The Permutation Topological Spaces and their Bases

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Abstract:-

Given a permutation β in the symmetric group on n letters, we present permutation topological space (Ω, t_n^{β}) and others new concepts in topology field such as $\beta-$sets, permutation subspace $(\Omega', t_m^{\gamma^{\lambda^{\beta}}})$, permutation continuous, $\beta-$decomposition, $\beta-$connected. In this paper we prove that every permutation space is an Lindelof space. Moreover, we prove that, if the spaces $\Omega_1, \Omega_2, \ldots$ are permutation topological spaces. Then the product permutation topology on $\Omega_1 \times \Omega_2 \times \ldots$ has a countable base, and we give a number of examples.

Keywords: Symmetric group, Lindelof space, continuity, permutation, connectedness

MSC: 54A05, 54B05, 54B10, 54C05

Introduction

The purpose of this work was to delve into the aspects of finite groups that has a link with topological space in terms of permutation in symmetric group is called permutation topological space. This would aid users to appreciate the role it plays in the theories and applications of topological space and deal with these in permutation topological space. In this work we use members in symmetric groups to structure permutation topological spaces and permutation continuous functions, the relation between groups and topological spaces was studied by many mathematicians like Flapan et al. (2006), Arhangel'skii and Tkachenko (2008), Pyrch (2011). For any permutation $\beta \in S_n$ can be decomposed essentially uniquely into the product of disjoint cycles. That means $\beta = (b_1^1, b_2^1, ..., b_{\alpha_1}^1)\ (b_1^2, b_2^2, ..., b_{\alpha_2}^2)$ $...(b_1^{c(\beta)}, b_2^{c(\beta)}, ..., b_{\alpha_{c(\beta)}}^{c(\beta)})$ where for each $i \neq j$ we have $\{b_1^i, b_2^i, ..., b_{\alpha_i}^i\} \cap \{b_1^j, b_2^j, ..., b_{\alpha_j}^j\} = \phi$ (Dixon, 1967). So we can write β as $\lambda_1\lambda_2...\lambda_{c(\beta)}$. With λ_i disjoint cycles of length $|\lambda_i| = \alpha_i$ and $c(\beta)$ is the number of disjoint cycle factors including the 1-cycle of β. We call the partition $\alpha = \alpha(\beta) = (\alpha_1(\beta), \alpha_2(\beta), ..., \alpha_{c(\beta)}(\beta)) =$ $(\alpha_1, \alpha_2, ..., \alpha_{c(\beta)})$ the cycle type of β (Zeindler, 2010). In this paper we introduce new two operations $\wedge$ and $\vee$ on $\beta-$sets in permutation topological space (Ω, t_n^β) where $\Omega = \{1,2,...,n\}$. Moreover, we introduce new concepts $\beta-$sets $\lambda_i^\beta = \{b_1^i, b_2^i, ..., b_{\alpha_i}^i\}$ and permutation topological space (Ω, t_n^β). We discuss closure and interior $\beta-$sets in permutation space Ω. In this paper we define permutation continuous function $\delta : (\Omega, t_n^\beta) \to (\Omega, t_n^\mu)$ by the rule $\delta(\lambda^\beta) = \{\delta(a_1), \delta(a_2), ..., \delta(a_k)\}$ for each $\beta-$set $\lambda^\beta = \{a_1, a_2, ..., a_k\} \subset \Omega$, where δ is permutation in symmetric group S_n. Also, we show in this work the identity permutation $e = (1)$ in symmetric group S_n is permutation continuous function on the permutation space Ω. Moreover, the composition of permutation continuous functions is permutation continuous function. For each $\beta-$set $\lambda^\beta \subset \Omega$ we define permutation subspace $(\Omega', t_m^{\gamma^{\lambda^\beta}})$ induced by λ^β of permutation topological space (Ω, t_n^β). Let $\Psi = \{\lambda_i^\beta\}_{i\in I}$ be collection of $\beta-$sets, then Ψ is said a $\beta-$decomposition of the set $\Omega = \{1,2,...,n\}$ if $\Omega = \bigvee_{i\in I} \lambda_i^\beta$ and if the members λ_i^β of Ψ are all nonempty different and pairwise disjoint. Finally, we define $\beta-$connected spaces and our work is supported by a number of examples.

2. $\beta-$Sets

Definition 2.1

Suppose β is permutation in symmetric group S_n on the set $\Omega = \{1,2,...,n\}$ and the cycle type of β is $\alpha(\beta) = (\alpha_1, \alpha_2, ..., \alpha_{c(\beta)})$, then β composite of pairwise disjoint cycles $\{\lambda_i\}_{i=1}^{c(\beta)}$ where $\lambda_i = (b_1^i, b_2^i, ..., b_{\alpha_i}^i)$, $1 \le i \le c(\beta)$. For any $k-$cycle $\lambda = (b_1, b_2, ..., b_k)$ in S_n we define $\beta-$set as $\lambda^\beta = \{b_1, b_2, ..., b_k\}$ and is called $\beta-$set of cycle λ. So the $\beta-$sets of $\{\lambda_i\}_{i=1}^{c(\beta)}$ are defined by $\{\lambda_i^\beta = \{b_1^i, b_2^i, ..., b_{\alpha_i}^i\} \mid 1 \le i \le c(\beta)\}$.

Remark 2.2

Suppose that λ_i^β and λ_j^β are $\beta-$sets in Ω, where $|\lambda_i| = \sigma$ and $|\lambda_j| = \upsilon$. We will give some definitions needed in this work.

Definition 2.3

We call λ_i^β and λ_j^β are disjoint $\beta-$sets in Ω, if and only if $\sum_{k=1}^{\sigma} b_k^i = \sum_{k=1}^{\upsilon} b_k^j$ and there exists $1 \le d \le \sigma$, for each $1 \le r \le \upsilon$ such that $b_d^i \ne b_r^j$.

Definition 2.4

We call λ_i^β and λ_j^β are equal $\beta-$sets in Ω, if and only if for each $1 \le d \le \sigma$ there exists $1 \le r \le \upsilon$ such that $b_d^i = b_r^j$.

Definition 2.5

We call λ_i^β is contained in λ_j^β, if and only if $\sum_{k=1}^{\alpha_i} b_k^i < \sum_{k=1}^{\alpha_j} b_k^j$.

Definition 2.6

We define the operations $\wedge$ and $\vee$ on $\beta-$sets in Ω as followers:

$$\lambda_i^\beta \wedge \lambda_j^\beta = \begin{cases} \lambda_i^\beta, if \ \sum_{k=1}^{\sigma} b_k^i < \sum_{k=1}^{\upsilon} b_k^j \\ \lambda_j^\beta, if \ \sum_{k=1}^{\sigma} b_k^i > \sum_{k=1}^{\upsilon} b_k^j \\ \lambda^\beta, if \ \lambda_i^\beta = \lambda_j^\beta = \lambda^\beta \\ \phi, if \ \lambda_i^\beta \ \& \ \lambda_j^\beta \ are \ disjoint \end{cases} \quad \text{and} \quad \lambda_i^\beta \vee \lambda_j^\beta$$

$$= \begin{cases} \lambda_i^\beta, if \ \sum_{k=1}^{\sigma} b_k^i > \sum_{k=1}^{\upsilon} b_k^j \\ \lambda_j^\beta, if \ \sum_{k=1}^{\sigma} b_k^i < \sum_{k=1}^{\upsilon} b_k^j \\ \lambda^\beta, if \ \lambda_i^\beta = \lambda_j^\beta = \lambda^\beta \\ \Omega, if \ \lambda_i^\beta \ \& \ \lambda_j^\beta \ are \ disjoint \end{cases}$$

Remarks 2.7

1. The intersection of λ_i^β and λ_j^β is $\lambda_i^\beta \wedge \lambda_j^\beta$.
2. The union of λ_i^β and λ_j^β is $\lambda_i^\beta \vee \lambda_j^\beta$.
3. The complement of λ_i^β is $\Omega - \lambda_i^\beta$.
4. The intersection and union of ϕ and λ_i^β are ϕ and λ_i^β, respectively.
5. The intersection and union of Ω and λ_i^β are λ_i^β and Ω, respectively.

Lemma 2.8

For any not disjoint $\beta-$sets $\lambda_i^\beta, \lambda_j^\beta$ and λ_h^β in Ω satisfying:

1. $\lambda_i^\beta \wedge (\lambda_j^\beta \vee \lambda_h^\beta) = (\lambda_i^\beta \wedge \lambda_j^\beta) \vee (\lambda_i^\beta \wedge \lambda_h^\beta)$.
2. $\lambda_i^\beta \vee (\lambda_j^\beta \wedge \lambda_h^\beta) = (\lambda_i^\beta \vee \lambda_j^\beta) \wedge (\lambda_i^\beta \vee \lambda_h^\beta)$.
3. $\Omega - (\lambda_i^\beta \vee \lambda_j^\beta) = (\Omega - \lambda_i^\beta) \wedge (\Omega - \lambda_j^\beta)$.
4. $\Omega - (\lambda_i^\beta \wedge \lambda_j^\beta) = (\Omega - \lambda_i^\beta) \vee (\Omega - \lambda_j^\beta)$.

Proof:

Firstly, we have to prove that $\lambda_i^\beta \wedge (\lambda_j^\beta \vee \lambda_h^\beta) = (\lambda_i^\beta \wedge \lambda_j^\beta) \vee (\lambda_i^\beta \wedge \lambda_h^\beta)$, since $\lambda_i^\beta, \lambda_j^\beta$ and λ_h^β are not disjoint $\beta-$sets, then we get ($\lambda_h^\beta \subseteq \lambda_j^\beta$ or $\lambda_j^\beta \subseteq \lambda_h^\beta$) and ($\lambda_h^\beta \subseteq \lambda_i^\beta$ or $\lambda_i^\beta \subseteq \lambda_h^\beta$) and ($\lambda_i^\beta \subseteq \lambda_j^\beta$ or $\lambda_j^\beta \subseteq \lambda_i^\beta$). Hence, there are eight cases cover all probabilities which are holed as following:

(1)- ($\lambda_h^\beta \subseteq \lambda_j^\beta$ and $\lambda_h^\beta \subseteq \lambda_i^\beta$ and $\lambda_i^\beta \subseteq \lambda_j^\beta$) $\Rightarrow \lambda_i^\beta \wedge (\lambda_j^\beta \vee \lambda_h^\beta) = \lambda_i^\beta = (\lambda_i^\beta \wedge \lambda_j^\beta) \vee (\lambda_i^\beta \wedge \lambda_h^\beta)$

(2)- ($\lambda_h^\beta \subseteq \lambda_j^\beta$ and $\lambda_h^\beta \subseteq \lambda_i^\beta$ and $\lambda_j^\beta \subseteq \lambda_i^\beta$) $\Rightarrow \lambda_i^\beta \wedge (\lambda_j^\beta \vee \lambda_h^\beta) = \lambda_j^\beta = (\lambda_i^\beta \wedge \lambda_j^\beta) \vee (\lambda_i^\beta \wedge \lambda_h^\beta)$

(3)- ($\lambda_h^\beta \subseteq \lambda_j^\beta$ and $\lambda_i^\beta \subseteq \lambda_h^\beta$ and $\lambda_i^\beta \subseteq \lambda_j^\beta$) $\Rightarrow \lambda_i^\beta \wedge (\lambda_j^\beta \vee \lambda_h^\beta) = \lambda_i^\beta = (\lambda_i^\beta \wedge \lambda_j^\beta) \vee (\lambda_i^\beta \wedge \lambda_h^\beta)$

(4)- ($\lambda_j^\beta \subseteq \lambda_h^\beta$ and $\lambda_h^\beta \subseteq \lambda_i^\beta$ and $\lambda_j^\beta \subseteq \lambda_i^\beta$) $\Rightarrow \lambda_i^\beta \wedge (\lambda_j^\beta \vee \lambda_h^\beta) = \lambda_h^\beta = (\lambda_i^\beta \wedge \lambda_j^\beta) \vee (\lambda_i^\beta \wedge \lambda_h^\beta)$

(5)- ($\lambda_j^\beta \subseteq \lambda_h^\beta$ and $\lambda_i^\beta \subseteq \lambda_h^\beta$ and $\lambda_i^\beta \subseteq \lambda_j^\beta$) $\Rightarrow \lambda_i^\beta \wedge (\lambda_j^\beta \vee \lambda_h^\beta) = \lambda_i^\beta = (\lambda_i^\beta \wedge \lambda_j^\beta) \vee (\lambda_i^\beta \wedge \lambda_h^\beta)$

(6)- ($\lambda_j^\beta \subseteq \lambda_h^\beta$ and $\lambda_i^\beta \subseteq \lambda_h^\beta$ and $\lambda_j^\beta \subseteq \lambda_i^\beta$) $\Rightarrow \lambda_i^\beta \wedge (\lambda_j^\beta \vee \lambda_h^\beta) = \lambda_i^\beta = (\lambda_i^\beta \wedge \lambda_j^\beta) \vee (\lambda_i^\beta \wedge \lambda_h^\beta)$

(7)- ($\lambda_h^\beta \subseteq \lambda_j^\beta$ and $\lambda_i^\beta \subseteq \lambda_h^\beta$ and $\lambda_j^\beta \subseteq \lambda_i^\beta$) $\Rightarrow \lambda_j^\beta \subseteq \lambda_i^\beta \subseteq \lambda_h^\beta \subseteq \lambda_j^\beta \Rightarrow \lambda_j^\beta = \lambda_i^\beta = \lambda_h^\beta \Rightarrow \lambda_i^\beta \wedge (\lambda_j^\beta \vee \lambda_h^\beta) = \lambda_j^\beta = \lambda_i^\beta = (\lambda_i^\beta \wedge \lambda_j^\beta) \vee (\lambda_i^\beta \wedge \lambda_h^\beta)$

(8) -($\lambda_j^\beta \subseteq \lambda_h^\beta$ and $\lambda_h^\beta \subseteq \lambda_i^\beta$ and $\lambda_i^\beta \subseteq \lambda_j^\beta$) $\Rightarrow \lambda_j^\beta \subseteq \lambda_h^\beta \subseteq \lambda_i^\beta \subseteq \lambda_j^\beta \Rightarrow \lambda_j^\beta = \lambda_h^\beta = \lambda_i^\beta \Rightarrow \lambda_i^\beta \wedge (\lambda_j^\beta \vee \lambda_h^\beta) = \lambda_h^\beta = \lambda_i^\beta = (\lambda_i^\beta \wedge \lambda_j^\beta) \vee (\lambda_i^\beta \wedge \lambda_h^\beta)$. Then for all eight cases we have $\lambda_i^\beta \wedge (\lambda_j^\beta \vee \lambda_h^\beta) = (\lambda_i^\beta \wedge \lambda_j^\beta) \vee (\lambda_i^\beta \wedge \lambda_h^\beta)$. Also, by the same way we show that $\lambda_i^\beta \vee (\lambda_j^\beta \wedge \lambda_h^\beta) = (\lambda_i^\beta \vee \lambda_j^\beta) \wedge (\lambda_i^\beta \vee \lambda_h^\beta)$. Next, we want to prove that $\Omega - (\lambda_i^\beta \vee \lambda_j^\beta) = (\Omega - \lambda_i^\beta) \wedge (\Omega - \lambda_j^\beta)$, since ($\lambda_i^\beta \subseteq \lambda_j^\beta$ or $\lambda_j^\beta \subseteq \lambda_i^\beta$). Hence $\lambda_i^\beta \vee \lambda_j^\beta = \begin{cases} \lambda_i^\beta, if \ \ \lambda_j^\beta \subseteq \lambda_i^\beta \\ \lambda_j^\beta, if \ \ \lambda_i^\beta \subseteq \lambda_j^\beta \end{cases}$

$$\Rightarrow \Omega - (\lambda_i^\beta \vee \lambda_j^\beta) = \begin{cases} \Omega - \lambda_i^\beta, if \ \ \lambda_j^\beta \subseteq \lambda_i^\beta \\ \Omega - \lambda_j^\beta, if \ \ \lambda_i^\beta \subseteq \lambda_j^\beta \end{cases}$$

But for each $\lambda_t^\beta \subseteq \lambda_g^\beta$ implies $\Omega - \lambda_g^\beta \subseteq \Omega - \lambda_t^\beta$.

Thus $\Omega - (\lambda_i^\beta \vee \lambda_j^\beta) = \begin{cases} \Omega - \lambda_i^\beta, if \ \ \Omega - \lambda_i^\beta \subseteq \Omega - \lambda_j^\beta \\ \Omega - \lambda_j^\beta, if \ \ \Omega - \lambda_j^\beta \subseteq \Omega - \lambda_i^\beta \end{cases}$. Moreover, λ_i^β and λ_j^β are not disjoint $\beta -$sets, then $\Omega - \lambda_i^\beta$ and $\Omega - \lambda_j^\beta$ are not disjoint $\beta -$sets. Hence ($\Omega - \lambda_i^\beta \subseteq \Omega - \lambda_j^\beta$ or $\Omega - \lambda_j^\beta \subseteq \Omega - \lambda_i^\beta$). Thus

$$(\Omega - \lambda_i^\beta) \wedge (\Omega - \lambda_j^\beta) = \begin{cases} \Omega - \lambda_i^\beta, if \ \ \Omega - \lambda_i^\beta \subseteq \Omega - \lambda_j^\beta \\ \Omega - \lambda_j^\beta, if \ \ \Omega - \lambda_j^\beta \subseteq \Omega - \lambda_i^\beta \end{cases}.$$

Then $\Omega - (\lambda_i^\beta \vee \lambda_j^\beta) = (\Omega - \lambda_i^\beta) \wedge (\Omega - \lambda_j^\beta)$. Moreover, by the same way we show that $\Omega - (\lambda_i^\beta \wedge \lambda_j^\beta) = (\Omega - \lambda_i^\beta) \vee (\Omega - \lambda_j^\beta)$.

Lemma 2.9

For any pair disjoint $\beta -$sets $\lambda_i^\beta, \lambda_j^\beta$ and for some λ_h^β $\beta -$set in Ω we have:

(1) If $\lambda_i^\beta \subset \lambda_h^\beta$ or $\lambda_j^\beta \subset \lambda_h^\beta$, then $\lambda_h^\beta \wedge (\lambda_i^\beta \wedge \lambda_j^\beta) = (\lambda_h^\beta \wedge \lambda_i^\beta) \wedge (\lambda_h^\beta \wedge \lambda_j^\beta)$.

(2) If $\lambda_h^\beta \subset \lambda_i^\beta$ or $\lambda_h^\beta \subset \lambda_j^\beta$, then $\lambda_h^\beta \wedge (\lambda_i^\beta \vee \lambda_j^\beta) = (\lambda_h^\beta \wedge \lambda_i^\beta) \vee (\lambda_h^\beta \wedge \lambda_j^\beta)$.

Proof: Suppose that λ_i^β and λ_j^β are disjoint $\beta-$sets, where $\lambda_i^\beta \subset \lambda_h^\beta$ or $\lambda_j^\beta \subset \lambda_h^\beta$. Let $\lambda_i^\beta = \{a_1, a_2, ..., a_\gamma\}$, $\lambda_j^\beta = \{b_1, b_2, ..., b_\sigma\}$, and $\lambda_h^\beta = \{c_1, c_2, ..., c_\upsilon\}$. If $\lambda_i^\beta \subset \lambda_h^\beta$ we have $\lambda_j^\beta \subset \lambda_h^\beta$, since $(\sum_{k=1}^{\sigma} b_k = \sum_{k=1}^{\gamma} a_k < \sum_{k=1}^{\upsilon} c_k)$. Also, if $\lambda_j^\beta \subset \lambda_h^\beta$ we have $\lambda_i^\beta \subset \lambda_h^\beta$, since $(\sum_{k=1}^{\gamma} a_k = \sum_{k=1}^{\sigma} b_k < \sum_{k=1}^{\upsilon} c_k)$ However, $(\lambda_i^\beta \wedge \lambda_j^\beta) = \phi$ and $(\lambda_i^\beta \vee \lambda_j^\beta) = \Omega$. Hence $\lambda_h^\beta \wedge (\lambda_i^\beta \wedge \lambda_j^\beta) = \lambda_h^\beta \wedge \phi = \phi = \lambda_i^\beta \wedge \lambda_j^\beta = (\lambda_h^\beta \wedge \lambda_i^\beta) \wedge (\lambda_h^\beta \wedge \lambda_j^\beta)$. And $\lambda_h^\beta \wedge (\lambda_i^\beta \vee \lambda_j^\beta) = \lambda_h^\beta \wedge \Omega = \lambda_h^\beta = \lambda_h^\beta \vee \lambda_h^\beta = (\lambda_h^\beta \wedge \lambda_i^\beta) \vee (\lambda_h^\beta \wedge \lambda_j^\beta)$.

Definition 2.10

For any collection of not disjoint $\beta-$sets $\{\lambda_i^\beta = \{b_1^i, b_2^i, ..., b_{\sigma_i}^i\}\}_{i\in I}$. We define the union (respectively, intersection) of $\{\lambda_i^\beta\}_{i\in I}$ by $\vee_{i\in I} \lambda_i^\beta = \lambda_j^\beta$, where $\sum_{k=1}^{\sigma_j} b_k^j = \sup\{\sum_{k=1}^{\sigma_i} b_k^i ;\, i \in I\}$ and $\wedge_{i\in I} \lambda_i^\beta = \lambda_j^\beta$, where $\sum_{k=1}^{\sigma_j} b_k^j = \inf\{\sum_{k=1}^{\sigma_i} b_k^i ;\, i \in I\}$.

Lemma 2.11

For any not disjoint $\beta-$sets $\{\lambda_i^\beta\}_{i\in I}$ and λ^β in Ω satisfying:

1. $\lambda^\beta \wedge (\vee_{i\in I} \lambda_i^\beta) = \vee_{i\in I} (\lambda^\beta \wedge \lambda_i^\beta)$.
2. $\lambda^\beta \vee (\wedge_{i\in I} \lambda_i^\beta) = \wedge_{i\in I} (\lambda^\beta \vee \lambda_i^\beta)$.
3. $\Omega - (\vee_{i\in I} \lambda_i^\beta) = \wedge_{i\in I} (\Omega - \lambda_i^\beta)$.
4. $\Omega - (\wedge_{i\in I} \lambda_i^\beta) = \vee_{i\in I} (\Omega - \lambda_i^\beta)$.

Proof:

Its clearly, we consider that all above equations are hold by using lemma (2.8).

3. Permutation Topological Spaces

Definition 3.1

Let β be permutation in symmetric group S_n, and β composite of pairwise disjoint cycles $\{\lambda_i\}_{i=1}^{c(\beta)}$, where $|\lambda_i| = \alpha_i, 1 \le i \le c(\beta)$, then (Ω, t_n^β) permutation topological space where $\Omega = \{1,2,...,n\}$ and t_n^β is a collection of $\beta-$set of the family $\{\lambda_i\}_{i=1}^{c(\beta)}$ union Ω and empty set.

Remark 3.2

If t_n^β and t_n^μ are two topology on the same set Ω, then for each $k-$cycle $\lambda = (b_1, b_2, ..., b_k)$ in S_n we have $\beta-$set of λ and $\mu-$set of λ are equal (that means $\lambda^\beta = \lambda^\mu$). Moreover, if λ belong to $\{\lambda_i\}_{i=1}^{c(\beta)}$ and $\{\lambda_j^\mu\}_{j=1}^{c(\mu)}$ are disjoint cycles decomposition of β and λ respectively, then the same set $\{b_1, b_2, ..., b_k\}$ is open $\beta-$set and open $\mu-$set in permutation spaces (Ω, t_n^β) and (Ω, t_n^μ), respectively. Let $\beta \in S_n$, $\Omega_1 = \{1,2,...,n\}$ and $\Omega_2 = \{1,2,...,n+1\}$, then for each $k-$cycle λ in S_n we have λ^β is $\beta-$subset of Ω_2 since λ is $k-$cycle in S_{n+1} too. However, this is not necessary true for any $k-$cycle λ in S_{n+1} to be $\beta-$subset of Ω_1 (i.e $\lambda^\beta \subset \Omega_1$), because λ is not necessary to be $k-$cycle in S_n (see example 3.4).

Example 3.3

Let $\beta \in S_3$, $\lambda_1 = (1,3,2)$ be $3-$cycle in S_3 and $\lambda_2 = (1,4)$ be $2-$cycle in S_4, the $\beta-$sets of λ_1 and λ_2 are $\lambda_1^\beta = \{1,3,2\}$ and $\lambda_2^\beta = \{1,4\}$ in $\Omega_1 = \{1,2,3\}$ and $\Omega_2 = \{1,2,3,4\}$ respectively. Finally, we have $\lambda_1^\beta \subset \{1,2,3,4\}$ (i.e λ_1^β is also $\beta-$subset of Ω_2). But λ_2^β is not $\beta-$subset of Ω_1, because $\lambda_2 = (1,4)$ is not cycle in S_3.

Permutation subspaces 3.4

Suppose (Ω, t_n^β) permutation space , $\lambda^\beta \subset \Omega$ and $T_i^\beta = \lambda^\beta \wedge \lambda_i^\beta$, for each proper $\lambda_i^\beta \in t_n^\beta$, then

$$T_i^\beta = \begin{cases} \{b_1^i, b_2^i, ..., b_{i_k}^i\}, & if\ \lambda^\beta\ \&\ \lambda_i^\beta\ are\ not\ disjoint \\ \phi, & if\ \lambda^\beta\ \&\ \lambda_i^\beta\ are\ disjoint \end{cases}$$

Let $\Re = \{T_i^\beta \mid T_i^\beta$ nonempty open $\beta-$set$\}$. For each $T_i^\beta \in \Re$, let $b_k^i = Max\{b_1^i, b_2^i, ..., b_{i_k}^i\}$ and $m = Max\{b_k^i; T_i^\beta \in \Re\}$. Suppose $\sum_{T_i^\beta \in \Re} |T_i| = s$, and $t = m - s$, then we have this set $B = \{b_1, b_2, ..., b_t\}$ has exactly t points where $B = \bigcap_{T_i^\beta \in \Re} (\Omega' - T_i^\beta)$ where $\Omega' = \{1,2,...,m\}$. Here we used normal intersection ($\cap$) between pairwise sets to find the set B. For each $T_i^\beta \in \Re$ we have

$T_i = (b_1^i, b_2^i, ..., b_{i_k}^i)$ is i_k – cycle in S_m. Then { $\{T_i\}_{T_i^\beta \in \Re}$, $\{(b_r)\}_{r=1}^t$ } are disjoint cycles decomposition of new permutation in symmetric group S_m induced by λ^β say γ^{λ^β}.

Definition 3.5

Let (Ω, t_n^β) be a permutation space and $\lambda^\beta \subset \Omega$, then we denote to permutation subspace of (Ω, t_n^β) by $(\Omega', t_m^{\gamma^{\lambda^\beta}})$ where $t_m^{\gamma^{\lambda^\beta}} = \{\Omega', \phi, \{T_i^\beta\}_{T_i^\beta \in \Re}, \{b_r\}_{r=1}^t\}$ and $\Omega' = \{1,2,...,m\}$.

Example 3.6

Find permutation space (Ω, t_n^β) and permutation subspace $(\Omega', t_m^{\gamma^{\lambda^\beta}})$, where $\beta = (2\ 3)(1\ 8)\ (6\ 9)(4\ 5)(7)$ in S_9 and $\lambda^\beta = \{1,8\}$.

Solution

$t_n^\beta = t_9^\beta = \{\Omega, \phi, \{2,3\}, \{1,8\}, \{6,9\}, \{4,5\}, \{7\}\}$ and $\Omega = \{1,2,3,4,5,6,7,8,9\}$, $\lambda^\beta = \{1,8\} \Rightarrow T_1^\beta = \{2,3\}$, $T_2^\beta = \{1,8\}$, $T_3^\beta = \{1,8\}$, $T_4^\beta = \phi$, $T_5^\beta = \{7\} \Rightarrow \Re = \{\{2,3\}, \{1,8\}, \{7\}\} \Rightarrow Max\{2,3\} = 3$, $Max\{1,8\} = 8$, $Max\{7\} = 7 \Rightarrow Max\{3,8,7\} = 8 = m \Rightarrow \Omega' = \{1,2,3,4,5,6,7,8\}$, $\sum_{T_i^\beta \in \Re} |T_i| = s \Rightarrow |(2\ 3)| + |(1\ 8)| + |(7)| = 2+2+1 = 5$. Let $t = m - s = 8 - 5 = 3 \Rightarrow B = \{b_1, b_2, b_3\}$ and $B = \bigcap_{T_i^\beta \in \Re} (\Omega' - T_i^\beta) = \{4,5,6\}$. Then $\gamma^{\lambda^\beta} = (2\ 3)(1\ 8)(7)(4)(5)(6)$ is a permutation in symmetric group S_8 induced by $\lambda^\beta = \{1,8\}$ and $(\Omega', t_8^{\gamma^{\lambda^\beta}})$ is a permutation subspace where $t_m^{\gamma^{\lambda^\beta}} = t_8^{\gamma^{\lambda^\beta}} = \{\Omega', \phi, \{2,3\}, \{1,8\}, \{7\}, \{4\}, \{5\}, \{6\}\}$.

Remark 3.7

A base for a permutation topological space (Ω_i, t_i^β) is a sub-collection D of t_i^β such that each member λ^β of t_i^β can be written as $\lambda^\beta = \vee_{i\in I} \lambda_i^\beta$, where each λ_i^β belong to D. So subbase for the product permutation topology on $(\Omega, t) = (\prod_{i\in I} \Omega_i, \prod_{i\in I} t_i^{\beta^i})$ is given by

$M=\{\pi_i^{-1}(\lambda_i^{\beta^i})\,|\,\lambda_i^{\beta^i}\in t_i^{\beta^i}, i\in I\}$, so that a base can be taken to be $D=\{\overset{d}{\underset{k=1}{\wedge}}\pi_{i_k}^{-1}(\lambda_{i_k}^{\beta^{i_k}})\,|\,\lambda_{i_k}^{\beta^{i_k}}\in t_{i_k}^{\beta^{i_k}}, i_k\in I,\, k=1,2,\ldots,d,\, d\in N\}$.

Definition 3.8

If $\lambda^\beta \in t_n^\beta$ is $\beta-$set in the space Ω, then $\Omega-\lambda^\beta$ is called closed $\beta-$set in the space Ω, and $\overline{\lambda^\beta}$ is smallest closed $\beta-$set containing λ^β, and any $\beta-$set $\lambda^\beta \subseteq \Omega$ is called closed $\beta-$set iff $\overline{\lambda^\beta}=\lambda^\beta$.

Example 3.9

Let $\beta=(1\ 2\ 3)(4\ 5)$ be a permutation in symmetric group S_5. Find permutation topological space on $\Omega=\{1,2,3,4,5\}$ and then find $\overline{\{1,2,3\}}$ in space Ω.

Solution

$c(\beta)=2,\ \alpha(\beta)=(\alpha_1(\beta),\alpha_2(\beta))=(\alpha_1,\alpha_2)=(3,2)\Rightarrow \beta=\lambda_1\lambda_2$, where $\lambda_1=(1\ 2\ 3)$, $\lambda_2=(4\ 5)$, $|\lambda_1|=\alpha_1=3$, and $|\lambda_2|=\alpha_2=2$. Then all the proper open $\beta-$sets in space Ω are $\lambda_1^\beta=\{1,2,3\}$, $\lambda_2^\beta=\{4,5\}$ $\Rightarrow t_5^\beta=\{\Omega,\phi,\lambda_1^\beta,\lambda_2^\beta\}$, in other words, $\overline{\{1,2,3\}}$ is the intersection of all closed $\beta-$set λ^β such that $\{1,2,3\}\subseteq\lambda^\beta \Rightarrow \overline{\{1,2,3\}}=\Omega\wedge\{4,5\}\wedge\{1,2,3\}=\{1,2,3\}$. Then $\{1,2,3\}$ is closed $\beta-$set.

Definition 3.10

The set $(\lambda^\beta)^o=\Omega-\overline{\Omega-\lambda^\beta}$ is called the interior of the $\beta-$set λ^β in the permutation space Ω.

Remarks 3.11

1. We call x belong to $\beta-$set $\lambda^\beta=\{b_1,b_2,\ldots,b_k\}$ iff $x=b_j$, for some $j\in\{1,2,\ldots,k\}$.
2. The condition $x\in\Omega-\overline{\Omega-\lambda^\beta}$ means that $x\notin\overline{\Omega-\lambda^\beta}$. Therefore, x is an interior point of $\beta-$set λ^β if and only if there is an open $\beta-$set λ_r^β containing x and such that $\lambda_r^\beta\wedge(\Omega-\lambda^\beta)=\phi$.

Example 3.12

Let $\beta=(4\ 2)(3\ 5)(6\ 1\ 7)$ be a permutation in symmetric group S_7. Find $(\lambda^\beta)^o$ in permutation space (Ω,t_7^β), where $\lambda^\beta=\{3,5\}$.

Solution

$t_7^\beta = \{\Omega,\phi,\{4,2\},\{3,5\},\{6,1,7\}\}$, where $\Omega=\{1,2,3,4,5,6,7\} \Rightarrow (\lambda^\beta)^o = \Omega - \overline{\Omega-\lambda^\beta} = \Omega - \overline{\{1,2,4,6,7\}}$ $= \Omega - (\Omega \wedge \{1,2,4,6,7\}) = \{3,5\}$. In other words, $(\lambda^\beta)^o$ is the union of all open $\beta-$set λ_r^β such that $\lambda_r^\beta \subseteq \lambda^\beta \Rightarrow (\lambda^\beta)^o = \{4,2\} \vee \{3,5\} = \{3,5\}$.

Lemma 3.13

A permutation topological space is an Lindelof space.

Proof

Let (Ω, t_n^β) be permutation topological space where $\beta \in S_n$, and $\alpha(\beta) = (\alpha_1,\alpha_2,\ldots,\alpha_{c(\beta)})$, then for each $1 \le i \le c(\beta)$ we have the proper open $\beta-$set $\lambda_i^\beta = \{b_1^i, b_2^i, \ldots, b_{\alpha_i}^i\}$ is a countable set, and for each base $D = \{\lambda_i^\beta\}_{i\in I}$ for permutation space Ω we have $\underset{i\in I}{\vee} \lambda_i^\beta = \lambda_j^\beta$ where $\sum_{k=1}^{\alpha_j} b_k^j = \sup\{\sum_{k=1}^{\alpha_i} b_k^i \mid i \in I\}$, but λ_j^β is a countable set (each finite set is a countable), (see Runde, 2005), so D is a countable base, since only the union of a countable collection of a countable sets is countable. Therefore permutation space Ω with countable base, then we have permutation space Ω is an Lindelof space (see Bourbaki; 1989. Page 144).

4. Functions and Permutation Continuity

Let β, μ and δ be three permutations in symmetric group S_n, and let $\delta : (\Omega, t_n^\beta) \to (\Omega, t_n^\mu)$ be a function, where for each $\beta-$set $\lambda^\beta = \{b_1, b_2, \ldots, b_k\}$, the image of λ^β under δ is called $\mu-$set and defined by the rule $\delta(\lambda^\beta) = \{\delta(b_1), \delta(b_2), \ldots, \delta(b_k)\}$. In another direction, let $\eta^\mu = \{a_1, a_2, \ldots, a_r\}$ be $\mu-$set, the inverse image of η^μ under δ is called $\beta-$set and defined by the rule $\delta^{-1}(\eta^\mu) = \{\delta^{-1}(a_1), \delta^{-1}(a_2), \ldots, \delta^{-1}(a_r)\}$. The usual properties relating images and inverse images of subsets of complements, unions, and intersections also hold for permutation sets.

Definition 4.1

Given permutation topological spaces (Ω, t_n^{β}) and (Ω, t_n^{μ}), a function $\delta : (\Omega, t_n^{\beta}) \to (\Omega, t_n^{\mu})$ is permutation continuous if the inverse image under δ of any open $\mu-$set in t_n^{β} is an open $\beta-$set in t_n^{μ} (i.e $\delta^{-1}(\lambda^{\mu}) \in t_n^{\beta}$ whenever $\lambda^{\mu} \in t_n^{\mu}$).

Example 4.2

Let $\beta = (1\ 2)(3\ 5)(4\ 6\ 7)(8)$, $\mu = (3)(1\ 5)(8\ 7)(4\ 6\ 2)$, and $\delta = (2\ 5\ 7)(3\ 8)$ in S_8, then $\delta : (\Omega, t_8^{\beta}) \to (\Omega, t_8^{\mu})$ is permutation continuous from permutation space (Ω, t_8^{β}) into permutation space (Ω, t_8^{μ}).

Lemma 4.3

The identity permutation $e = (1)$ in symmetric group S_n is a permutation continuous on a permutation space (Ω, t_n^{β}).

Proof

Let $\delta = (1) : (\Omega, t_n^{\beta}) \to (\Omega, t_n^{\beta})$ and $\lambda^{\beta} \in t_n^{\beta}$, then $\delta^{-1}(\lambda^{\beta}) = (1)^{-1}(\lambda^{\beta}) = (1)(\lambda^{\beta}) = \lambda^{\beta}$.

Lemma 4.4

A composition of permutation continuous functions is permutation continuous.

Proof

Let $\delta_1 : (\Omega, t_n^{\beta}) \to (\Omega, t_n^{\mu})$ and $\delta_2 : (\Omega, t_n^{\mu}) \to (\Omega, t_n^{\eta})$ be permutation continuous functions. For $\lambda^{\eta} \in t_n^{\eta}, (\delta_2\delta_1)^{-1}(\lambda^{\eta}) = \delta_1^{-1}\delta_2^{-1}(\lambda^{\eta}) = \delta_1^{-1}(\delta_2^{-1}(\lambda^{\eta}))$. But $\delta_2^{-1}(\lambda^{\eta}) \in t_n^{\mu}$ since δ_2 is permutation continuous, and so $(\delta_2\delta_1)^{-1}(\lambda^{\eta}) = \delta_1^{-1}(\delta_2^{-1}(\lambda^{\eta})) \in t_n^{\beta}$ since δ_1 is permutation continuous.

Definition 4.5

Let $(\Omega_i, t_i^{\beta^i})$ be permutation topological space for each index $i \in I$. The product permutation topology $t = \prod_{i \in I} t_i^{\beta^i}$ on the set $\Omega = \prod_{i \in I} \Omega_i$ is the coarsest permutation topology on Ω making all the projection mappings $\pi_i : \Omega \to \Omega_i$ permutation continuous.

Lemma 4.6

If the spaces $\Omega_1, \Omega_2, \ldots$ are permutation topological spaces, then $\Omega_1 \times \Omega_2 \times \ldots$ have a countable base.

Proof

Since $\Omega_1, \Omega_2, ...$ permutation spaces, then each one of them has countable base. Let $h_{k,1}, h_{k,2}, ...$ are permutation topological spaces denote the base of Ω_k. The base of $\Omega_1 \times \Omega_2 \times ...$ is composed of sets of the form $g_1 \times g_2 \times ...$. Because, if λ is open in Ω_k, then $\pi_k^{-1}(\lambda)$ is generated by the sets $\pi_k^{-1}(g)$, where g belong to the subbase G for space Ω_k, in the same way that λ is generated by the sets g in G (with the aid of $\vee$ operation and the finite $\wedge$ operation). So the base of $\Omega_1 \times \Omega_2 \times ...$ is composed of sets of the form $g_1 \times g_2 \times ...$ where for each index k, except a finite number of indices, $g_k = \Omega_k$, while for the exceptional indices g_k is a term of the sequence $h_{k,1}, h_{k,2}, ...$. This base is obviously countable since the set of all finite sequences with terms belonging to a given countable set is countable.

5. $\beta -$ Connectedness

Let (Ω, t_n^β) be permutation topological space. The collection of $\beta -$sets $\Psi = \{\lambda_i^\beta\}_{i \in I}$ is said to be a $\beta -$decomposition of the set $\Omega = \{1,2,...,n\}$ if $\Omega = \underset{i \in I}{\vee} \lambda_i^\beta$ and if the members λ_i^β of Ψ are all nonempty and $\{\lambda_i\}_{i \in I}$ pairwise disjoint cycles in S_n. Then Ψ is called $\beta -$decomposition of Ω we also say that Ω has been $\beta -$decomposed into the $\beta -$sets of Ψ. Assume the permutation topological space (Ω, t_n^β) has been $\beta -$decomposed into two open $\beta -$sets λ_k^β and λ_j^β. Then the neighborhoods filter of Ω is completely determined by its traces on λ_k^β and λ_j^β. This means that no relation or connection exists between the behavior of the underlying permutation topology on λ_k^β and its behavior on λ_j^β in this form the permutation space is called $\beta -$disconnected.

Definition 5.1

A permutation space (Ω, t_n^β) and its topology are both said to be $\beta -$connected if Ω cannot be $\beta -$decomposed into two open $\beta -$sets. A $\beta -$subset λ^β of Ω is said to be $\beta -$connected whenever the permutation subspace $(\Omega', t_m^{\gamma^{\lambda^\beta}})$ is $\gamma^{\lambda^\beta} -$connected, and λ^β is said to be $\beta -$disconnected if Ω' is $\gamma^{\lambda^\beta} -$decomposed into two open $\gamma^{\lambda^\beta} -$sets.

Example 5.2

See example (3.6) the permutation space (Ω, t_9^{β}) is $\beta-$disconnected, where $t_9^{\beta} = \{\Omega, \phi, \{2,3\}, \{1,8\}, \{6,9\}, \{4,7\}, \{5\}\}$ and $\Omega = \{1,2,3,4,5,6,7,8,9\}$, since there are two open $\beta-$sets

$t_8^{\gamma^{\lambda\beta}} = \{\Omega', \phi, \{2,3\}, \{1,8\}, \{5\}, [4\}, \{6\}, \{7\}\}$ and $\Omega' = \{1,2,3,4,5,6,7,8\}$.

Example 5.3

See example (3.10) the permutation space (Ω, t_5^{β}) is $\beta-$connected, where $t_i^{\beta} = \{\Omega, \phi,$ $\{1,2,3\}, \{4,5\}\}$ and $\Omega = \{1,2,3,4,5\}$, since Ω cannot be $\beta-$decomposed into two open $\beta-$sets.

Remark 5.4

If Ω is a permutation space and Ω is $\beta-$decomposed into two open $\beta-$sets λ_1^{β} and λ_2^{β}, then we have:

1) $\overline{\lambda_1^{\beta}} = \overline{\lambda_2^{\beta}}$.

2) $\lambda_1^{\beta} \wedge \overline{\lambda_2^{\beta}} = \lambda_1^{\beta}$ and $\overline{\lambda_1^{\beta}} \wedge \lambda_2^{\beta} = \lambda_2^{\beta}$.

Lemma 5.5

Let $(\Omega', t_m^{\gamma^{\lambda\beta}})$ be $\gamma^{\lambda\beta}-$connected subspace of permutation space Ω. If Ω is $\beta-$decomposed into two open $\beta-$sets λ_1^{β}, λ_2^{β}, where $\{\lambda_i\}_{i=1}^{2}$ are cycles in S_m and $\{(\Omega' \wedge \lambda_i^{\beta})\}_{i=1}^{2}$ are disjoint $\gamma^{\lambda\beta}-$ sets, then either $\Omega' \cap \lambda_1^{\beta}$ or $\Omega' \cap \lambda_2^{\beta}$ is not open $\{2,3\}$ and $\{5\}$ in Ω, where Ω
$\gamma^{\lambda\beta}-$set.

Proof

By definition disjoint permutation sets in permutation space we have for each member in $\{(\Omega' \wedge \lambda_i^{\beta})\}_{i=1}^{2}$ is nonempty. Hence, Ω' and λ_i^{β}, $(i = 1,2)$ are not disjoint, since $(\Omega' \wedge \lambda_i^{\beta}) \neq \phi$, $i = 1,2$. Now we want to prove that $\lambda_1^{\beta} \subset \Omega'$ and $\lambda_2^{\beta} \subset \Omega'$. Assume $\Omega' \subset \lambda_1^{\beta} = \{b_1^1, b_2^1, ..., b_{k_1}^1\}$ or $\Omega' \subset \lambda_2^{\beta} = \{b_1^2, b_2^2, ..., b_{k_2}^2\}$, when $\Omega' \subset \lambda_1^{\beta}$ we have $\Omega' \subset \lambda_2^{\beta}$ too, since $\sum_{k=1}^{k_1} b_k^1 = \sum_{k=1}^{k_2} b_k^2$. Moreover, $(\Omega' \wedge \lambda_1^{\beta}) \wedge (\Omega' \wedge \lambda_2^{\beta}) =$ $\Omega' \wedge \Omega' = \Omega' \neq \phi$. But this contradiction, since $\{(\Omega' \wedge \lambda_i^{\beta})\}_{i=1}^{2}$ are disjoint, so $\lambda_1^{\beta} \subset \Omega'$ and $\lambda_2^{\beta} \subset \Omega'$. Then $\lambda_i^{\beta} = \lambda_i^{\gamma^{\lambda\beta}}$, $(i = 1,2)$ and they are disjoint $\gamma^{\lambda\beta}-$sets too. Cleary the union for $\{(\Omega' \wedge \lambda_i^{\beta})\}_{i=1}^{2}$ of $\gamma^{\lambda\beta}-$sets in permutation space Ω' is Ω' [i.e $(\Omega' \wedge \lambda_1^{\beta}) \vee (\Omega' \wedge \lambda_2^{\beta}) = \lambda_1^{\gamma^{\lambda\beta}} \vee \lambda_2^{\gamma^{\lambda\beta}} = \Omega'$]. If we assume that both $(\Omega' \wedge \lambda_1^{\beta})$ and $(\Omega' \wedge \lambda_2^{\beta})$ are open $\gamma^{\lambda\beta}$ – sets. We have Ω' is $\gamma^{\lambda\beta}$ – decomposed into two open $\gamma^{\lambda\beta}$ – sets $(\Omega' \wedge \lambda_1^{\beta})$ and $(\Omega' \wedge \lambda_2^{\beta})$. But this is a contradiction with our hypothesis that Ω' is $\gamma^{\lambda\beta}$ – connected. Hence, either $\Omega' \cap \lambda_1^{\beta}$ or $\Omega' \cap \lambda_2^{\beta}$ is not open $\gamma^{\lambda\beta}$ – set.

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فضاءات التبديل التبولوجية و قواعدها

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الخلاصة

ليكن β تبديل في زمرة التناظر n، قدمنا هنا فضاء التبديل (Ω, t_n^{β}) و مفاهيم جديدة أخرة في مجال التبولوجيا مثل مجموعات- β، فضاء التبديل الجزئي $(\Omega', t_m^{\gamma^{\lambda\beta}})$، مستمر التبديل، تحلل- β، ترابط - β. في هذا البحث برهنا ان كل فضاء تبديل هو فضاء لندلوف بالأضافة الى ذلك برهنا على انه اذا كان لدينا مجموعة من فضاءات التبديل فأن الجداء لها يمتلك قاعدة معدودة كذلك عززنا هذا العمل بعدد من الأمثلة.