Approximate solutions of He's Variational iteration method for solving space-time fractional wave like and heat like equations

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Abstract:-

In this paper, we will consider He's variational iteration method for solving space-time fractional wave like and heat like equations with appropriate data. This method is using linear operator with putting approximate value for time fractional derivative. Six examples are presented to show the application of the variational iteration method. This application show important and efficient the method, which show the approximate solution is near to exact solution.

Keywords: Variational iteration method, space-time fractional wave like and heat like equations, Caputo derivative.

1.Introduction

The variational iteration method has been extensively worked out for many years by numerous authors. Starting from the pioneer ideas of the Inokuti -Sekine-Mura method [1978], JI-Huan He [1999] developed the variational iteration method . The variational iteration method, has been widely applied to solve nonlinear problems, more and more merits have been discovered and some modifications are suggested to overcome the demerits arising in the solution procedure. For example Jassim [2012] apply of variational iteration method by choosing linear operators for higher dimensional initial value boundary problems, Schrödinger and Laplace problems . We introduce the basic idea underlying the variational iteration method for solving nonlinear equations. Consider the general equation:

$$Lu(x,y,z,t) + Nu(x,y,z,t) = g(x,y,z,t), \quad (1)$$

where L is a linear differential operator, N is a nonlinear operator, and g is a given analytical function. The essence of the method is to construct a correction functional of the form

$$u_{n+1}(x,y,z,t) = u_n + \int_0^t \lambda(s,t)[Lu_n(x,y,z,s) + N\tilde{u}_n(x,y,z,s) - g(x,y,z,s)]ds, \quad (2)$$

where λ is aLagrange multiplier which can be identified optimally via the variational theory Inokuti and Sekine [1978], u_n is the approximate solution and $\tilde{u}_n$ denotes the restricted variation, i.e.$\delta\tilde{u}_n = 0$. After determining the Lagrange multiplier λ and selecting an appropriate initial function u_0, the successive approximations $u_n$of the solution u can be readily obtained.

Definition 1.1. Fractional integral operator of order $\alpha \geq 0$ is defined as

$$I_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} f(\tau) d\tau \quad \alpha > 0, \tag{3}$$

Γ is a gamma function.

Definition 1.2. Fractional derivative of $f(x)$ in the Caputo sense [Caputo,1967] is defined as

$$D_x^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-\tau)^{m-\alpha-1} \frac{d^m f(\tau)}{d\tau^m} d\tau, \; m-1 < \alpha \leq m, m \in \mathbb{N}, x > 0 \tag{4}$$

α is the order of the derivative. For the Caputo's derivative we have:

$1 - D^{\alpha} C = 0$, C is constant,

$2 - D^{\alpha} x^{\beta} = 0$, $\beta \leq \alpha - 1$,

$3 - D^{\alpha} x^{\beta} = \frac{\Gamma(1+\beta)}{\Gamma(1-\alpha+\beta)} x^{\beta-\alpha}$, $\beta > \alpha - 1$,

In this paper, we consider, the three dimensional space-time fractional wave like and heat like

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = f(x,y,z) \frac{\partial^{\beta} u}{\partial x^{\beta}} + g(x,y,z) \frac{\partial^{\beta} u}{\partial y^{\beta}} + h(x,y,z) \tag{5}$$

$0 < x < a$, $0 < y < b$, $0 < z < c$, $0 < \alpha < 2$, $1 < \beta \leq 2$, $t > 0$.

Subject to the boundary conditions

$u(0,y,z,t) = f_1(y,z,t)$, $\quad u_x(a,y,z,t) = f_2(y,z,t)$,

$u(x,0,z,t) = g_1(x,z,t)$, $\quad u_y(x,b,z,t) = g_2(x,z,t)$,

$u(x,y,0,t) = h_1(x,y,t)$, $\quad u_z(x,y,c,t) = h_2(x,y,t)$,

and the initial conditions

$u(x,y,z,0) = \psi(x,y,z)$, $\quad u_t(x,y,z,0) = \eta(x,y,z)$, where α and β are parameters describing the order of the fractional time- and space derivatives, respectively. In case $0 < \alpha \leq 1,\ 1 < \beta \leq 2$, equation (5) reduce to the fractional heat like equation with variable coefficients, and in case $1 < \alpha \leq 2,\ 1 < \beta \leq 2$, Equation(5) reduce to the fractional wave like equation. In this paper, we apply $He's$ variational iteration method to find approximate solutions for fractional wave like and heat like equations.

2. $He's$ variational iteration method for solving space-time fractional wave like and heat like.

First, to convey the basic ideal for variational iteration method to solve fractional heat like equation, equation (5) can be written by using linear operator in the form:

$$\frac{\partial^\alpha u(x,y,z,t)}{\partial t^\alpha} = ku(x,y,z,t) - ku(x,y,z,t) + [f(x,y,z)\frac{\partial^\beta u(x,y,z,t)}{\partial x^\beta} + g(x,y,z)\frac{\partial^\beta u(x,y,z,t)}{\partial y^\beta}$$
$$+h(x,y,z)\frac{\partial^\beta u(x,y,z,t)}{\partial z^\beta}], 0 < \alpha \leq 1, 1 < \beta \leq 2, \text{ k is constant,} \quad (6)$$

using the standard variational iteration method, we construct the following correction functional as

$$u_{n+1}(x,y,z,t) = u_n(x,y,z,t) + \int_0^t \lambda(\xi,t)[\frac{\partial^\alpha u_n}{\partial \xi^\alpha} - ku_n(x,,y,z,\xi) + ku_n(x,y,z,\xi) - f(x,y,z)\frac{\partial^\beta u_n}{\partial x^\beta} -$$
$$g(x,y,z)\frac{\partial^\beta u_n}{\partial y^\beta} - h(x,y,z)\frac{\partial^\beta u_n}{\partial z^\beta}]\, d\xi, \quad (7)$$

Now, we assume that

$$\frac{\partial^\alpha u}{\partial \xi^\alpha} \cong \Gamma(1+\alpha)\frac{\partial u}{\partial \xi},\ 0 < \alpha \leq 1, \quad (8)$$

$if\ \alpha = 1,\ Eq$uation (8) becomes $\frac{\partial u}{\partial \xi} = \Gamma(2)\frac{\partial u}{\partial \xi}$,

Substituting (8) in (7), we obtain

$$u_{n+1}(x,y,z,t) = u_n(x,y,z,t) + \int_0^t \lambda(\xi,t)\,[\,\Gamma(1+\alpha)\frac{\partial u}{\partial \xi} - ku_n(x,,y,z,\xi)$$
$$+ku_n(x,y,z,\xi) - f(x,y,z)\frac{\partial^\beta u_n}{\partial x^\beta} - g(x,y,z)\frac{\partial^\beta u_n}{\partial y^\beta} - h(x,y,z)\frac{\partial^\beta u_n}{\partial z^\beta}]\, d\xi,$$

$$\delta u_{n+1}(x,y,z,t) = \delta u_n(x,y,z,t) + \delta \int_0^t \lambda(\xi,t)\ [\Gamma(1+\alpha)\frac{\partial u_n}{\partial \xi} - k u_n(x,y,z,\xi) +$$

$$k\tilde{u}_n(x,y,z,s\,) - f(x,y,z)\frac{\partial^\beta \tilde{u}_n}{\partial x^\beta} - g(x,y,z)\frac{\partial^\beta \tilde{u}_n}{\partial y^\beta} - h(x,y,z)\frac{\partial^\beta \tilde{u}_n}{\partial z^\beta}]\ d\xi, \qquad (9)$$

$$\delta u_{n+1}(x,y,z,t) = \delta u_n(x,y,z,t) + \Gamma(1+\alpha)\ \lambda(\xi,t)\delta u_n(x,y,z,\xi)\ -\ \Gamma(1+\alpha)\ \int_0^t \frac{\partial \lambda(\xi,t)}{\partial \xi}$$

$$\delta u_n(x,y,z,\xi)d\xi - k\int_0^t \lambda(\xi,t)\ \delta u_n\,(x,y,z,\xi)d\xi. \qquad (10)$$

Moreover, the stationary conditions are as follow;

$$\Gamma(1+\alpha)\ \left.\frac{\partial \lambda(\xi,t)}{\partial \xi}\right|_{\xi=t} + \left.k\lambda(\xi,t)\right|_{\xi=t} = 0,$$

$$1 + \Gamma(1+\alpha)\lambda(\xi,t)|_{\xi=t} = 0,$$

therefore, the general Lagrange multiplier can be readily identified by

$$\lambda(\xi,t) = -\frac{1}{\Gamma(1+\alpha)}e^{-\frac{1}{\Gamma(1+\alpha)}k(\xi-t)}, \qquad (11)$$

Substituting (11) for correction functional (7), we have the following iteration formula:

$$u_{n+1}(x,y,z,t) = u_n(x,y,z,t) - \frac{1}{\Gamma(1+\alpha)}\int_0^t e^{-\frac{1}{\Gamma(1+\alpha)}k(\xi-t)}[\frac{\partial^\alpha u_n}{\partial \xi^\alpha} - f(x,y,z)\frac{\partial^\beta u_n}{\partial x^\beta}$$

$$-g(x,y,z)\frac{\partial^\beta u_n}{\partial y^\beta} - h(x,y,z)\frac{\partial^\beta u_n}{\partial z^\beta}]\ d\xi, \qquad (12)$$

by the variational iteration formula (12) and initial approximation, we will get that first iterative step is the exact solution., when $\alpha = 1, \beta = 2$, as shows in this paper.

Second, to convey the basic ideal for variational iteration method to solve fractional wave like equation, Equation (5) can be written by using linear operator in the form:

$$\frac{\partial^\alpha u(x,y,z,t)}{\partial t^\alpha} = ku(x,y,z,t) - ku(x,y,z,t) + [f(x,y,z)\frac{\partial^\beta u(x,y,z,t)}{\partial x^\beta} + g(x,y,z)\frac{\partial^\beta u(x,y,z,t)}{\partial y^\beta}$$

$$+h(x,y,z)\frac{\partial^\beta u(x,y,z,t)}{\partial z^\beta}], 1 < \alpha \leq 2, 1 < \beta \leq 2, \text{ k is constant,} \quad (13)$$

using the standard variational iteration method, we construct the following correction functional as

$$u_{n+1}(x,y,z,t) = u_n(x,y,z,t) + \int_0^t \lambda(\xi,t)[\frac{\partial^\alpha u_n}{\partial \xi^2} - ku_n(x,,y,z,\xi) + ku_n(x,y,z,\xi)$$

$$f(x,y,z)\frac{\partial^\beta u_n}{\partial x^\beta} - g(x,y,z)\frac{\partial^\beta u_n}{\partial y^\beta} - h(x,y,z)\frac{\partial^\beta{}_{u_n}}{\partial z^\beta}]\, d\xi, \quad (14)$$

Now, we assume that

$$\frac{\partial^\alpha u}{\partial \xi^\alpha} \cong \Gamma(\alpha)\frac{\partial^2 u}{\partial \xi^2},\ 1 < \alpha \leq 2, \quad (15)$$

$if\ \alpha = 2,$ Equation (15) becomes:

$$\frac{\partial^2 u}{\partial \xi^2} = \Gamma(2)\frac{\partial^2 u}{\partial \xi^2}.$$

Substituting (15) in (14), we obtain

$$u_{n+1}(x,y,z,t) = u_n(x,y,z,t) + \int_0^t \lambda(\xi,t)[\Gamma(\alpha)\frac{\partial^2 u}{\partial \xi^2} - ku_n(x,,y,z,\xi) + ku_n(x,y,z,\xi) +$$

$$f(x,y,z)\frac{\partial^\beta u_n}{\partial x^\beta} + g(x,y,z)\frac{\partial^\beta u_n}{\partial y^\beta} + h(x,y,z)\frac{\partial^\beta{}_{u_n}}{\partial z^\beta}]\, d\xi,$$

$$\delta u_{n+1}(x,y,z,t) = \delta u_n(x,y,z,t) + \delta \int_0^t \lambda(\xi,t)\ [\Gamma(\alpha)\frac{\partial^2 u}{\partial \xi^2} - ku_n(x,y,z,\xi) +$$

$$k\tilde{u}_n(x,y,z,\xi\ ) - f(x,y,z)\frac{\partial^\beta \tilde{u}_n}{\partial x^\beta} - g(x,y,z)\frac{\partial^\beta \tilde{u}_n}{\partial y^\beta} - h(x,y,z)\frac{\partial^\beta{}_{\tilde{u}_n}}{\partial z^\beta}]\ d\xi, \qquad (16)$$

$$\delta u_{n+1}(x,y,z,t) = \delta u_n(x,y,z,t) - \Gamma(\alpha)\frac{\partial \lambda(\xi,t)}{\partial \xi}\delta u_n(x,y,z,\xi) + \Gamma(\alpha)\ \lambda(\xi,t)\frac{\partial \delta u_n(x,y,z,\xi)}{\partial \xi} +$$

$$\Gamma(\alpha)\ \int_0^t \frac{\partial^2 \lambda(\xi,t)}{\partial \xi^2}\delta u_n(x,y,z,\xi)d\xi - k\int_0^t \lambda(\xi,t)\ \delta u_n\ (x,y,z,\xi)d\xi, \qquad (17)$$

moreover, the stationary conditions are as follow

$$\Gamma(\alpha)\ \frac{\partial^2 \lambda(\xi,t)}{\partial \xi^2}\Big|_{\xi=t} -\ k\lambda(\xi,t)\Big|_{\xi=t} = 0,$$

$$1 - \Gamma(\alpha)\frac{\partial \lambda(\xi,t)}{\partial \xi}\Big|_{\xi=t} = 0,$$

$$\Gamma(\alpha)\lambda(\xi,t)|_{\xi=t} = 0,$$

therefore, the general Lagrange multiplier can be readily identified by

$$\lambda(\xi,t) = \frac{1}{(\Gamma\alpha)^{1/2}}\sinh(\frac{k\xi - kt}{(\Gamma\alpha)^{\frac{1}{2}}}), \qquad (18)$$

Substituting (18) for correction functional (14), we have the following iteration formula:

$$u_{n+1}(x,y,z,t) = u_n(x,y,z,t) + \frac{1}{(\Gamma\alpha)^{1/2}}\int_0^t \sinh(\frac{k\xi - kt}{(\Gamma\alpha)^{\frac{1}{2}}})[\frac{\partial^\alpha u_n}{\partial \xi^\alpha} - f(x,y,z)\frac{\partial^\beta u_n}{\partial x^\beta}$$

$$-g(x,y,z)\frac{\partial^\beta u_n}{\partial y^\beta} - (x,y,z)\frac{\partial^\beta{}_{u_n}}{\partial z^\beta}]\ d\xi, \qquad (19)$$

by the variational iteration formula (19) and initial approximation, we will get that first iterative step is the exact solution., when $\alpha = 2, \beta = 2$, as shows in this paper.

3.Applications and results.

Example 3.1.We consider the one –dimensional space-time fractional heat -like equation Molliq, at el.[2009]:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{1}{2} x^2 \frac{\partial^{\beta} u}{\partial x^{\beta}}, \quad 0 < x < 1, 0 < \alpha \leq 1, 1 < \beta \leq 2, t > 0, \qquad (20)$$

the initial condition

$$u(x,0) = x^2, \qquad (21)$$

The exact solution $\alpha = 1, \beta = 2$

$$u(x,t) = x^2 e^t, \qquad (22)$$

from Equation(6), we can be written (20) in the form

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \frac{1}{2} \frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}} + u(x,t) - u(x,t) \text{ , where } k = 1. \qquad (23)$$

We make the correction functional and the stationary conditions for Equation(23), the Lagrange multiplier can be determined as :

$$\lambda(\xi,t) = -\frac{1}{\Gamma(1+\alpha)} e^{-\frac{(\xi-t)}{\Gamma(1+\alpha)}} \qquad (24)$$

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \frac{1}{\Gamma(1+\alpha)} e^{-\frac{(\xi-t)}{\Gamma(1+\alpha)}} \left[\frac{\partial^{\alpha} u_n(x,\xi)}{\partial \xi^{\alpha}} - \frac{1}{2} x^2 \frac{\partial^{\beta} u_n(x,\xi)}{\partial x^{\beta}}\right] d\xi ,$$

$$u_1(x,t) = u_0(x,t) - \int_0^t \frac{1}{\Gamma(1+\alpha)} e^{-\frac{(\xi-t)}{\Gamma(1+\alpha)}} \left[\frac{\partial^{\alpha} u_0(x,\xi)}{\partial \xi^{\alpha}} - \frac{1}{2} x^2 \frac{\partial^{\beta} u_0(x,\xi)}{\partial x^{\beta}}\right] d\xi,$$

$$u_1(x,t) = x^2 - \int_0^t \frac{1}{\Gamma(1+\alpha)} e^{-\frac{(\xi-t)}{\Gamma(1+\alpha)}} \left[\frac{\partial^{\alpha} x^2}{\partial \xi^{\alpha}} - \frac{1}{2} x^2 \frac{\partial^{\beta} x^2}{\partial x^{\beta}}\right] d\xi,$$

$$u_1(x,t) = x^2 + \left[-\frac{1}{2} x^2 \frac{\Gamma(3)}{\Gamma(3-\beta)} x^{2-\beta} + \frac{1}{2} x^2 \frac{\Gamma(3)}{\Gamma(3-\beta)} x^{2-\beta} e^{\frac{t}{\Gamma(1+\alpha)}}\right],$$

$$u_1(x,t) = x^2 + \left[\frac{-x^{\beta}}{\Gamma(3-\beta)} + \frac{x^{4-\beta}}{\Gamma(3-\beta)} e^{\frac{t}{\Gamma(1+\alpha)}}\right],$$

when $\alpha = 1, \beta = 2$, $u_1(x,t) = x^2 e^t$, is the exact solution.

Table.1. Approximate solutions for example (1) of $u_1(x,t)$.

t	x	$\alpha = 0.5, \beta = 2$	$\alpha = 0.5, \beta = 1.7$	$\alpha = 1, \beta = 1.5$	$\alpha = 1, \beta = 2$
0.25	0.3	0.1193313959	0.1127744990	0.1057984836	0.1155622875
0.25	0.6	0.4773255836	0.4721547886	0.4493697188	0.4622491501
0.25	0.9	1.0739825630	1.0949887500	1.0562739860	1.0400605880
0.5	0.3	0.1582220228	0.1429713072	0.1260841380	0.1483849144
0.5	0.6	0.6328880910	0.6208613162	0.5641227091	0.5935396576
0.5	0.9	1.4239982050	1.4728565880	1.3724960430	1.3354642300
0.75	0.3	0.2097872760	0.1830093884	0.1521314338	0.1905300015
0.75	0.6	0.8391491039	0.8180319553	0.7114684653	0.7621200061
0.75	0.9	1.8880854840	1.9738732160	1.7785332010	1.7147700140
1.0	0.3	0.2781578720	0.2360960560	0.1855768236	0.2446453645
1.0	0.6	1.1126314880	1.0794613730	0.9006641610	0.9785814581
1.0	0.9	2.5034208480	2.6381733670	2.2998952310	2.2018082810

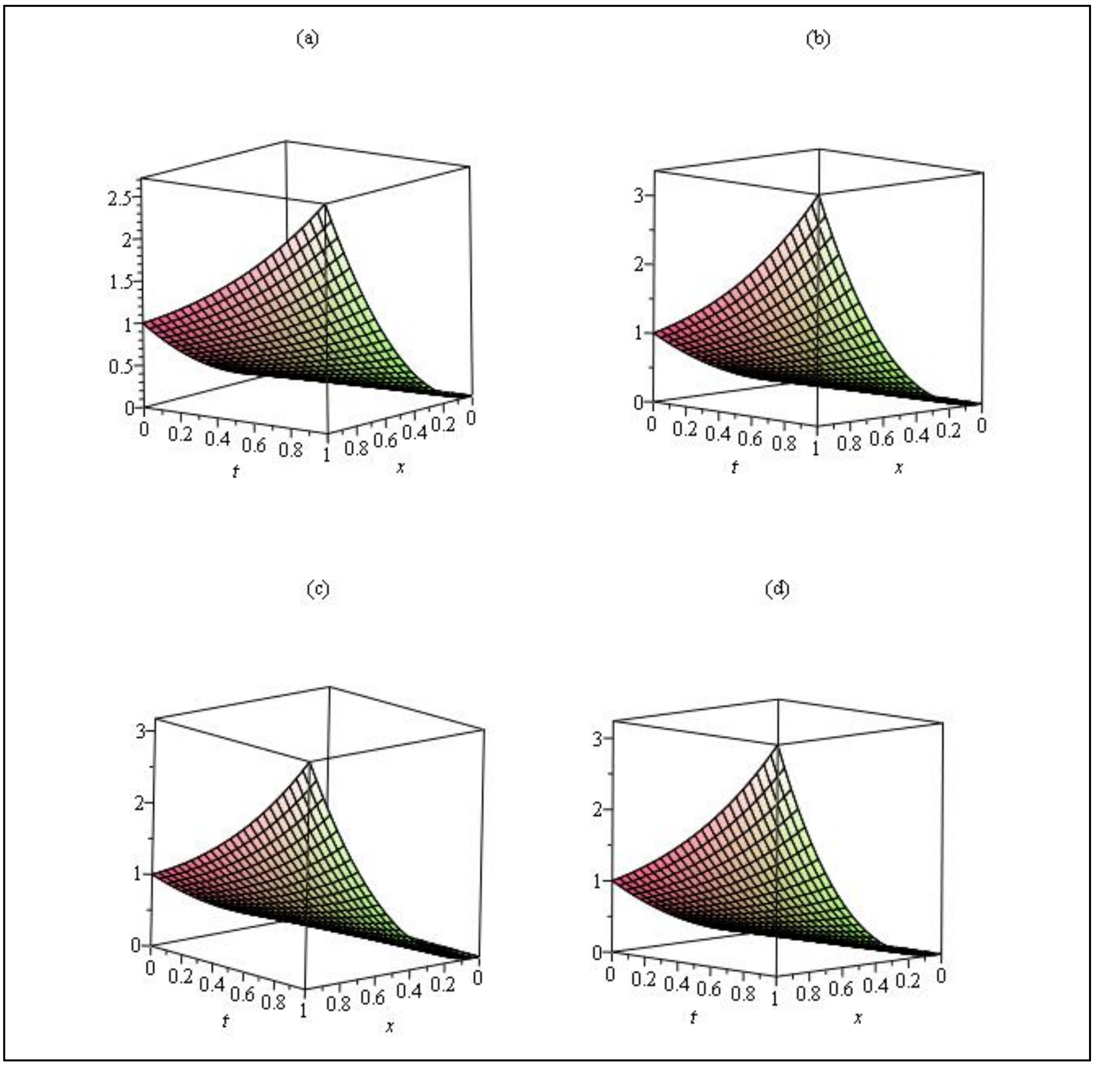

Fig.1 . The surface show the solution $u_1(x,t)$ for example (1)

(a) $\alpha = 1$, $\beta = 2$ (b) $\alpha = 0.5$, $\beta = 1.5$

(c) $\alpha = 0.75$, $\beta = 1.75$ (d) $\alpha = 0.4$, $\beta = 1.2$

Example.2. We consider the two–dimensional space-time fractional heat -like equation . Batiha, at el.[2009]:

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^\beta u}{\partial x^\beta}+\frac{\partial^\beta u}{\partial y^\beta},\quad 0<x,y<2\pi,\ 0<\alpha\leq 1, 1<\beta\leq 2, t>0, \qquad (25)$$

with the initial condition

$$\mathrm{u}(x,\mathrm{y},0)=\sin x\ \sin\mathrm{y}. \qquad (26)$$

The exact solution when($\alpha=1,\beta=2$),

$u(x,y,t)=sinx\ siny\ e^{-2t}$,

we can be written (25) in the form:

$$\frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha}=\frac{\partial^\beta u(x,y,t)}{\partial x^\beta}+\frac{\partial^\beta u(x,y,t)}{\partial y^\beta}-2u(x,y,t)+2u(x,y,t)\ \text{, where }\ k=-2, \qquad (27)$$

we make the correction functional and the stationary conditions for Equation(27), the Lagrange multiplier can be determined as :

$$\lambda(\xi,t)=-\frac{1}{\Gamma(1+\alpha)}\mathrm{e}^{\frac{2(\xi-t)}{\Gamma(1+\alpha)}}\ , \qquad (28)$$

$$u_{n+1}(x,y,t)=u_n(x,\mathrm{y},\mathrm{t})-\int_0^t\frac{1}{\Gamma(1+\alpha)}\mathrm{e}^{\frac{2(\xi-t)}{\Gamma(1+\alpha)}}\left[\frac{\partial^\alpha u_n(x,y,\xi)}{\partial \xi^\alpha}-\frac{\partial^\beta u_n(x,y,\xi)}{\partial x^\beta}-\frac{\partial^\beta u_n(x,y,\xi)}{\partial y^\beta}\right]d\xi\ ,$$

$$u_1(x,y,t)=u_0(x,\mathrm{y},\mathrm{t})-\int_0^t\frac{1}{\Gamma(1+\alpha)}\mathrm{e}^{\frac{2(\xi-t)}{\Gamma(1+\alpha)}}\left[\frac{\partial^\alpha u_0(x,y,\xi)}{\partial \xi^\alpha}-\frac{\partial^\beta u_0(x,y,\xi)}{\partial x^\beta}-\frac{\partial^\beta u_n(x,y,\xi)}{\partial y^\beta}\right]d\xi,$$

$$u_1(x,y,t)=\sin x\ \sin\mathrm{y}-\int_0^t\frac{1}{\Gamma(1+\alpha)}\mathrm{e}^{\frac{2(\xi-t)}{\Gamma(1+\alpha)}}\left[-\sin\mathrm{y}\sin\left(x+\frac{\pi\beta}{2}\right)-\ \sin\mathrm{x}\sin\left(\mathrm{y}+\frac{\pi\beta}{2}\right)\right]d\xi,$$

$u_1(x,y,t) = \sin x \sin y + \frac{1}{2}\left[\sin y \sin\left(x + \frac{\pi\beta}{2}\right) + \sin x \sin\left(y + \frac{\pi\beta}{2}\right)\right](1 - e^{\frac{-2t}{\Gamma(1+\alpha)}}),$

when $\alpha = 1, \beta = 2,\ u_1(x,y,t) = sinx\ siny\ e^{-2t}$, is the exact solution.

Table.2.Approximate solutions for example (2) of $u_1(x,y,t)$.

t	x	y	$\alpha = 0.5$, β =1.75	$\alpha = 0.7$, β =1.5	$\alpha = 1$, $\beta = 2$
0.1	$\pi/4$	2 $\pi/3$	0.5080808599	0.5449013478	0.5013681456
0.1	$\pi/4$	4 $\pi/3$	-0.5354144104	-0.5942936123	-0.5013681456
0.2	$\pi/4$	2 $\pi/3$	0.4248586083	0.4907604590	0.4104855193
0.2	$\pi/4$	4 $\pi/3$	-0.4740036962	-0.5797866048	-0.4104855193
0.3	$\pi/4$	2 $\pi/3$	0.3584491875	0.4473161347	0.3360771183
0.3	$\pi/4$	4 $\pi/3$	-0.4249993749	-0.5681457332	-0.3360771183
0.4	$\pi/4$	2 $\pi/3$	0.3054560146	0.4124550646	0.2751566721
0.4	$\pi/4$	4 $\pi/3$	-0.3858950665	-0.5588047377	-0.2751566721
0.5	$\pi/4$	2 $\pi/3$	0.2631686956	0.3844814633	0.2252792295
0.5	$\pi/4$	4 $\pi/3$	-0.3546907382	-0.5513092338	-0.2252792295

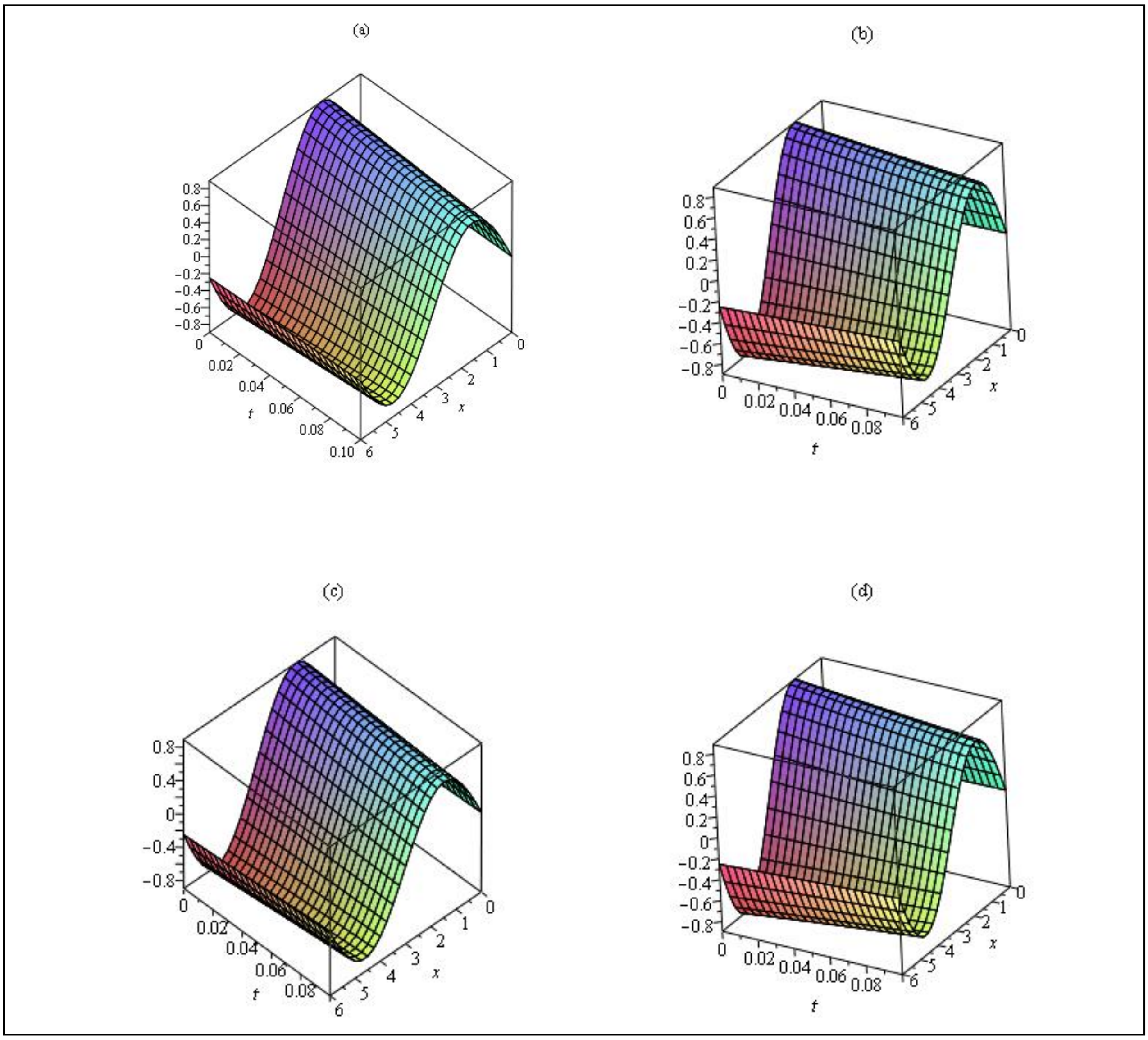

Fig.2 . The surface show the solution $u_1(x, y, t)$ for example (2)

(a) $\alpha = 1$, $\beta = 2$ (b) $\alpha = 0.75$, $\beta = 1.75$

(c) $\alpha = 0.5$, $\beta = 1.5$ (d) $\alpha = 0.5$, $\beta = 1.75$

Example.3. We consider the three–dimensional time fractional heat -like equation Molliq, at el [2009]:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = x^4y^4z^4 + \frac{1}{36}\left[x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + z^2\frac{\partial^2 u}{\partial z^2}\right],\ 0 < x, y, z < 1,\ 0 < \alpha \le 1,\ t > 0, \quad (29)$$

with the initial condition

$$\mathrm{u}(x, \mathrm{y}, \mathrm{z}, 0) = 0. \quad (30)$$

The exact solution when($\alpha = 1$),

$u(x, y, z, t) = x^4y^4z^4(\ e^t - 1)$,

we can be written (29) in the form:

$$\frac{\partial^\alpha u(x,y,z,t)}{\partial t^\alpha} = x^4y^4z^4 + \frac{1}{36}\left[x^2\frac{\partial^2 u(x,y,z,t)}{\partial x^2} + y^2\frac{\partial^2 u(x,y,z,t)}{\partial y^2} + z^2\frac{\partial^2 u(x,y,z,t)}{\partial z^2}\right]$$

$$+u(x, y, z, t) - u(x, y, z, t),\ where\ k = 1. \quad (31)$$

We make the correction functional and the stationary conditions for Equation(31), the Lagrange multiplier can be determined as :

$$\lambda(\xi, t) = -\frac{1}{\Gamma(1+\alpha)}\mathrm{e}^{-\frac{(\xi-\mathrm{t})}{\Gamma(1+\alpha)}}, \quad (32)$$

$$u_{n+1}(x, y, z, t) = u_\mathrm{n}(x, \mathrm{y}, \mathrm{z}, \mathrm{t}) - \int_0^\mathrm{t}\frac{1}{\Gamma(1+\alpha)}\mathrm{e}^{-\frac{(\xi-\mathrm{t})}{\Gamma(1+\alpha)}}\left[\frac{\partial^\alpha u}{\partial \xi^\alpha} - x^4y^4z^4 - \frac{1}{36}(x^2\frac{\partial^2 u_n(x,y,z,\xi)}{\partial x^2}\right.$$

$$\left.+y^2\frac{\partial^2 u_n(x,y,z,\xi)}{\partial y^2} + z^2\frac{\partial^2 u_n(x,y,z,\xi)}{\partial z^2})\right]d\xi,$$

$$u_1(x, y, z, t) = u_0(x, \mathrm{y}, \mathrm{z}, \mathrm{t}) + \left[x^4y^4z^4\mathrm{e}^{\frac{\mathrm{t}}{\Gamma(1+\alpha)}} - x^4y^4z^4\right],$$

$$u_1(x, y, z, t) = x^4y^4z^4\left[\mathrm{e}^{\frac{\mathrm{t}}{\Gamma(1+\alpha)}} - 1\right],$$

when $\alpha = 1$,

$u_1(x, y, z, t) = x^4y^4z^4(\ e^t - 1)$, is the exact solution.

Table 3.Approximate solutions for example (3) of $u_1(x, y, z, t)$.

t	x	y	z	$\alpha = 0.7$	$\alpha = 0.9$	$\alpha = 1$
0.2	0.25	0.1	0.3	7.79033826610^{-10}	7.31381362010^{-10}	7.00532164010^{-10}
0.2	0.215	0.1	0.6	1.24645412210^{-8}	1.17021017910^{-8}	1.12085146210^{-8}
0.2	0.25`	0.5	0.3	4.86896141510^{-7}	4.57113351210^{-7}	4.37832602410^{-7}
0.2	0.25	0.5	0.6	0.000007790338266	0.000007313813620	0.000007005321640
0.5	0.25	0.1	0.3	2.32155960210^{-9}	2.15732389710^{-9}	2.05259464710^{-9}
0.5	0.25	0.1	0.6	3.71449536210^{-8}	3.45171823510^{-8}	3.28415143410^{-8}
0.5	0.25	0.5	0.3	0.000001450974751	0.000001348327435	0.000001282871654
0.5	0.25	0.5	0.6	0.000023215596020	0.000021573238970	0.000020525946470
0.8	0.25	0.1	0.3	4.46751617410^{-9}	4.10523829810^{-9}	3.87768809210^{-9}
0.8	0.25	0.1	0.6	7.14802587810^{-8}	6.56838127710^{-8}	6.20430094810^{-8}
0.8	0.25	0.5	0.3	0.000002792197608	0.000002565773936	0.000002423555057
0.8	0.25	0.5	0.6	0.000044675161740	0.000041052382980	0.000038776880920

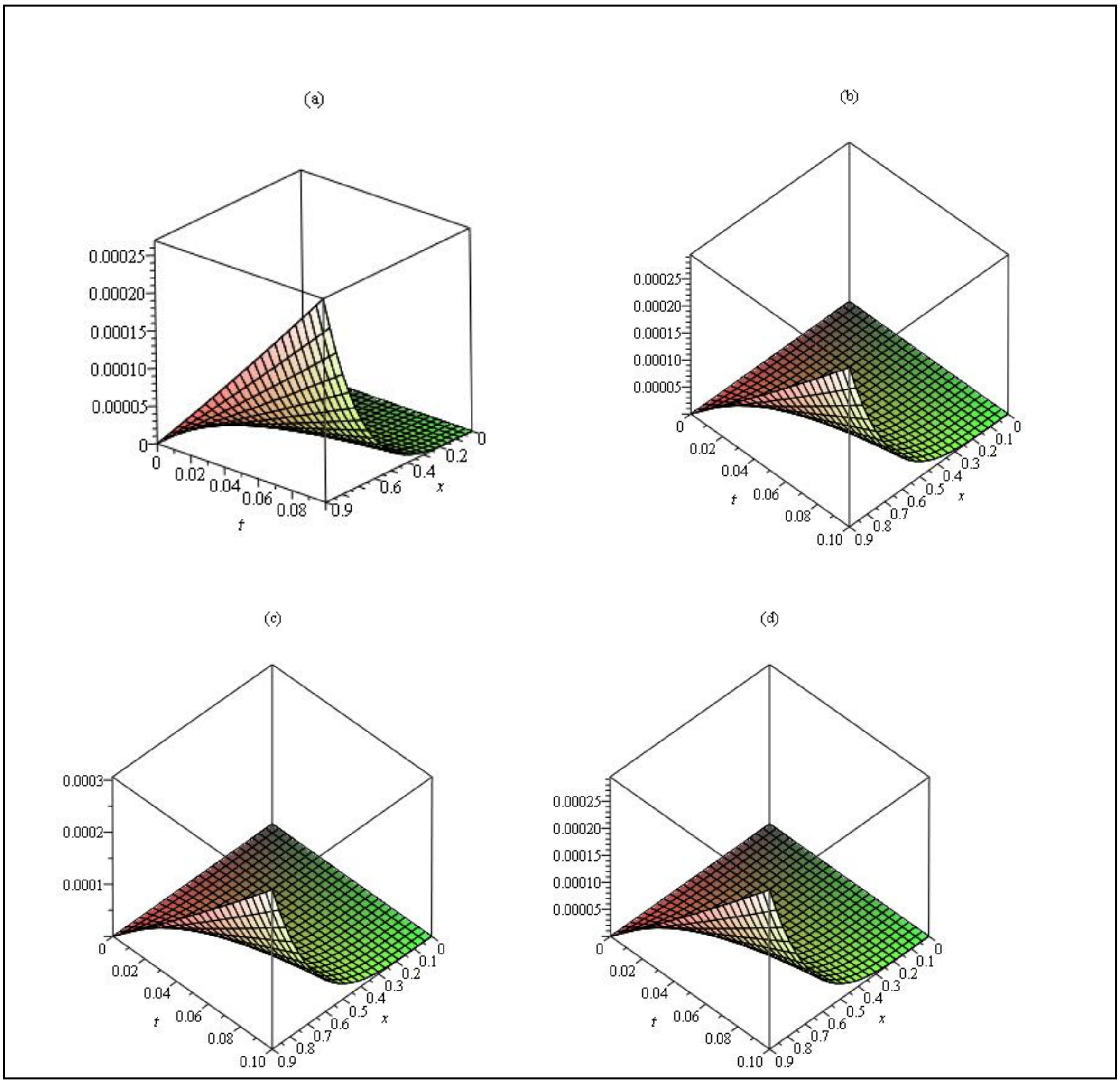

Fig.3 . The surface show the solution $u_1(x, y, z, t)$ for example(3),y=z=0.5

$(a)\ \ \alpha = 1, \beta = 2\ (b)\ \ \alpha = 0.75, \beta = 1.75$

$(c)\ \ \alpha = 0.5, \beta = 1.5\ \ (d)\ \ \alpha = 0.5, \beta = 1.75$

***Example* .4.** Consider the one-dimensional fractional wave-like equation Noor and Mohyud [2008]:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{1}{2} x^2 \frac{\partial^{\beta} u}{\partial x^{\beta}}, \quad 0 < x < 1, 1 < \alpha, \beta \leq 2, t > 0, \tag{33}$$

with the initial condition

$$u(x,0) = x, \quad \frac{\partial u(x,0)}{\partial t} = x^2, \quad 0 < x < 1, \tag{34}$$

the exact solution when ($\alpha = 2, \beta = 2$)

$u(x,t) = x + x^2 \sinh t,$

we can be written (33) in the form

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \frac{1}{2} x^2 \frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}} + u(x,t) - u(x,t), \text{where } k = 1. \tag{35}$$

We make the correction functional and the stationary conditions for Equation(35), the Lagrange multiplier can be determined as

$$\lambda(\xi, t) = \frac{1}{(\Gamma\alpha)^{1/2}} \sinh\left(\frac{\xi - t}{(\Gamma\alpha)^{\frac{1}{2}}}\right), \tag{36}$$

$$u_1(x,t) = u_0(x,\mathrm{t}) + \int_0^{\mathrm{t}} \frac{1}{(\Gamma\alpha)^{\frac{1}{2}}} \sinh \frac{(\xi - \mathrm{t})}{(\Gamma\alpha)^{\frac{1}{2}}} \left[\frac{\partial^\alpha u_0(x,\xi)}{\partial \xi^\alpha} - \frac{1}{2} x^2 \frac{\partial^\beta u_0(x,\xi)}{\partial x^\beta} \right] d\xi,$$

$$u_1(x,t) = x + x^2 t + \int_0^{\mathrm{t}} \frac{1}{(\Gamma\alpha)^{1/2}} \sinh \frac{(\xi - \mathrm{t})}{(\Gamma\alpha)^{\frac{1}{2}}} \left[\frac{\partial^\alpha (x + x^2\xi)}{\partial \xi^\alpha} - \frac{1}{2} x^2 \frac{\partial^\beta (x + x^2\xi)}{\partial x^\beta} \right] d\xi,$$

$$u_1(x,t) = x + x^2 t - \frac{x^{4-\beta}}{\Gamma(3-\beta)} \mathrm{t} + \frac{x^{4-\beta} (\Gamma\alpha)^{1/2}}{\Gamma(3-\beta)} \sinh \left(\frac{\mathrm{t}}{(\Gamma\alpha)^{1/2}} \right).$$

when $\alpha = 2, \beta = 2,$ $u_1(x,t) = x + x^2 \sinh t$, is the exact solution.

Table.4.Approximate solutions $for\ example$ (4) of $u_1(x,t)$

t	x	$\alpha = 1.5\ ,\beta = 2$	$\alpha = 1.6\ ,\beta$ =1.75	$\alpha = 1.5\ ,\beta = 2$	$\alpha = 2\ ,\beta = 2$
0.25	0.3	0.3227653980	0.3351007893	0.3227653980	0.3227351085
0.25	0.6	0.6910615919	0.6988538720	0.6910615919	0.6909404340
0.25	0.9	1.1048885820	1.0597371790	1.1048885820	1.1046159770
0.5	0.3	0.3471457537	0.3549947719	0.3471457537	0.3468985775
0.5	0.6	0.7885830149	0.7934861345	0.7885830149	0.7875943100
0.5	0.9	1.3243117830	1.2953747600	1.3243117830	1.3220871980
0.75	0.3	0.3748705886	0.3775961165	0.3748705886	0.3740085059
0.75	0.6	0.89948235465	0.9009968539	0.8994823546	0.8960340235
0.75	0.9	1.5738352980	1.5630801410	1.5738352980	1.5660765530
1.0	0.3	0.4079066787	0.4044949888	0.4079066787	0.4057681075
1.0	0.6	1.0316267150	1.0289501750	1.0316267150	1.0230724300
1.0	0.9	1.8711601080	1.8816883020	1.8711601080	1.8519129670

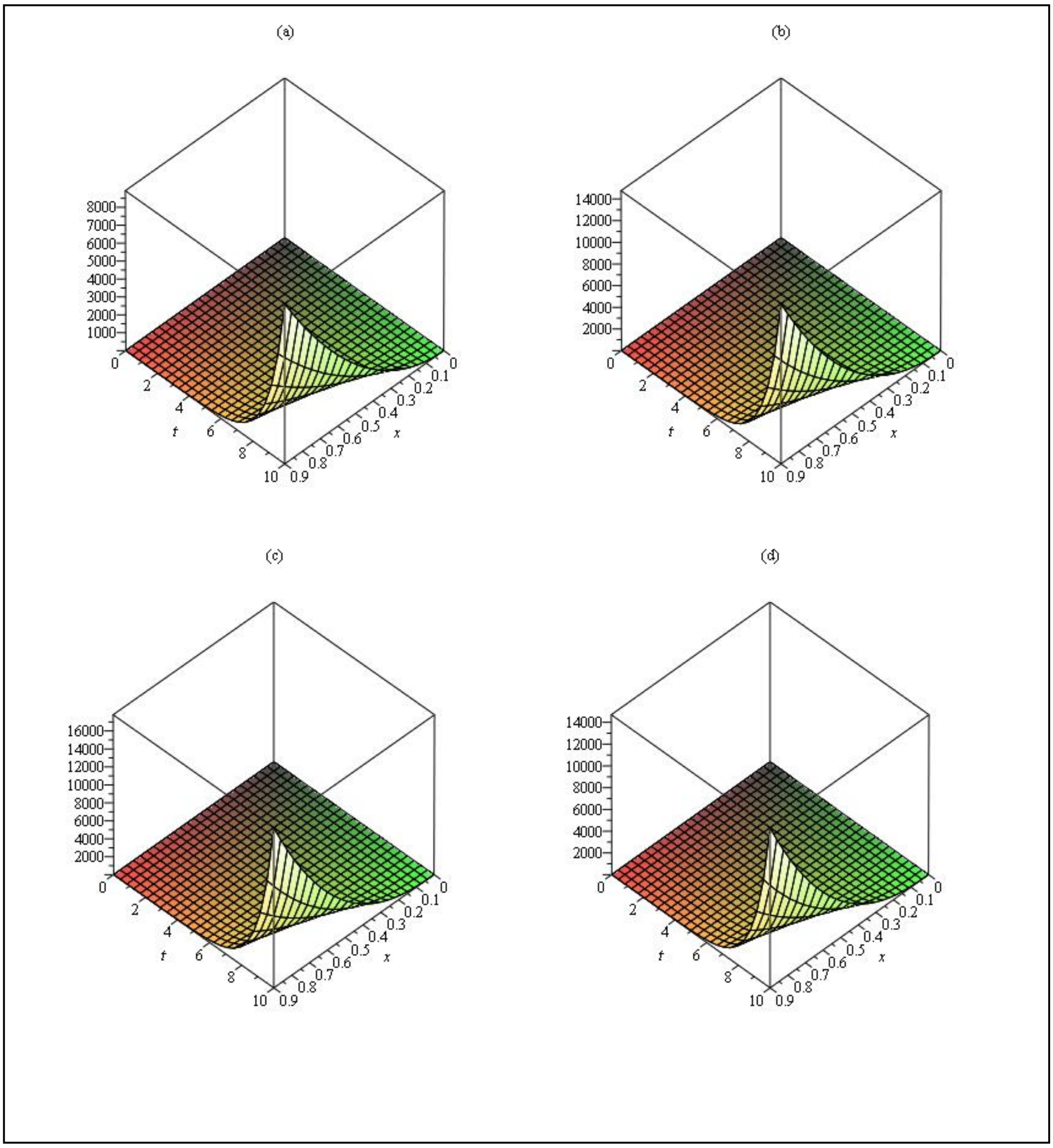

Fig.4 .The surface show the solution $u_1(x,t)$ for example (4).
$(a)\ \ \alpha = 2, \beta = 2$, $(b)\ \ \alpha = 1.75, \beta = 1.75$

$(c)\ \ \alpha = 1.5, \beta = 1.5$, $(d)\ \ \alpha = 1.5, \beta = 1.75$,

Example.5. We consider the three–dimensional space-time fractional wave -like equation Molliq, at el. [2009]:

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{1}{12}\left[x^2\frac{\partial^\beta u}{\partial x^\beta}+y^2\frac{\partial^\beta u}{\partial y^\beta}\right],\ 0<x,y,z<1,\ 1<\alpha,\beta\le 2,t>0, \qquad (37)$$

with the initial condition

$$\mathrm{u}(\mathrm{x},\mathrm{y},0)=\mathrm{x}^4\ , \qquad \mathrm{u_t}(\mathrm{x},\mathrm{y},0)=\mathrm{y}^4, \qquad (38)$$

the exact solution when ($\alpha=2,\beta=2$),

$$u(x,y,t)=x^4cosht+y^4\sinh t,$$

we can be written (37) in the form

$$\frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha}=\frac{1}{12}x^2\frac{\partial^\beta u(x,y,t)}{\partial x^\beta}+\frac{1}{12}y^2\frac{\partial^\beta u(x,y,t)}{\partial y^\beta}+u(x,y,t)-u(x,y,t),\ \text{where}\ \ k=1. \qquad (39)$$

We make the correction functional and the stationary conditions for Equation(39), the Lagrange multiplier can be determined as :

$$\lambda(\xi,t)=\frac{1}{(\Gamma\alpha)^{1/2}}\sinh(\frac{\xi-\mathrm{t}}{(\Gamma\alpha)^{\frac{1}{2}}}), \qquad (40)$$

$$u_1(x,y,t)=u_0+\int_0^{\mathrm{t}}\frac{1}{(\Gamma\alpha)^{1/2}}\sinh\frac{(\xi-\mathrm{t})}{(\Gamma\alpha)^{\frac{1}{2}}}\left[\frac{\partial^\alpha u_0(x,y,\xi)}{\partial \xi^\alpha}-\frac{1}{12}(x^2\frac{\partial^\beta u_0(x,y,\xi)}{\partial x^\beta}+y^2\frac{\partial^\beta u_0(x,y,\xi)}{\partial y^\beta})\right]d\xi,$$

$$u_1=x^4+y^4t+\int_0^{\mathrm{t}}\frac{1}{(\Gamma\alpha)^{1/2}}\sinh\frac{(\xi-\mathrm{t})}{(\Gamma\alpha)^{\frac{1}{2}}}\left[\frac{\partial^\alpha(x^4+y^4\xi)}{\partial \xi^\alpha}-\frac{1}{12}(x^2\frac{\partial^\beta(x^4+y^4\xi)}{\partial x^\beta}+y^2\frac{\partial^\beta\ (x^4+y^4\xi)}{\partial y^\beta})\right]d\xi,$$

$$u_1=x^4+y^4t+\int_0^{\mathrm{t}}\frac{1}{(\Gamma\alpha)^{1/2}}\sinh\frac{(\xi-\mathrm{t})}{(\Gamma\alpha)^{1/2}}\left[0-\frac{1}{12}(\frac{4\Gamma(4)}{\Gamma(5-\beta)}x^{6-\beta}+\xi\frac{4\Gamma(4)}{12\Gamma(5-\beta)}y^{6-\beta})\right]d\xi,$$

$$\mathrm{u}_1=\mathrm{x}^4+\mathrm{y}^4\mathrm{t}-\left[\frac{\Gamma 3\ \mathrm{x}^{6-\beta}}{\Gamma(5-\beta)}-\frac{\Gamma 3\ \mathrm{x}^{6-\beta}}{\Gamma(5-\beta)}\cosh(\frac{\mathrm{t}}{(\Gamma\alpha)^{\frac{1}{2}}})+\frac{\Gamma(3)}{\Gamma(5-\beta)}\mathrm{y}^{6-\beta}\mathrm{t}-\frac{\Gamma(3)(\Gamma\alpha)^{1/2}}{\Gamma(5-\beta)}\mathrm{y}^{6-\beta}\sinh(\frac{\mathrm{t}}{(\Gamma\alpha)^{\frac{1}{2}}})\right],$$

when $\alpha=2,\beta=2,\ u_1(x,y,t)=x^4cosht+y^4\sinh t$, is the exact solution.

Table.5.Approximate solutions $for\ example$ (5) of $u_1(x, y, t)$.

t	x	y	$\alpha = 1.5$, $\beta = 2$	$\alpha = 1.75$, β =*1.6*	$\alpha = 1.7$, β =1.9	$\alpha = 2$, $\beta = 2$
0.2	0.01	0.25	0.0007871504787	*0.0007834483526*	*0.0007858114192*	0.000786478961
0.2	0.01	0.75	0.0637583706500	0.0635563319100	*0.0636927241700*	0.063703979740
0.2	0.05	0.25	0.0007935318309	0.0007897159584	*0.0007921444216*	0.000792844177
0.2	0.05	0.75	0.0637647520000	0.0635625995100	*0.0636990571700*	0.063710344960
0.5	0.01	0.25	0.0020462681160	0.0019877205200	*0.0020250759750*	0.002035539814
0.5	0.01	0.75	0.1657468018000	0.1625504995000	*0.1647069571000*	0.164877822800
0.5	0.05	0.25	0.0020534091370	0.0019941363700	*0.0020319085130*	0.002042576200
0.5	0.05	0.75	0.1657539429000	0.1625569154000	*0.1647137895000*	0.164884859200
0.8	0.01	0.25	0.0035149573230	0.0032697062880	*0.1647137895000*	0.003469177366
0.8	0.01	0.75	0.2847104364000	0.2713132109000	*0.2803418546000*	0.281002296800
0.8	0.05	0.25	0.0035235893720	0.0032764125210	*0.0034338751830*	0.003477522960
0.8	0.05	0.75	0.2847190685000	0.2713199172000	*0.2803496662000*	0.281010642400

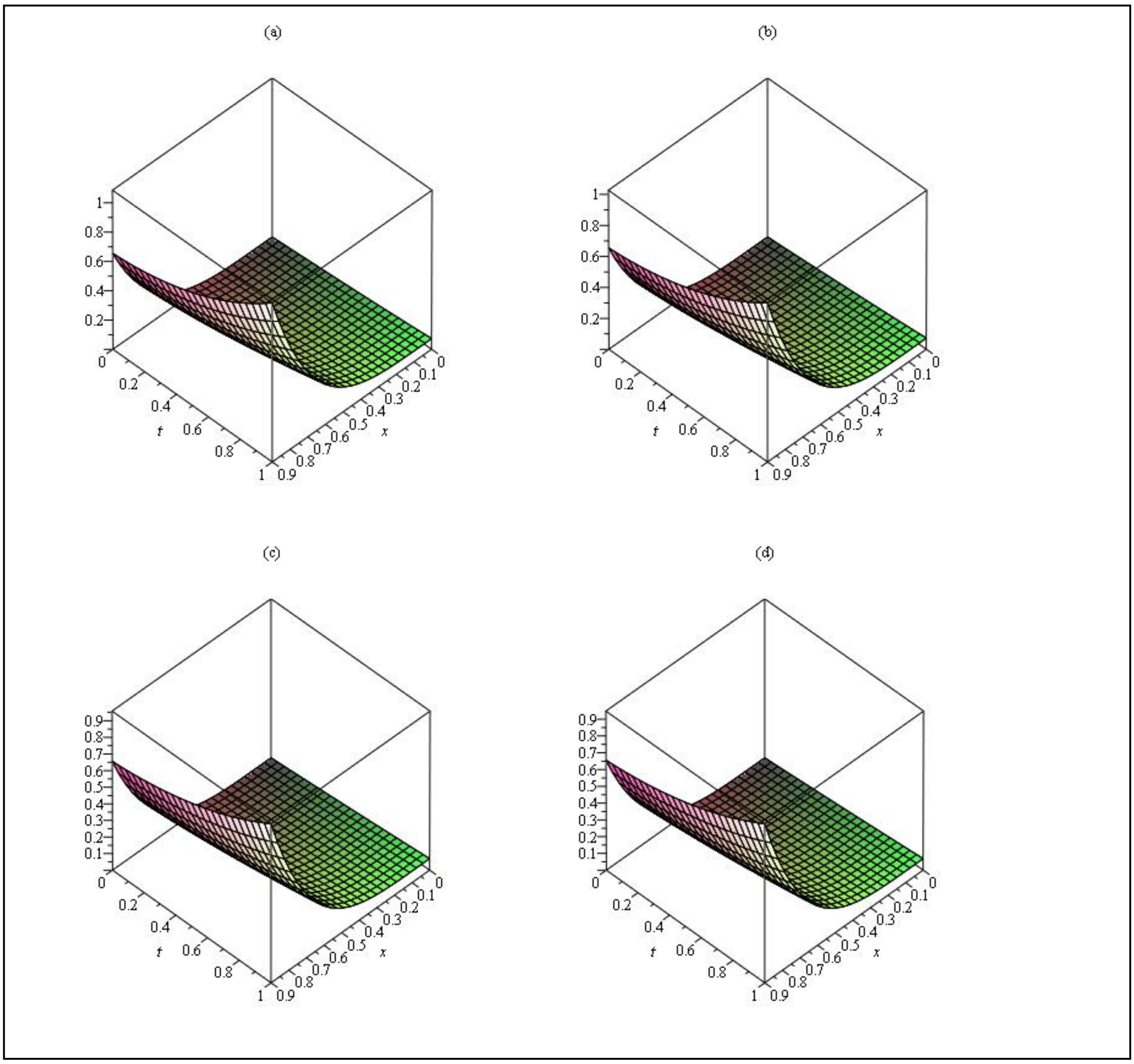

Fig.5 .The surface show the solution $u_1(x, y, t)$ for example (5), y =0.5

(a) $\alpha = 2, \beta = 2$, (b) $\alpha = 1.75, \beta = 1.75$

(c) $\alpha = 1.5, \beta = 1.5$, (d) $\alpha = 1.75, \beta = 1.5$,

Example.6. We consider the three–dimensional space-time fractional wave -like equation Molliq, at el[2009]
$\frac{\partial^\alpha u}{\partial t^\alpha} = x^2 + y^2 + z^2 + \frac{1}{2}\left[x^2 \frac{\partial^\beta u}{\partial x^\beta} + y^2 \frac{\partial^\beta u}{\partial y^\beta} + z^2 \frac{\partial^\beta u}{\partial z^\beta}\right]$, -:

$0 < x, y, z < 1,\ 1 < \alpha, \beta \le 2, t > 0,$ (41)

the initial condition

$u(x, y, z, 0) = 0 ,\qquad u_t(x, y, z, 0) = x^2 + y^2 - z^2,$ (42)

the exact solution when ($\alpha = 2, \beta = 2$),

$u(x, y, t) = -(x^2 + y^2 + z^2) + (x^2 + y^2)e^t + z^2 e^{-t},$

we can be written (41) in the form

$\frac{\partial^\alpha u(x,y,z,t)}{\partial t^\alpha} = x^2 + y^2 + z^2 + \frac{1}{2}x^2 \frac{\partial^\beta u(x,y,z,t)}{\partial x^\beta} + \frac{1}{2}y^2 \frac{\partial^\beta u(x,y,z,t)}{\partial y^\beta} + \frac{1}{2}z^2 \frac{\partial^\beta u(x,y,z,t)}{\partial z^\beta} +$

$u(x, y, z, t) - u(x, y, z, t) ,$ (43)

We make the correction functional and the stationary conditions for Equation(43), the Lagrange multiplier can be determined as :

$\lambda(\xi, t) = \frac{1}{(\Gamma\alpha)^{1/2}} \sinh(\frac{\xi - t}{(\Gamma\alpha)^{\frac{1}{2}}}),$ (44)

$$u_{n+1}(x, y, z, t) = u_n(x, y, z, t) + \int_0^t \frac{1}{(\Gamma\alpha)^{1/2}} \sinh(\frac{\xi - t}{(\Gamma\alpha)^{\frac{1}{2}}}) \left[\frac{\partial^\alpha u_n(x,y,z,\xi)}{\partial \xi^\alpha} - x^2 - y^2 - z^2 - \frac{1}{2}x^2 \frac{\partial^\beta u_n(x,y,z,\xi)}{\partial x^\beta} - \frac{1}{2}y^2 \frac{\partial^\beta u_n(x,y,z,\xi)}{\partial y^\beta} - \frac{1}{2}z^2 \frac{\partial^\beta u_n(x,y,z,\xi}{\partial z^\beta}\right] d\xi,$$

$$u_1(x, y, z, t) = u_0(x, y, z, t) + \int_0^t \frac{1}{(\Gamma\alpha)^{1/2}} \sinh \frac{(\xi - t)}{(\Gamma\alpha)^{1/2}} \left[\frac{\partial^\alpha u_0(x,y,z,\xi)}{\partial \zeta^\alpha} - x^2 - y^2 - z^2 - \frac{1}{2}x^2 \frac{\partial^\beta u_0(x,y,z,\xi)}{\partial x^\beta} - \frac{1}{2}y^2 \frac{\partial^\beta u_0(x,y,z,\xi)}{\partial y^\beta} - \frac{1}{2}z^2 \frac{\partial^\beta u_0(x,y,z,\xi)}{\partial z^\beta}\right] d\xi,$$

$$u_1 = (x^2 + y^2 - z^2)t + \int_0^t \frac{1}{(\Gamma\alpha)^{\frac{1}{2}}} \sinh\left(\frac{\xi - t}{(\Gamma\alpha)^{\frac{1}{2}}}\right)\left[-\frac{x^{4-\beta}}{\Gamma(3-\beta)}\xi - x^2 - y^2 - z^2 - \frac{y^{4-\beta}}{\Gamma(3-\beta)}\xi + \frac{z^{4-\beta}}{\Gamma(3-\beta)}\xi\right] d\xi,$$

$$u_1 = (x^2 + y^2 - z^2)t - (x^2 + y^2 + z^2) + (x^2 + y^2 + z^2)\cosh(\frac{t}{(\Gamma\alpha)^{\frac{1}{2}}}) - \frac{x^{4-\beta}}{\Gamma(3-\beta)}t - \frac{y^{4-\beta}}{\Gamma(3-\beta)}t + \frac{z^{4-\beta}}{\Gamma(3-\beta)}t + \left(x^{4-\beta} + y^{4-\beta} - z^{4-\beta}\right)\frac{(\Gamma\alpha)^{\frac{1}{2}}}{\Gamma(3-\beta)}\sinh\left(\frac{t}{(\Gamma\alpha)^{\frac{1}{2}}}\right),$$

when $\alpha = 2, \beta = 2,\ u_1(x, y, z, t) == -(x^2 + y^2 + z^2) + (x^2 + y^2 + z^2)e^{-t}$, is the exact solution.

Table.6.Approximate solutions for example (6) of $u_1(x, y, z, t)$.

t	x	y	z	$\alpha = 1.75$, β =1.6	$\alpha = 1.7$, β =1.9	$\alpha = 2$, $\beta = 2$
0.2	0.025	0.04	0.04	0.0002087734958	0.0002101692817	0.0002025903436
0.2	0.025	0.04	0.08	-0.003204049930	-0.000649825270	-0.000667502040
0.2	0.025	0.08	0.04	0.0038312642390	0.0012821025050	0.0012653235840
0.2	0.025	0.08	0.08	0.0004184408056	0.0004222506927	0.0003952312020
0.5	0.025	0.04	0.04	0.0008483322835	0.0008614353879	0.0008138538820
0.5	0.025	0.04	0.08	-0.004126013960	-0.000956096830	-0.001074798900
0.5	0.025	0.08	0.04	0.0071582236540	0.0040301836960	0.0039277159800
0.5	0.025	0.08	0.08	0.0021838774140	0.0022126514700	0.0020390631460
0.8	0.025	0.04	0.04	0.0019263764320	0.0019722092520	0.0018457549070
0.8	0.025	0.04	0.08	-0.004179587650	-0.000465537360	-0.000797466060
0.8	0.025	0.08	0.04	0.0115734036800	0.0079939994390	0.0077283513630
0.8	0.025	0.08	0.08	0.0054674395830	0.0055562528380	0.005081303900

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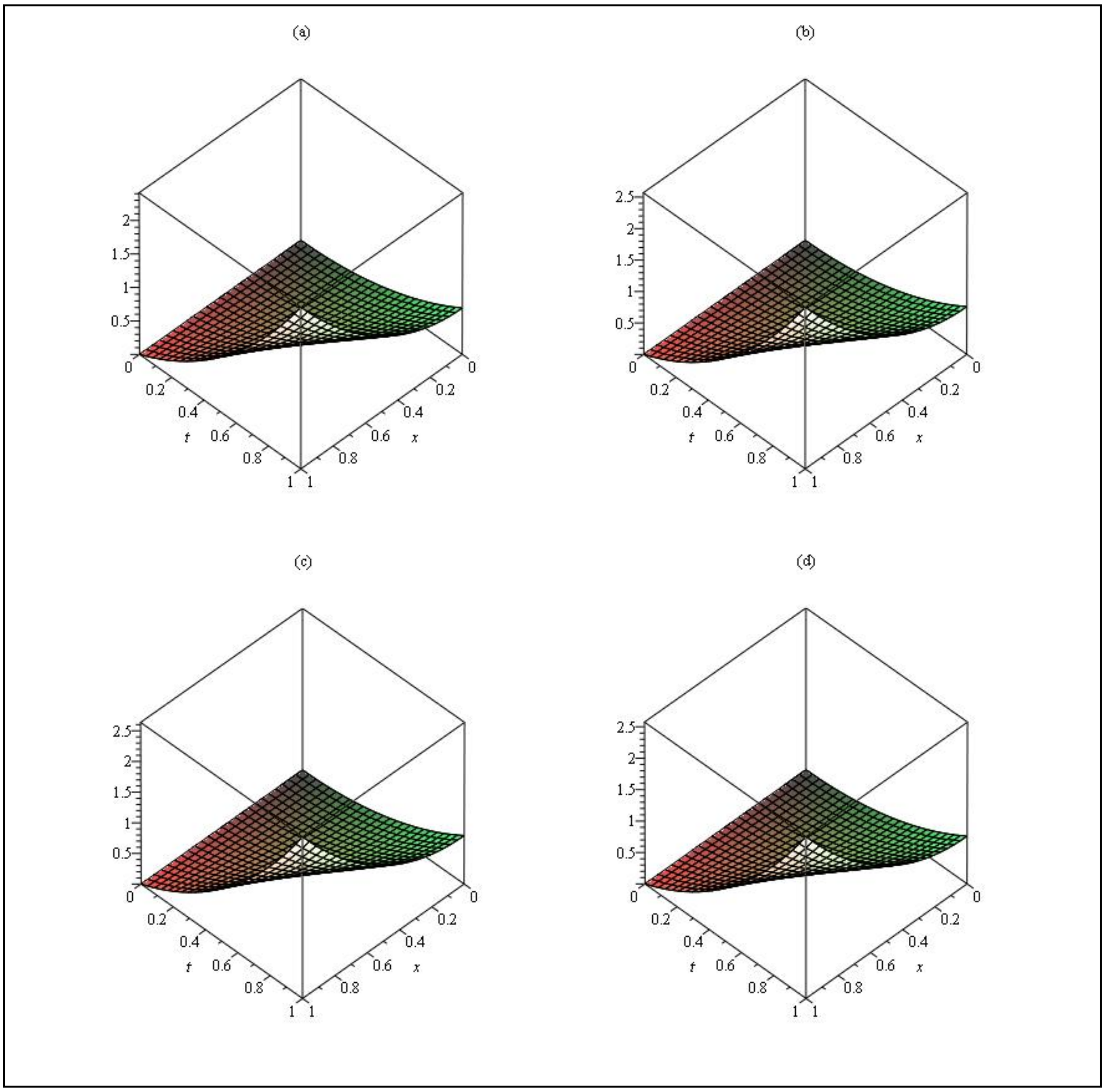

Fig.6 .The surface show the solution $u_1(x, y, z, t)$ for example (6), y =0.8,z=0.8

(a) $\alpha = 2, \beta = 2$, (b) $\alpha = 1.75, \beta = 1.75$

(c) $\alpha = 1.5, \beta = 1.5$, (d) $\alpha = 1.75, \beta = 1.5$,

4. conclusion

In this work, the variational iteration method is use to solve the fractional heat like and wave like. This technique provides the solution with first step i-e $u_1(x,t)$is the exact solution in case $\alpha = \beta = 2$ in equation wave like and $\alpha = 1, \beta = 2$ in equation heat like. Results show the ability and efficiency of this method.

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طريقة أسلوب التغاير التكراري لحل معادلتي الحرارة والموجة الخاصة الكسرية للزمن والفضاء

في هذا البحث سوف نطبق طريقة أسلوب التغاير التكراري لحل معادلة الحرارة الخاصة والموجة الخاصة الكسرية الزمن والفضاء. وفي هذه الطريقة تستخدم معامل خطى مع وضع قيمة تقريبية للمشتقة الكسرية الزمنية . ستة أمثلة تبرهن تطبيق هذه الطريقة. هذه التطبيقات بينت اهمية وكفاءة الطريقة لان الحلول التقريبية ظهرت قريبة جدا من الحل الحقيقي.

الكلمات المفتاحية: طريقة اسلوب التغاييرالتكراري, معادلة الحرارة الخاصة الكسرية والموجة الخاصة الكسرية ,مشتقة كباتو.