B}$, for all $\mathbf{Z} \in \mathbf{B}$. Using theorems (3.8), (2.7) and corollary (2.14) the proof is complete.

References

[1] A. Takemura, and S. Aoki, "Some characterizations of minimal Markov basis for sampling from discrete conditional distributions", Annals of the Institute of Statistical Mathematics, to appear, 2003.

[2] W. J. Gilbert," Modern Algebra With Application ",John Wiley, Inc., New Jersey, U.S.A, 2004.

[3] S. Aoki, and A. Takemura," Minimal Invariant Markov Basis for Sampling Contingency Tables with Fixed Marginals", Annals of the Institut of Statistical Mathematics, pp.230-256, 2006.

[4] S. Aoki, A. Takemura, "The largest group of invariance for Markov bases and toric ideals", J. Symbolic Computation 43 (5) pp 342–358, 2008.

[5] S. Aoki, H.Hara, and A. Takemura," Markov basis and Algebraic Statistics", Springer series in Statistics, Vol 199, New York, 2012.

[6] S. Sullivant, J. Rauh, T.Kahle. "Positive Margins and Primary Decomposition", North Carolina State, pp. 1-22, 2012.

[7] H. H. Abbass, and H. S. Mohammed Hussein " On Toric Ideals for $\mathbf{3} \times \frac{\boldsymbol{n}}{\mathbf{3}}$ - Contingency Tables with Fixed Two Dimensional Marginals, n is a multiple of 3", European Journal of Scientific Research, Vol. 123, pp. 83-98, 2014.