Some adjoint pair of covariant functors between the category of profinite crossed modules and some related categories

Ra'ad S. Mahdi and Shaimaa A. Abdullah

Department of Mathematics, College of Science, University of Basra, Basra- Iraq

raadalzurkani64@yahoo.com

Abstract.

In this paper, we introduce and study some adjoint pairs of covariant functors. The first adjoint pair of covariant functors is constructed between the category of profinite crossed modules and the category of profinite transformation groups, which explains how to construct a profinite crossed module from a given profinite transformation group. The second adjoint pair of covariant functors is constructed between the category of free profinite crossed modules and the category of continuous maps (from profinite spaces into profinite groups) which explains how to construct a free profinite crossed module on a continuous map from a profinite space into a profinite group.

Keywords: Adjoint pair of covariant functors; profinite transformation group; profinite crossed module.

1. **Introduction**

Crossed modules were introduced by [J.H.C. Whitehead 1949] in his study on combinatorial homotopy theory. During the research work (1984-1987) of F.J. Korkes and T. Porter, they entered a topological structure on the theory of crossed modules and they chose the category of profinite groups that occur in problems relating to number theory, commutative algebra, algebraic geometry and algebraic topology, and they introduced the definition of a profinite crossed module.

Although the category of profinite groups form a natural extension to the category of finite groups; it carries a richer structure in that it has categorical objects and notions which do not exist in finite case such as projective (inverse) limits and free products. The existence of such notions in the extended category leads to the definition of the profinite analogues as free groups and presentation of groups by generators and relations.

Profinite groups are compact, Hausdorff and totally disconnected topological groups. The open normal subgroups of such a group constitute a neighbourhood basis of the identity element. From this, it was impliesed that any profinite group can be represented as a projective (inverse) limit of a projective (inverse) system of finite groups where each finite group is given the discrete topology.

[F.J. Korkes and T. porter 1987] were pointed that many results in combinatorial group theory are fail to have a nice profinite analogues, which are required to be re-worked completely by using profinite technique.

2. An adjoint pair of covariant functors between the category of profinite crossed modules and the category of profinite transformation groups

Let us consider the category of topological transformation groups, $i.e.$ triples (X, G, θ) where X is a topological space, G is a topological group and $\theta: G \times X \longrightarrow X$ is a continuous map (usually written $\theta(g, x) = {}^{g}x$) such that:

(i) ${}^{1_G}x = x$ for all $x \in X$ and (ii) ${}^{gg'}x = {}^{g}({}^{g'}x)$ for all $x \in X$ and $g, g' \in G$. Such a map θ is called a continuous left action of G on X. A continuous morphism in this category, say $(\mu, \eta): (X, G, \theta) \longrightarrow (Y, H, \varphi)$ consists of a continuous map $\mu: X \longrightarrow Y$ and a continuous homomorphism $\eta: G \longrightarrow H$ such that $\mu\theta = \varphi(\eta \times \mu), i.e. \mu({}^{g}x) = {}^{\eta(g)}\mu(x)$ for all $x \in X$ and $g \in G$.

Definition 2.1. [Haran and Jarden 1985] A *profinite transformation group* is a topological transformation group (X, G, θ) in which X is a profinite space and G is a profinite group.

Definition 2.2. [Haran and Jarden 1985] A *continuous morphism* $(\mu, \eta): (X, G, \theta) \rightarrow (Y, H, \varphi)$ of profinite transformation groups is the same data as a continuous morphism of topological transformation groups.

Profinite transformation groups and their continuous morphisms as defined above form a category, $ProfTGrps$. There is a subcategory $\overline{ProfTGrps}$ of $ProfTGrps$ which has as objects those profinite transformation groups (X, G, θ) with X as a profinite group and having as continuous morphisms from (X, G, θ) to (Y, H, φ) just those (μ, η) in $ProfTGrps$ in which $\mu: X \rightarrow Y$ is a continuous homomorphism from a profinite group X to a profinite group Y. We shall remark her that in case $(X, G, \theta) \in \overline{ProfTGrps}$, the continuous left action θ of G on X necessarily satisfies the condition ${}^{g}(xx') = {}^{g}x\, {}^{g}x'$ for all $x, x' \in X$ and $g \in G$.

Definition 2.3. [Korkes and Porter 1986] A *profinite crossed module* (N, G, ∂, θ) consists of a profinite transformation group $(N, G, \theta) \in \overline{ProfTGrps}$ and a continuous homomorphism $\partial: N \longrightarrow G$ such that the following conditions are satisfied:

$(CM1)$ for all $n \in N, g \in G$,

$$\partial({}^{g}n) = g\partial(n)g^{-1} \quad (i.e.\, \partial \text{ is G-equivariant})$$

$(CM2)$ for all $n, n' \in N$,

$${}^{\partial(n_1)}n_2 = n_1 n_2 n_1^{-1} \quad \text{(Peiffer identity)}.$$

Examples 2.4.

(1) Let H be a closed normal subgroup of a profinite group G with a continuous left action $\theta: G \times H \longrightarrow H$ of G on H by conjugations. It is obvious that (H, G, θ) is a profinite transformation group. If $i: H \rightarrow G$ is the inclusion map, therefore i is a continuous homomorphism and (H, G, i, θ) is a profinite crossed module.

In the above example, if either $H = G$ or $= \{1_G\}$, then (G, G, I_G, θ) and $(\{1_G\}, G, i, \theta)$ are profinite crossed modules. Thus each profinite group G can be viewed as a profinite crossed module by any one of the above two cases.

Definition 2.5. [Korkes and Porter 1986] A *continuous morphism* $(\mu,\eta):(N,G,\partial,\theta)\to(C,H,\delta,\varphi)$ of profinite crossed modules (N,G,∂,θ) and (C,H,δ,φ) is a continuous morphism $(\mu,\eta):(N,G,\theta)\to(C,H,\varphi)$ of profinite transformation groups in $\overline{ProfTGrps}$ such that $\delta\mu=\eta\partial$.

Profinite crossed modules and their continuous morphisms form a category, $ProfCMod$.

There is for fixed profinite group G, a subcstegory $ProfCMod/G$ of $ProfCMod$ consists of all profinite crossed modules with G as the "base", $i.e.$ all (N,G,∂,θ) for this fixed G and continuous morphism from (N,G,∂,θ) to (C,H,δ,φ) just those (μ,η) in $ProfCMod$ in which $\eta=I_G:G\longrightarrow G$ is the identity homomorphism on G.

A structure with the same data as a profinite crossed module and satisfying $(CM1)$ but not $(CM2)$ is called a profinite precrossed module. A continuous morphism of profinite precrossed modules is the same data as a continuous morphism of profinite crossed modules. Profinite precrossed modules and their continuous morphisms from a category, $ProfPreCMod$. In this case, $ProfPreCMod$ is a full subcategory of $ProfCMod$.

There are two covariant functors, the forgetful functor $F:ProfCMod\longrightarrow ProfPreCMod$ and the left adjoint functor $L:ProfPreCMod\longrightarrow ProfCMod$, which assigns to each profinite precrossed module (N,G,∂,θ), a profinite crossed module $(N^{cr},G,\partial^{cr},\theta^{cr})$ where $N^{cr}={N}/{[\![N,N]\!]}$, $\partial^{cr}:N^{cr}\longrightarrow G$ is a continuous homomorphism induced by $\partial, i.e. \partial^{cr}(n[\![N,N]\!])=\partial(n)$ and $\theta^{cr}:G\times N^{cr}\longrightarrow N^{cr}$ is a continuous left action of G on N^{cr} induced by $\theta, i.e. \theta({}^{g}(n[\![N,N]\!]))={}^{g}n[\![N,N]\!]$. In the above description $[\![N,N]\!]$ is a closed G-invariant normal subgroup of N generated by all peiffer elements of the form $[\![n,n']\!]=nn'n^{-1}\,{}^{\partial(n)}n'^{-1}$ for all $n,n'\in N$, and assigns to each continuous morphism $(\mu,\eta):(N,G,\partial,\theta)\to(C,H,\delta,\varphi)$ of profinite precrossed modules (N,G,∂,θ) and (C,H,δ,φ) a continuous morphism of profinite crossed modules $(\mu^{cr},\eta):(N^{cr},G,\partial^{cr},\theta^{cr})\longrightarrow(C^{cr},G,\delta^{cr},\varphi^{cr})$, where $\mu^{cr}:N^{cr}\longrightarrow C^{cr}$ induced by $\mu, i.e. \mu^{cr}(n[\![N,N]\!])=\mu(n)[\![C,C]\!]$.

To construct an adjoint pair of covariant functors between $ProfCMod$ and $\overline{ProfTGrps}$, we establish the following five results whose proofs are straight forward.

Theorem 2.6. If $(N,G,\theta)\in\overline{ProfTGrps}$, then we can form $(N,N\rtimes G,\partial_N,\bar{\theta})\in ProfCMod$, where $\partial_N:N\to N\rtimes G$ is a continuous group homomorphism defined by $\partial_N(n)=(n,1_G)$ for all $n\in N$ and $\overline{\theta}:(N\rtimes G)\times N\to N$ is a continuous left action of $N\rtimes G$ (the semidrect product of N and G) on the left of N by ${}^{(n,g)}n'=n\,{}^{g}n'n^{-1}$ for all $n,n'\in N$ and $g\in G$. Furthermore, if $(f,l):(N,G,\theta)\to(C,H,\varphi)$ is in $\overline{ProfTGrps}$, then $(f,f\rtimes l):\left(N,N\rtimes G,\partial_N,\overline{\theta}\right)\to(C,C\rtimes$

$H, \lambda_c, \overline{\varphi})$ is in $ProfCMod$, where $f \rtimes l: N \rtimes G \to C \rtimes H$ is a continuous homomorphism defined by $(f, l)(n, g) = \big(f(n), l(g)\big)$ for all $n \in N$ and $g \in G$.

Corollary 2.7. Let (N, G, θ) be in $\overline{ProfTGrps}$ and (C, H, λ, φ) be in $ProfCMod$. If $(l, k): \big(N, N \rtimes G, \partial_N, \overline{\theta}\big) \to (C, H, \lambda, \varphi)$ *is* in $ProfCMod$, then the mapping $k^*: G \to H$ defined by $k^*(g) = k(I_N, g)$ for all $g \in G$ is a continuous homomorphism, and $(l, k^*): (N, G, \theta) \to (C, H, \varphi)$ is in $\overline{ProfTGrps}$.

Corollary 2.8. Let (N, G, θ) be in $\overline{ProfTGrps}$ and (C, H, λ, φ) be in $ofCMod$. If $(l, k): (N, G, \theta) \longrightarrow (C, H, \varphi)$ is in $\overline{ProfTGrps}$, then the mapping $\mathrm{V}: C \rtimes H \longrightarrow H$ defined by $\mathrm{V}(c, h) = \lambda(c)h$ is a continuous homomorphism and $\big(l, \mathrm{V}\,(l \rtimes k)\big): \big(N, N \rtimes G, \partial_N, \overline{\theta}\big) \longrightarrow (C, H, \lambda, \varphi)$ is in $ProfCMod$.

Lemma 2.9. There is a covariant functor (forgetful functor)
$F: ProfCMod \longrightarrow \overline{ProfTGrps}$ defined by:
(F1) $F: (N, G, \partial, \theta) = (N, G, \theta)$ for all (N, G, ∂, θ) in $ProfCMod$,
(F2) $F(\mu, \eta) = (\mu, \eta): (N, G, \theta) \longrightarrow (C, H, \varphi)$ for all
$(\mu, \eta) \in Mor_{ProfCMod}\big((N, G, \partial, \theta), (C, H, \lambda, \varphi)\big)$.

Lemma 2.10. There is a covariant functor $L: \overline{ProfTGrps} \longrightarrow ProfCMod$ defined by:
(L1) $L(N, G, \theta) = \big(N, N \rtimes G, \partial_N, \overline{\theta}\big)$ for all (N, G, θ) in $\overline{ProfTGrps}$,
(L2) $L(f, l) = (f, f \rtimes l): \big(N, N \rtimes G, \partial_N, \overline{\theta}\big) \longrightarrow (C, C \rtimes H, \lambda_H, \overline{\varphi})$ for all
$(f, l) \in Mor_{\overline{ProfTGrps}}\big((N, G, \theta), (C, H, \varphi)\big)$.

Theorem 2.11. The covariant functors F and L as defined above represent an adjoint pair of functors, $i.e.$ L is a left adjoint functor of F.

Proof: We shall show that there is a natural isomorphism $\phi: Mor_{ProfCMod}(L-, -) \longrightarrow Mor_{\overline{ProfTGrps}}(-, F\,-)$,where $Mor_{ProfCMod}(L-, -), Mor_{\overline{ProfTGrps}}(-, F\,-): \overline{ProfTGrps}^{op} \times ProfCMod \longrightarrow \mathrm{S}$ are bifunctors, the notation $\overline{ProfTGrps}^{op}$ denotes the opposite (or dual) category of $\overline{ProfTGrps}$, and S is the category of sets, defined respectively by the following compositions;

$$\overline{ProfTGrps}^{op} \times ProfCMod \xrightarrow{L^{op} \times I_{ProfCMod}} ProfCMod^{op} \times ProfCMod \xrightarrow{E_{ProfCMod}} \mathrm{S}$$ and

$$\overline{ProfTGrps}^{op} \times ProfCMod \xrightarrow{I_{\overline{ProfTGrps}^{op}} \times F} \overline{ProfTGrps}^{op} \times \overline{ProfTGrps} \xrightarrow{E_{\overline{ProfTGrps}}} \mathrm{S},$$

where $L^{op}: \overline{ProfTGprs}^{op} \longrightarrow ProfCMod^{op}$ is a functor assigns to each profinite transformation group $(N,G,\theta)^{op}$, a profinite crossed module $L^{op}(N,G,\theta)^{op} = \big(L(N,G,\theta)\big)^{op} = \big(N, N \rtimes G, \partial_N, \overline{\theta}\big)^{op}$ and to each continuous morphism $(l,k)^{op}: (N,G,\theta)^{op} \longrightarrow (C,H,\varphi)^{op}$ of a profinite transformation groups, a continuous morphism $L^{op}(l,k)^{op} = \big(L(l,k)\big)^{op} = (l, l \rtimes k)^{op}: \big(N, N \rtimes G, \partial_N, \overline{\theta}\big)^{op} \longrightarrow (C, C \rtimes H, \lambda_c, \overline{\varphi})^{op}$.

Moreover, $E_{ProfCMod}$ is a bifunctor assigns to each object $\big((N,G,\partial,\theta)^{op},(C,H,\lambda,\varphi)\big)$ a set $Mor_{ProfCMod}\big((N,G,\partial,\theta),(C,H,\lambda,\varphi)\big)$ and to each morphism $\big((f,t)^{op},(\mu,\eta)\big): \big((N,G,\partial,\theta)^{op},(C,H,\lambda,\varphi)\big) \longrightarrow \big((N',G',\partial',\theta'),(C',H',\lambda',\varphi')\big)$ a function $E_{ProfCMod}\big((f,t)^{op},(\mu,\eta)\big): Mor_{ProfCMod}\big((N,G,\partial,\theta),(C,H,\lambda,\varphi)\big) \longrightarrow Mor_{ProfCMod}\big((N',G',\partial',\theta'),(C',H',\lambda',\varphi')\big)$ which is defined by $E_{ProfCMod}\big((f,t)^{op},(\mu,\eta)\big)(l,k) = (\mu,\eta)(l,k)(f,t)$ for all $(l,k) \in Mor_{ProfCMod}\big((N,G,\partial,\theta),(C,H,\lambda,\varphi)\big)$. Also, $E_{\overline{ProfTGrps}}$ can be similarly defined. Now Define a function $\phi: Mor_{ProfCMod}(L-,-) \longrightarrow Mor_{\overline{ProfTGrps}}(-,F-)$ as follows: for all $A = \big((N,G,\theta)^{op},(C,H,\lambda,\varphi)\big) \in \overline{ProfTGrps}^{op} \times ProfCMod$, $\phi(A): Mor_{ProfCMod}\Big(\big(N, N \rtimes G, \partial_N, \overline{\theta}\big),(C,H,\lambda,\varphi)\Big) \longrightarrow$ $Mor_{\overline{ProfTGrps}}\big((N,G,\theta),(C,H,\varphi)\big)$ is a function defined by $\phi(A)(l,k) = (l,k^*)$ for all continuous morphism $(l,k): \big(N, N \rtimes G, \partial_N, \overline{\theta}\big) \longrightarrow (C,H,\lambda,\varphi)$ of profinite crossed modules, where (l,k^*) is a continuous morphism of profinite transformation groups as defined in corollary (2.7). We shall show that ϕ is a natural transformation. To do this, let $\big((f,t)^{op},(\mu,\eta)\big) \in Mor_{\overline{ProfTGrps}^{op} \times ProfCMod}(A,B)$, where $A = \big((N,G,\theta)^{op},(C,H,\lambda,\varphi)\big)$ and $B = \big((N',G',\partial',\theta')^{op},(C',H',\lambda',\varphi')\big)$. It is enough to show the commutativity of the following diagram with respect to ϕ*:*

$$\begin{array}{ccccc}
A & Mor_{ProfCMod}\Big(\big(N, N \rtimes G, \partial_N, \overline{\theta}\big),(C,H,\lambda,\varphi)\Big) & \underset{\Psi(A)}{\overset{\Phi(A)}{\rightleftarrows}} & Mor_{\overline{ProfTGrps}}\big((N,G,\theta),(C,H,\varphi)\big) \\
\downarrow \big((f,t)^{op},(\mu,\eta)\big) & \downarrow Mor_{ProfCMod}(L-,-)\big((f,t)^{op},(\mu,\eta)\big) = E_{ProfCMod}\big((f, f \rtimes t)^{op},(\mu,\eta)\big) & & \downarrow Mor_{\overline{ProfTGrps}}(-,F-)\big((f,t)^{op},(\mu,\eta)\big) = E_{\overline{ProfTGrps}}\big((f,t)^{op},(\mu,\eta)\big) \\
B & Mor_{ProfCMod}\Big(\big(N', N' \rtimes G', \partial'_N, \overline{\theta}'\big),(C',H',\lambda',\varphi')\Big) & \underset{\Psi(A)}{\overset{\Phi(B)}{\rightleftarrows}} & Mor_{\overline{ProfTGrps}}\big((N',G',\theta'),(C',H',\varphi')\big)
\end{array}$$

Let $(l,k) \in Mor_{ProfCMod}(N, N \rtimes G, \partial_N, \overline{\theta}) \longrightarrow (C,H,\lambda,\varphi)$. Therefore

$E_{\overline{ProfTGrps}}\big((f,t)^{op},(\mu,\eta)\big)\Phi(A)(l,k) = (\mu l f, \eta k^{*} t)$. On the other hand,

$\Phi(B)E_{ProfCMod}\big((f, f \rtimes t)^{op},(\mu,\eta)\big)(l,k) = \big(\mu l f, (\eta k(f \rtimes t))^{*}\big)$.

But $\eta k^{*} t = \big(\eta k(f \rtimes t)\big)^{*}$ on G', therefore ϕ is a natural transformation.

Likewise, define $\Psi: Mor_{\overline{ProfTGrps}}(-, F\,-) \longrightarrow Mor_{ProfCMod}(L-,-)$ as follows;

for all $A = \big((N,G,\theta)^{op},(C,H,\lambda,\varphi)\big) \in \overline{ProfTGrps}^{op} \times ProfCMod$,

$\Psi(A): Mor_{\overline{ProfTGrps}}\big((N,G,\theta),(C,H,\varphi)\big) \longrightarrow$

$Mor_{ProfCMod}\Big(\big(N, N \rtimes G, \partial_N, \overline{\theta}\big),(C,H,\lambda,\varphi)\Big)$, is a function defined by $\Psi(A)(l,k) = \big(l, \mathrm{V}\,(l \rtimes k)\big)$, for all $(l,k) \in Mor_{\overline{ProfTGrps}}(N,G,\theta) \to (C,H,\varphi)$,where $\big(l, \mathrm{V}\,(l \rtimes k)\big): \big(N, N \rtimes G, \partial_N, \overline{\theta}\big) \longrightarrow (C,H,\lambda,\varphi)$ is a continuous morphism of profinite crossed modules as defined in corollary (2.8). Ψ is a natural transformation according to the commutativity of the above diagram w.r.t. Ψ as follows:

Let $(l,k) \in Mor_{\overline{ProfTGrps}}\big((N,G,\theta),(C,H,\lambda)\big)$. Therefore, we have

$E_{ProfCMod}\big((f, f \rtimes g)^{op},(\mu,\eta)\big)\Psi(A)(l,k) = \big(\mu l f, \eta \mathrm{V}\,(l \rtimes k)(f \rtimes g)\big)$. On the other hand; $\Psi(B)E_{\overline{ProfTGrps}}\big((f,g)^{op},(\mu,\eta)\big)(l,k) = \big(\mu l f, \mathrm{V}'(\mu \rtimes \eta)(l \rtimes k)(f \rtimes g)\big)$. Since $\mathrm{V}'(\mu \rtimes \eta) = \eta \mathrm{V}$ on $C \rtimes H$. Therefore Ψ is a natural transformation.

To complete the proof, we show that

$\Psi\Phi = I_{Mor_{ProfCMod}(L-,-)}$ and $\Phi\Psi = I_{Mor_{\overline{ProfTGrps}}(-,F-)}$.

Let $A = \big((N,G,\theta)^{op},(C,H,\lambda,\varphi)\big) \in \overline{ProfTGrps}^{op} \times ProfCMod$ and let

$(l,k) \in Mor_{ProfCMod}\Big(\big(N, N \rtimes G, \partial_N, \overline{\theta}\big),(C,H,\lambda,\varphi)\Big)$, we have

$\big((\Psi\Phi)(A)\big)(l,k) = \big(f, \mathrm{V}(f \rtimes k^{*})\big)$. But $\mathrm{V}(f \rtimes k^{*}) = k$ on $N \rtimes G$, therefore

$$\big((\Psi\Phi)(A)\big)(l,k) = (l,k) = \left(I_{Mor_{ProfCMod}(L-,-)}(A)\right)(l,k).$$

Likewise, let $(r,s) \in Mor_{\overline{ProfTGrps}}\big((N,G,\theta),(C,H,\lambda)\big)$,

$\big((\Phi\Psi)(A)\big)(r,s) = \big(r, (\mathrm{V}(r \rtimes s))^{*}\big)$. In fact $\big(\mathrm{V}(r \rtimes s)\big)^{*} = s$ on G, therefore

$$\big((\Phi\Psi)(A)\big)(r,s) = (r,s) = \left(I_{Mor_{\overline{ProfTGrps}}(-,F-)}(A)\right)(r,s).$$

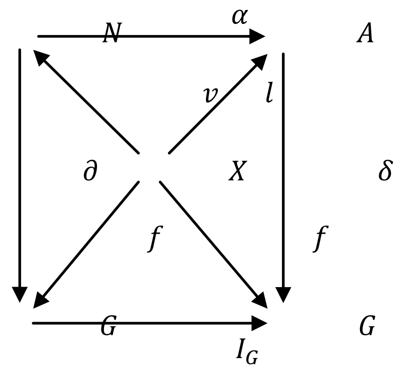
Hence L is a left adjoint functor of F.

3. An adjoint pair of covariant functors between the category of free profinite crossed modules and the category of continuous maps from profinite spaces to profinite groups

Definition 3.1. [Ribes and Zalesskii 2000] Let X be a profinite space. A *free profinite group* on X is a profinite group $F(X)$ together with a continuous map $f: X \longrightarrow F(X)$ satisfying the following universal property: if $h: X \longrightarrow G$ is any continuous map from X to a profinite group G, there exists a unique continuous homomorphism $\phi: F(X) \longrightarrow G$ such that $\phi f = h$.

Now we define a free profinite crossed module in an analogous manner.

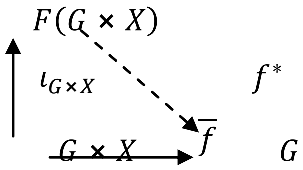
Definition 3.2. [Korkes and Porter 1986**]** Let (N, G, ∂, θ) in $ProfCMod$ and $f: X \to G$ be a continuous function, where X is a profinite space, we say that (N, G, ∂, θ) is a *free profinite crossed module* on f if there is a continuous function $\nu: X \to N$ such that $f = \partial\nu$ together with the following universal property: Given any (A, G, δ, φ) in $ProfCMod$ and any continuous function $l: X \to A$ such that $\delta l = f$, there exists a unique continuous morphism $(\alpha, I_G): (N, G, \partial, \theta) \longrightarrow (A, G, \delta, \varphi)$ in $ProfCMod$ such that $\alpha\nu = l$.



Free profinite crossed modules and their continuous morphisms from a category, $FProfCMod$.

Free construction 3.3. [Korkes and Porter 1986**]** Suppose given a continuous function $f: X \longrightarrow G$ from a profinite space X into a profinite group G. Let $F(G \times X)$ be the free profinite group on the profinite space $G \times X$. Define a continuous left action $\sigma: G \times F(G \times X) \longrightarrow F(G \times X)$ of G on $F(G \times X)$ by ${}^{g}(g', x) = (gg', x)$ for all $x \in X$ and $g, g' \in G$.
The continuous function f induces a continuous function $\overline{f}: G \times X \longrightarrow G$ defined on generators by $\overline{f}(g, x) = gf(x)g^{-1}$. The continuity of $\overline{f}$ is impliesed from the continuity of f as well as from the continuity of the left action of G on itself by conjugations. Now, as $\overline{f}: G \times X \longrightarrow G$ is a continuous

function, therefore from the universal property of the free profinite group $F(G \times X)$ on $G \times X$, there is a continuous homomorphism f^* extending $\overline{f}$, $f^* \iota_X = \overline{f}$,



From above, $(F(G \times X), G, f^*, \sigma)$ is in $ofPreCMod$. Let $[\![F,F]\!]$, where $F = F(G \times X)$ be the peiffer subgroup of F generated by the peiffer elements $[u,v] = uvu^{-1}\,{}^{f^*(u)}v^{-1}$, $u, v \in F(G \times X)$. In fact $[\![F,F]\!]$ is a closed G-invariant normal subgroup of $F(G \times X)$. Hence the quotient group $F(G \times X)^{cr} = {F(G \times X)}/{[\![F,F]\!]}$ is a profinite group. The continuous left action σ induces a continuous left action σ^{cr} of G on $F(G \times X)^{cr}$ defined by: ${}^{g}\big((g',x)[\![F,F]\!]\big) = (gg',x)[\![F,F]\!]$. Also, the continuous homomorphism f^* induces a continuous homomorphism ${f^*}^{cr}: F(G \times X)^{cr} \longrightarrow G$ defined by

${f^*}^{cr}\big((g,x)[\![F,F]\!]\big) = gf(x)g^{-1} = f^*(g,x)$. From above $\big(F(G \times X)^{cr}, G, {f^*}^{cr}, \sigma^{cr}\big)$ is in $ofCMod$.

Definition 3.4. [Korkes and Porter 1986] Let G be a fixed profinite group. If we take as objects the continuous functions $f: X \longrightarrow G$, where X a profinite space, which we denote it by (X, G, f), and as continuous morphisms between these objects the pairs $(l, I_G): (X, G, f) \longrightarrow (X', G, f')$ where $l: X \longrightarrow X'$ is a continuous function such that $f'l = f$, then the above data consititute a category, ${ContMap}/{G}$, called the category of continuous functions.

In the following, we shall give an interpretation to the description of the above free construction (3.3) by constructing an adjoint pair of covariant functors between the categories ${FProfCMod}/{G}$ and ${ContMap}/{G}$. To do this let us first establish the following five results whose proofs are straight forwards.

Proposition 3.5. Let X and X' be profinite spaces and G be a profinite group. If $f: X \longrightarrow G$ and $f': X' \longrightarrow G$ are continuous functions such that $f'\ell = f$, then there is a unique continuous homomorphism ${\ell^*}^{cr}: F(G \times X)^{cr} \longrightarrow F'(G \times X')^{cr}$ defined by ${\ell^*}^{cr}\big((g,x)[\![F,F]\!]\big) = \big(g, \ell(x)\big)[\![F',F']\!]$ such that

$\big({\ell^*}^{cr}, I_G\big): \big(F(G \times X)^{cr}, G, {f^*}^{cr}, \sigma^{cr}\big) \longrightarrow \Big(F'(G \times X')^{cr}, G, {f'^*}^{cr}, \sigma'^{cr}\Big)$ is in ${ProfCMod}/{G}$.

Proposition 3.6. Let (N, G, λ, θ) be in ${}^{ProfCMod}/_{G}$ and $f: X \longrightarrow G$ be a continuous function, where X a profinite space. If $(\alpha^{cr}, I_G): \left(F(G \times X)^{cr}, G, f^{*^{cr}}, \sigma^{cr}\right) \longrightarrow (N, G, \lambda, \theta)$ is in ${}^{ProfCMod}/_{G}$, then there is a continuous function

$\gamma: X \longrightarrow F(G \times X)^{cr}$ defined by $\gamma(x) = (1_G, x)[\![F, F]\!]$ such that $\lambda\, \alpha^{cr}\, \gamma = f$.

Proposition 3.7. Let (N, G, λ, θ) be in ${}^{ProfCMod}/_{G}$ and let $f: X \longrightarrow G$ and $\beta: X \longrightarrow N$ are continuous functions, where X is a profinite space, such that $\lambda\beta = f$, then there is a unique continuous homomorphism $\beta^{*^{cr}}: F(G \times X)^{cr} \longrightarrow N$ defined by $\beta^{*^{cr}}\left((g, x)[\![F, F]\!]\right) = {}^{g}\beta(x)$ such that $(\beta^{*^{cr}}, I_G): \left(F(G \times X)^{cr}, G, f^{*^{cr}}, \sigma^{cr}\right) \longrightarrow (N, G, \lambda, \theta)$ is in ${}^{ProfCMod}/_{G}$.

Lemma 3.8. There is a forgetful functor $F: {}^{ProfCMod}/_{G} \longrightarrow {}^{ContMap}/_{G}$ defined by:

(F1) $F(N, G, \lambda, \theta) = (N, G, \lambda)$ for all $(N, G, \lambda, \theta) \in {}^{ProfCMod}/_{G}$.

(F2) $F(\mu, I_G) = (\mu, I_G): (N, G, \lambda) \longrightarrow (N', G, \lambda')$

for all $(\mu, I_G) \in Mor_{{}^{ProfCMod}/_{G}}\left((N, G, \lambda, \theta), (N', G, \lambda', \theta')\right)$.

Lemma 3.9. There is a covariant functor $L: {}^{ContMap}/_{G} \longrightarrow {}^{ProfCMod}/_{G}$ defined by:

(L1) $L(X, G, f) = \left(F(G \times X)^{cr}, G, f^{*^{cr}}, \sigma^{cr}\right)$ for all $(X, G, f) \in {}^{ContMap}/_{G}$.

(L2) $L(\ell, I_G) = (\ell^{*^{cr}}, I_G): \left(F(G \times X)^{cr}, G, f^{*^{cr}}, \sigma^{cr}\right) \longrightarrow \left(F'(G \times X')^{cr}, G, f'^{*^{cr}}, \sigma'^{cr}\right)$ for all $(\ell, I_G) \in Mor_{{}^{ContMap}/_{G}}\left((X, G, f), (X', G, f')\right)$.

Theorem 3.10. The covariant functors F and L as defined above represent an adjoint pair of functors, $i.e.$ L is a left adjoint functor of F.

Proof. We shall follow the same procedure described in the proof of theorem (2.11).

Define two functions $\Phi: Mor_{{}^{ProfCMod}/_{G}}(L-, -) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} Mor_{{}^{ContMap}/_{G}}(-, F-)$ as follow: For any $= \left((X, G, f)^{op}, (N, G, \lambda, \theta)\right) \in {}^{ContMap^{op}}/_{G} \times {}^{ProfCMod}/_{G}$,

$\Phi(A): Mor_{{}^{ProfCMod}/_{G}}\left(\left(F(G \times X)^{cr}, G, f^{*^{cr}}, \sigma^{cr}\right), (N, G, \lambda, \theta)\right) \longrightarrow$

$Mor_{ContMap/_G}\big((X,G,f),(N,G,\lambda)\big)$ is a function defined by: $\Phi(A)(\alpha^{cr},I_G)=(\alpha^{cr}\gamma,I_G)$ for all $(\alpha^{cr},I_G):\big(F(G\times X)^{cr},G,f^{*^{cr}},\sigma^{cr}\big)\longrightarrow(N,G,\lambda,\theta)$, where $(\alpha^{cr}\gamma,I_G):(X,G,f)\longrightarrow(N,G,\lambda)$ is in $^{ContMap}/_G$ as defined in proposition (3.6). Also, $\Psi(C):Mor_{ContMap/_G}\big((X,G,f),(N,G,\lambda)\big)$

$\longrightarrow$ $Mor_{ProfMod/_G}\Big(\big(F(G\times X)^{cr},G,f^{*^{cr}},\sigma^{cr}\big),(N,G,\lambda,\theta)\Big)$ is a function defined by: $\Psi(C)(\beta,I_G)=\big(\beta^{*^{cr}},I_G\big)$, for all $(\beta,I_G):(X,G,f)\longrightarrow(N,G,\lambda)$ in $^{ContMap}/_G$, where $\big(\beta^{*^{cr}},I_G\big):\big(F(G\times X)^{cr},G,f^{*^{cr}},\sigma^{cr}\big)\longrightarrow(N,G,\lambda,\theta)$ is in $^{ProfCMod}/_G$ as defined in proposition (3.7).

Let $\big((\ell,I_G)^{op},(\mu,I_G)\big):A=\big((X,G,f)^{op},(N,G,\lambda,\theta)\big)\longrightarrow$

$B=\big((X',G,f')^{op},(N',G,\lambda',\theta')\big)$ in $Mor_{ContMap/_G{}^{op}\times ProfCMod/_G}$. To show Φ and Ψ are natural transformations, we need to show the commutativity of the following diagram:

$$
\begin{array}{cccc}
A & Mor_{ProfCMod/_G}\Big(\big(F(G\times X)^{cr},G,f^{*^{cr}},\sigma^{cr}\big),U\Big) & \underset{\Psi(A)}{\overset{\Phi(A)}{\rightleftarrows}} & Mor_{ContMap/_G}\big((X,G,f),J\big) \\
\Big\downarrow \big((\ell,I_G)^{op},(\mu,I_G)\big) & \Big\downarrow E_{ProfCMod/_G}\Big(\big(\ell^{*^{cr}},I_G\big)^{op},(\mu,I_G)\Big) & & \Big\downarrow E_{ContMap/_G}\big((\ell,I_G)^{op},(\mu,I_G)\big) \\
B & Mor_{ProfCMod/_G}\Big(\Big(\big(F'(G\times X')^{cr},G,f'^{*^{cr}},\sigma'^{cr}\big)\Big),Y\Big) & \underset{\Psi(A)}{\overset{\Phi(B)}{\rightleftarrows}} & Mor_{Contmap/_G}\big((X',G,f'),Q\big)
\end{array}
$$

Where $U=(N,G,\lambda,\theta), J=(N,G,\lambda), Y=(N',G,\lambda',\theta')$, and $Q=(N',G,\lambda')$. Let $(\alpha^{cr},I_G):\big(F(G\times X)^{cr},G,f^{*^{cr}},\sigma^{cr}\big)\longrightarrow(N,G,\lambda,\theta)$ be in $^{ProfCMod}/_G$.

Therefore, $E_{ContMap/_G}\big((\ell,I_G)^{op},(\mu,I_G)\big)\Phi(A)(\alpha^{cr},I_G)=(\mu\alpha^{cr}\gamma\ell,I_G)$. On the other hand $\Phi(B)E_{ProfCMod/_G}\Big(\big(\ell^{*^{cr}},I_G\big)^{op},(\mu,I_G)\Big)(\alpha^{cr},I_G)=\big(\mu\,\alpha^{cr}\ell^{*^{cr}}\gamma',I_G\big)$. But $\gamma\ell=\ell^{*^{cr}}\gamma'$ on X', therefore Φ is a natural transformation. Also, let $(\beta,I_G):(X,G,f)\longrightarrow(N,G,\lambda)$ be in $^{ContMap}/_G$, therefore

$E_{ProfCMod/_G}\Big(\big(\ell^{*^{cr}},I_G\big)^{op},(\mu,I_G)\Big)\Psi(A)(\beta,I_G)=\big(\mu\,\beta^{*^{cr}}\ell^{*^{cr}},I_G\big)$. On the other hand,

$\Psi(B)E_{ContMap/_G}\big((\ell,I_G)^{op},(\mu,I_G)\big)(\beta,I_G)=\big((\mu\beta\ell)^{*^{cr}},I_G\big)$. Since

$(\mu\beta\ell)^{*^{cr}} = \mu\, \beta^{*^{cr}} \ell^{*^{cr}}$ on $F'(G \times X')^{cr}$. Thus Ψ is a natural transformation. Finally, we show that $\Psi\phi = I_{Mor_{ProfCMod/_G}(L-,-)}$ and $\phi\Psi = I_{Mor_{ContMap/_G}(-,F-)}$. Let

$A = \left((X,G,f)^{op}, (N,G,\lambda,\theta)\right) \in {}^{ContMap^{op}}/_G \times {}^{ProfCMod}/_G$, and

$(\alpha^{cr}, I_G)\colon \left(F(G \times X)^{cr}, G, f^{*^{cr}}, \sigma^{cr}\right) \longrightarrow (N,G,\lambda,\theta)$. Thus $\left((\Psi\Phi)(A)\right)(\alpha^{cr}, I_G) = \left((\alpha^{cr}\gamma)^{*^{cr}}, I_G\right)$. In fact $(\alpha^{cr}\gamma)^{*^{cr}} = \alpha^{cr}$ on $F(G \times X)^{cr}$, therefore,

$\left((\Psi\phi)(A)\right)(\alpha^{cr}, I_G) = \left(I_{Mor_{ProfCMod/_G}(L-,-)}(A)\right)(\alpha^{cr}, I_G)$.

Likewise, let $(\beta, I_G)\colon (X,G,f) \longrightarrow (N,G,\lambda)$, therefore

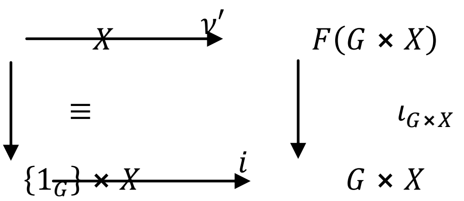
$\left((\Phi\Psi)(A)\right)(\beta, I_G) = \left(\beta^{*^{cr}}\gamma, I_G\right)$. But $\beta^{*^{cr}}\gamma = \beta$ on X, therefore

$\left((\Phi\Psi)(A)\right)(\beta, I_G) = (\beta, I_G)\left(I_{Mor_{ContMap/_G}(-,F-)}(A)\right)(\beta, I_G)$.

Hence L is a left adjoint functor of F.

Theorem 3.11. If (X,G,f) in ${}^{ContMap}/_G$, then $\left(F(G \times X)^{cr}, G, f^{*^{cr}}, \sigma^{cr}\right)$ is a free profinite crossed module on (X,G,f).

Proof. Let $\nu'\colon X \longrightarrow F(G \times X)$ be a function define by $\nu'(x) = (1_G, x)$ for all $x \in X$. Therefore, ν' is a continuous map of profinite spaces according to the following commutative diagram:



since the quotient homomorphism $q\colon F(G \times X) \longrightarrow F(G \times X)^{cr}$ is also continuous, therefore $\nu = q\,\nu'\colon X \longrightarrow F(G \times X)^{cr}$ is a continuous map defined by $\nu(x) = (1_G, x)[\![F,F]\!]$ for all $x \in X$. Note that $f^{*^{cr}}\nu = f$ on X.

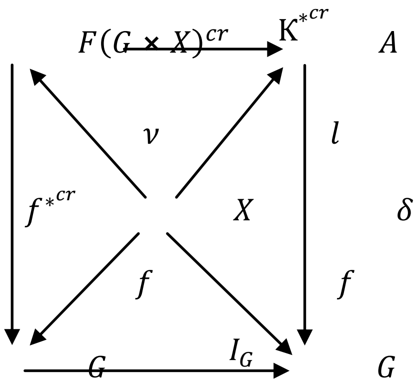
Now, we prove the universal property of the free profinite crossed module. Given a profinite crossed module (A,G,δ,φ) and a continuous function $l\colon X \longrightarrow A$ such that $\delta l = f$. The continuous function l and the continuous left action φ induce a continuous map $\mathrm{K}\colon G \times X \longrightarrow A$ defined by $\mathrm{K}(g,x) = {}^{g}\left(l(g)\right)$ for all $x \in X$ and $g \in G$. Therefore, from the universal property of the free profinite group $F(G \times X)$, there is a unique continuous homomorphism $\mathrm{K}^*\colon F(G \times X) \longrightarrow A$ such that $\mathrm{K}^*\iota_{G\times X} = \mathrm{K}$, *i.e.* $\mathrm{K}^*(g,x) = \mathrm{K}(g,x) = {}^{g}\left(l(g)\right)$.

Since $K\left((g,x)(g',x')(g,x)^{-1}\left({}^{f^*(g,x)}(g',x')\right)^{-1}\right) = 1_A, i.e. [\![F,F]\!] \subseteq KerK$,

Therefore, from the universal property of the quotient homomorphism $q: F(G \times X) \longrightarrow F(G \times X)^{cr}$, there is a unique continuous homomorphism $K^{*^{cr}}: F(G \times X)^{cr} \longrightarrow A$

such that $K^{*^{cr}}q = K^*$, $i.e.$ $K^{*^{cr}}\left((g,x)[\![F,F]\!]\right) = K^*(g,x) = {}^g l(x)$.

Now, Consider the following diagram:



We need only to show that $\left(K^{*^{cr}}, I_G\right): \left(F(G \times X)^{cr}, G, f^{*^{cr}}, \sigma^{cr}\right) \longrightarrow (A, G, \delta, \varphi)$ is in ${}^{ProfCNod}/_G$

, where, the uniqueness of $\left(K^{*^{cr}}, I_G\right)$ is followed from the uniqueness of $K^{*^{cr}}$ and I_G .

For all $(g,x)[\![F,F]\!] \in F(G \times X)^{cr}$,

$$\begin{aligned}
\delta K^{*^{cr}}\left((g,x)[\![F,F]\!]\right) &= \delta\left({}^g l(x)\right) \\
&= g\delta l(x)g^{-1} \\
&= gf(x)g^{-1} \\
&= f^*(g,x) \\
&= f^{*^{cr}}\left((g,x)[\![F,F]\!]\right).
\end{aligned}$$

Also, for any $g \in G$ and $(g',x)[\![F,F]\!] \in F(G \times X)^{cr}$,

$$\begin{aligned}
K^{*^{cr}}\left({}^g\left((g',x)[\![F,F]\!]\right)\right) &= K^{*^{cr}}\left((gg',x)[\![F,F]\!]\right) \\
&= {}^{gg'}l(x) \\
&= {}^g\left(K^{*^{cr}}\left((g',x)[\![F,F]\!]\right)\right) \\
&= {}^{I_G(g)}K^{*^{cr}}\left((g',x)[\![F,F]\!]\right).
\end{aligned}$$

The following result can be deduced from lemmas (3.8), (3.9) and theorems (3.10),(3.11).

Theorem 3.12. The covariant functor L: ${}^{ContMap}/_G \longrightarrow {}^{FProfCMod}/_G$ is a left adjoint functor of the forgetful functor : ${}^{FProfCMod}/_G \longrightarrow {}^{ContMap}/_G$.

References:

D. Haran and M.Jarden, (1985)The Absolute Galois of A Pseudo Real Closed Field, Scuola Normal Superiore, Pisa , P. 449-489.

F.J. Korkes, (1987) The Cohomological and Combinatoral Group Theory of Profinite groups, Ph.D. Thesis, U.C.N.W..

F.J. Korkes and T. Porter, (1986)Profinite crossed modules**,** U.C.N.W. Pure Mathematics preprint, No.11,.

L. Ribes and P. Zalesskii, (2000) Profinite Groups**,** Ergebnisse der Math. Und ihrer Grenzgebiete, Vol. 40, Springer-verlag, Berlin.

J.H.C. Whitehead, (1949)Combinatorial Homotopy I and II, Bull. Amer. Math. Soc., 55: 231-245 and 453-496.

بعض الأزواج المترافقة للمقرنات التغايرية بين فصيلة الموديولات المتصالبة المنتهية أسقاطياً وبعض الفصائل ذات العلاقة

رعد صالح مهدي الزركاني شيماء عبد الرزاق عبدالله

قسم الرياضيات- كلية العلوم-جامعة البصره-البصرة-العراق

raadalzurkani64@yahoo.com

الخلاصة :-

في هذا البحث تم تقديم ودراسة بعض الأزواج المترافقة من المقرنات التغايرية. الزوج المترافق الأول من المقرنات التغايرية تم بناءه بين فصيلة الموديولات المتصالبة المنتهية أسقاطيا وفصيلة زمر التحويلات المنتهية أسقاطيا، والذي نبين فيه كيفية بناء موديول متصالب منتهي أسقاطيا من زمرة تحويلات منتهية أسقاطيا معطاة. الزوج المترافق الثاني تم بناءه بين فصيلة الموديولات المتصالبة المنتهية أسقاطيا الحرة وبين فصيلة التطبيقات المستمرة (من فضاءات منتهية أسقاطيا إلى زمر منتهية أسقاطيا) والذي نبين فيه كيفية بناء موديول متصالب منتهي أسقاطيا حر على تطبيق مستمر من فضاء منتهي أسقاطيا إلى زمرة منتهية أسقاطيا.

المفاتيح: الزوج المرافق للمقرنات التغايرية، زمرة التحويلات المنتهية، الموديول المتصالب المنتهي أسقاطياً