On Complete Convex Bornological Vector Spaces

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Abstract: The structure of the complete bornological vector space has been studied and related concepts, such as quotient, product and direct sum of completion of bornological vector spaces. The study also implied concepts of Cauchy net in the basis of converge net in convex bornological vector space.

Introduction:

In (1971), H.Hogbe-Nlend introduced the concepts of bornology on a set. In(1981) M.D. Patwardhan extended this idea to a space of entire functions. However, The study also implied concepts of Cauchy net in the basis of converge net in convex bornological vector space (*cbvs*) and the bornological completion of bornological vector space and some main results related to this concept.

1. Basic Notations:

In this section we introduce the basic definitions, notions and the theories of bornological vector spaces and construction of bornological vector space.

Definition 1.1.[3] :- Let A and B be two subsets of a vector space E. We say that:

(i) A is circled if $\lambda A \subset A$ whenever $\lambda \in K$ and $|\lambda| \leq 1$;

(ii) A is convex if $\lambda A + \mu A \subset A$ whenever λ and μ are positive real numbers such that $\lambda + \mu = 1$;

(iii) A is disked, or a disk, if A is both convex and circled;

(iv) A absorbs B if there exists $\alpha \in R$, $\alpha > 0$, such that $\lambda A \subset B$ whenever $\alpha \leq |\lambda|$;

(v) A is an absorbent in E if A absorbs every subset of E consisting of a single point.

Remark 1.2.[3]:- (i) If A and B are convex and $\lambda, \mu \in K$, then $\lambda A + \mu B$ is convex.

(ii) Every intersection of circled (resp. convex, disked) sets is circled (resp. convex, disked).

(iii) Let E and F be vector spaces and let $u : E \to F$ be a linear map. Then the image, direct or inverse, under u of a circled (resp. convex, disked) subset is circled (resp. convex, disked).

Remark 1.3.[6]:- A subset A is bounded if it is absorbed by any neighborhood of the origin.

Definition 1.4.[1]:- A bornology on a set X is a family $\boldsymbol{\beta}$ of subsets of X satisfying the following axioms:

(i) $\boldsymbol{\beta}$ is a covering of X, i.e. $X = \bigcup_{B \in \beta} B$;

(ii) $\boldsymbol{\beta}$ is hereditary under inclusion i.e. if $A \in \boldsymbol{\beta}$ and B is a subset of X contained in A, then $B \in \boldsymbol{\beta}$;

(iii) $\boldsymbol{\beta}$ is stable under finite union.

A pair $(X, \boldsymbol{\beta})$ consisting of a set X and a bornology $\boldsymbol{\beta}$ on X is called a bornological space, and the elements of $\boldsymbol{\beta}$ are called the bounded subsets of X.

Definition 1.5.[5]:- A base of a bornology *ß* on X is any subfamily $ß_0$ of *ß* such that every element of *ß* is contained in an element of $ß_0$.

Definition 1.6.[3]:- Let E be a vector space over the field K (the real or complex field). A bornology *ß* on E is said to be a bornology compatible with a vector space structure of E or to be a vector bornology on E, if *ß* is stable under vector addition, homothetic transformations and the formation of circled hulls, in other words, if the sets $A+B$, λA, $\bigcup_{|\alpha|\leq 1} \alpha A$ belongs to *ß* whenever A and B belong to *ß* and $\lambda \in K$.

Definition 1.7.[2]:- A convex bornological space is a bornological vector space for which the disked hull f every bounded set is bounded i.e. it is stable under the formation of disked hull.

Definition 1.8.[3]:- A separated bornological vector space (E, *ß*) is one where {0} is the only bounded vector subspace of E.

Example 1.9.[3]:- The Von–Neumann bornology of a topological vector space, Let E be a topological vector space. The collection

ß ={$A \subseteq E$: A is a bounded subset of a topological vector space E}

forms a vector bornology on E called the Von–Neumann bornology of E. Let us verify that *ß* is indeed a vector bornology on E, if

$ß_0$ is a base of circled neighborhoods of zero in E, it is clear that a subset A of E is bounded if and only if for every $B \in ß_0$ there exists $\lambda>0$ such that $A \subseteq \lambda B$.

Since every neighborhood of zero is absorbent, *ß* is a covering of E.

ß is obviously hereditary and we shall show that its also stable under vector addition. Let $A_1, A_2 \in ß$ and $B_0 \in ß_0$; there exists $B_0^{/}$

such that $B_0^{'} + B_0^{'} \subseteq B_0$. Since A_1 and A_2 are bounded in *E*, there exists positive scalars λ and μ such that $A_1 \subseteq \lambda B_0^{'}$ and $A_2 \subseteq \mu B_0^{'}$. with α= max(λ, μ), we have:

$A_1 + A_2 \subseteq \lambda B_0^{'} + \mu B_0^{'} \subset \alpha B_0^{'} + \alpha B_0^{'} \subset \alpha(B_0^{'} + B_0^{'}) \subset \alpha B_0$.

Finally, since β_0 is stable under the formation of circled hulls (resp. under homothetic transformations). Then so is ***ß***, and we conclude that ***ß*** is a vector bornology on *E*. If *E* is locally convex, then clearly ***ß*** is a convex bornology. Moreover, since every topological vector space has a base of closed neighborhoods of 0, the closure of each bounded subset of *E* is again bounded.

Definition 1.10.[6]:- Let $(X_i, \beta_i)_{i\in I}$ be a family of bornological space indexed by a non-empty set I and let $X = \prod_{i\in I} X_i$ be the product of the sets X_i . For every i∈I, let $P_i : X \to X_i$ be the canonical projection then. The product bornology on X is the initial bornology on X for the maps P_i. The set X endowed with the product bornology is called the bornological product of the space (X_i, β_i).

Definition 1.11.[6]:- Let F be a vector subspace of a bornological vector space *E*, the vector bornological quotient space is the quotient space *E*/*F* with the quotient bornology on *E*/*F*

i.e. $\beta_\theta = \{\theta(B):\theta(B)=B+F, B \text{ is bounded in } E\}$.

Definition 1.12.[3]:- Let (X_i, v_{ji}) be an inductive system of bornological vector spaces (resp., of convex bornological space) and let X be the set (resp. The vector space) which is the inductive limit of the system $(X_i, v_{ji}$

). For every i∈I, denote by $ß_i$ the bornology of X_i and by v_i the canonical map of X_i into X. The inductive limit bornology on X with respect to bornologyies $ß_i$ is the final bornology on X for the maps v_i. For every i∈I, let $v_i(ß_i)=\{v_i(A): A \in ß_i\}$.

Then the family $ß = \bigcup_{i\in I} v_i(ß_i)$ is precisely the final bornology on X. $X = \bigcup_{i\in I} v_i(ß_i)$ and $ß$ is indeed a bornology. It follows that, if (X_i, v_{ji}) is an inductive system of bornological vector spaces (resp. of convex bornological space), then the inductive limit bornology on X is necessarily a vector (resp. convex) bornology. When giving the inductive limit of the bornology, X is called the bornological inductive limit of the bornological inductive system (X_i, v_{ji}) and is denoted by: $X = \underset{i\in I}{\underset{\longrightarrow}{\lim}}(X_i, v_{ji})$.

Proposition 1.13.[3] :- Let $E=\underset{i\in\Gamma}{\underset{\to}{\lim}}(E_i, v_{ji})$be a bornological inductive limit of separated convex bornogical vector space . If all the maps V_{ji} are injective, then E is separated.

Proof: Indeed every bounded subset of E is then a bounded subset of one of the space E_i, which is separated.

Definition 1.14.[3]:- Let J(I) be the set of all finite subsets of I ordered by inclusion and, for every J∈ J(I), Let $E_J = \bigoplus_{i\in J} E_i$ be the bornological direct sum of the spaces $(E_i)_{i\in J}$. For J⊆ J′ denoted by $u_{j'j}$ the canonical embedding of E_J in $E_{J'}$. Then $(E_J, u_{j'j})$ is an inductive system of bornological vector spaces and $E = \underset{\to}{\lim}(E_J, u_{j'j})$.

Definition 1.15.[7]:- Let X and Y be two bornological spaces and *u*: $X \to$ Y is a map of X into Y. We say that *u* is a bounded map if the image under *u* of every bounded subset of X is bounded in Y i.e. $u(A) \in \mathcal{B}_y$, $\forall A \in \mathcal{B}_x$.

Definition 1.16.[3]:- Let *E* be a bornological vector space. *A* subset $A \subseteq E$ is said to be bornologically closed (briefly, b- closed) if the conditions $(x_\gamma)_{\gamma\in\Gamma} \subset A$ and $x_\gamma \longrightarrow x$ in *E* imply that $x \in A$.

Remark 1.17.[3] : Let *E* and F be bornological vector spaces and let $u: E \to F$ be a bounded linear map. The inverse image under u of b-closed subset of F is b-closed in *E*, since $(x_\gamma) \longrightarrow x$ in *E* implies $u(x_\gamma) \to u(x)$ in F.

The following theorems are given in [3] with out proof:-

Theorem 1.18.[3] :- A bornological vector space *E* is separated if and only if the vector subspace {0} is b- closed in *E*

***Proof*:-** Necessity: Suppose *E* to be separated and let *A*={0} and let (x_γ) be a net in *A* which converges bornologically to an element x in *E*. since $x_\gamma=0$ for every γ, this net also converges to 0 in *E* and the uniqueness of limits x=0.

Sufficiency: Suppose that {0} is b- closed in *E* and that (x_γ) is a net having limits x and y in *E*. The net $x_\gamma - x_\gamma = 0$ converges to x-y, hence x-y=0 and *E* is separated.

Theorem 1.19.[3] :-Let *E* be a convex bornological vector cpace and let *M* be a subspace of *E*. the quotient E/M is separated if and only if *M* is bornologically closed in *E*.

Proof:- If E/M is separated, then 0 is b-closed in E/M. If $\theta: E \to E/M$

is the canonical map, then $M = \theta^{-1}(0)$ is b- closed in E remark 1.17 Conversely, suppose M is b- closed in E and let H be a bounded subspace of E/M. We show that $H=\{0\}$ let $\theta(x)\in H$, $x\in E$; we can fined a bounded and absolutely convex set $A\subset E$ such that $K_{\theta(x)} \subset \theta(A)$, and hence $K_x \subset A+M$. Thus, for every $\gamma \in \Gamma, \gamma x \in A+M$ and hence there exists $(x_\gamma) \in M$ such that $\gamma x - x_\gamma \in A$. It follows that $(x - y_\gamma) \in (\frac{1}{\gamma})A$, Where $y_\gamma = (x_\gamma/\gamma) \in M$ and therefore $y_\gamma \longrightarrow x$.

Since M is b- closed, which implies $\theta(x)=0$.

2. Bornological Cauchy Net

In convex bornological vector space(*cbvs*), a notation of cauchy net can be introduced. The main results are of considerable interest in many situations.

Definition 2.1.[6] :- A directed system is an index set I together with an ordering < which satisfies:

i. If $\alpha, \beta \in I$, then exists $\gamma \in I$ such that $\gamma>\alpha$ and $\gamma>\beta$.
ii. < is a partial ordering (a reflexive transitive and antisymmetric relation on I).

Definition 2.2.[6] :- A net $(x_\gamma)_{\lambda\in\Gamma}$ in a set x is a function $q:\Gamma \to X$ where Γ is some ordered set. The point $q(\lambda)$ is usually denoted x_λ.

Definition 2.3 :- Let (x_γ) be a net in a convex bornological vector space E.

We say that (x_γ) converges bornologically to 0 ($(x_\gamma) \to 0$) if there exists a bounded and absolutely convex set $B\subset E$ and a net (λ_γ) in K converging to 0, such that $x_\gamma \in \lambda_\gamma B$, for every $\gamma \in \Gamma$. We then say that a net (x_γ) converges bornologically to a point $x\in E$ and $x_\gamma \to x$ when $x_\gamma - x \to 0$.

Definition 2.4 :- Let E be a convex bornological vector space. $(x_\gamma)_{\gamma \in \Gamma}$ is called a bornological Cauchy net in E if there exists a bounded and absolutely convex set $B \subseteq E$ and a null net of postive scalars $(\mu_{\gamma,\gamma'})_{(\gamma,\gamma') \in \Gamma \times \Gamma}$ such that $(x_\gamma - x_{\gamma'}) \in \mu_{\gamma,\gamma'} B$.

Theorem 2.5:- Every bornological Cauchy net is bounded.

Proof:- Suppose $(x_\gamma)_{\gamma \in \Gamma}$ is a Cauchy net in a bornological vector space E. by definition 2.4, there exists a bounded and absolutely convex set $B \subseteq E$ and a null net of postive scalars $(\mu_{\gamma,\gamma'})_{(\gamma,\gamma') \in \Gamma \times \Gamma}$ such that $(x_\gamma - x_{\gamma'}) \in \mu_{\gamma,\gamma'} B$. Since $x_\gamma \in B$ for all $\gamma \in \Gamma$. since B is bounded subset of E.Then (x_γ) is bounded.

Theorem 2.6 :- A bornological convergent net in a convex bornological vector space is a Cauchy net **.**

Proof:- Let (x_γ) in E converges bornologically to a point $x \in E$ by definition 2.3, there exists a bounded and absolutely convex set $B \subseteq E$ and a net (μ_γ) of scalars tends to 0 such that $(x_\gamma - x) \in \mu_\gamma B$.for every $\gamma \in \Gamma$ hence if γ' is a positive integer such that $\gamma \geq \gamma'$. $x_{\gamma'} - x \in \mu_{\gamma'} B$ for every integer γ' with $\gamma \geq \gamma'$.

Then $x_{\gamma'} - x \to 0$ implies $x_\gamma - x_{\gamma'} = x_\gamma - x - (x_{\gamma'} - x) \in \mu_{\gamma,\gamma'} B$ for all $\gamma, \gamma' \in \Gamma$ with $\gamma \geq \gamma'$, then (x_γ) is a bornological Cauchy net **.**

Theorem 2.7 : - Every Cauchy net in a convex bornological vector space E which has a bornologically convergent subnet , it is bornologically convergent.

Proof:- Let (x_γ) be a Cauchy net in a bornological vector space E and let $(x_{k_{\gamma'}})$ be a subnet of (x_γ) converges to $x \in E$, then there exists a bounded and absolutely convex set $B_1 \subseteq E$ and a net $(\mu_{\gamma'})$ of scalars tends to 0 such that

$x_{k_{\gamma'}} - x \in \mu_{\kappa_{\gamma'}} B_1$ for every integer $k_{\gamma'} \in \Gamma$. Since (x_γ) is a Cauchy net . By definition 2.4, there exists a bounded and absolutely convex set $B_2 \subseteq E$ and a null net of positive scalars (μ_γ) such that:-

$x_\gamma - x_{k_{\gamma'}} \in \mu_{\kappa_{\gamma'}} B_2$

for every $k_{\gamma'}, \gamma \in \Gamma$. with $\gamma \geq k_{\gamma'}$ let $B = B_1 \bigcup B_2$,

then $x_\gamma - x - (x_{k_{\gamma'}} - x) \in \mu_{k_{\gamma'}} B$ for every $k_{\gamma'}, \gamma \in \Gamma$ with $\gamma \geq k_{\gamma'}$,

Since $x_{k_{\gamma'}} - x \to 0$ then $x_\gamma - x \in \mu_{k_{\gamma'}} B$ for every $k_{\gamma'}, \gamma \in \Gamma$ with $\gamma \geq k_{\gamma'}$,

then $x_\gamma - x \in \mu_\gamma B$ for every $\gamma \in \Gamma$.

i.e. (x_γ) converges bornologically to a point $x \in E$.

3. Completeness

In this section the fundamental bornological complete concepts such as product, quotient and direct sum of complete bornology have been studied on the basis of converge net and Cauchy net in convex bornological vector space.

Definition 3.1 : A separated *cbvs* is called a complete bornological vector space if every bornological Cauchy net in E converges in E.

Theorem 3.2 :-Let E be a separated *cbvs* and let F be a bornological subspace of E. Then:-

(i) If F is complete, it is b-closed in E;

(ii) If E is complete and F is b-closed, then F is complete.

Proof: (i) Let (x_γ) be a net in F which converges bornologically to $x \in E$; by Proposition 2.6 (x_γ) is a Cauchy net in F. Since F is complete then (x_γ) converges bornologically to y in F but, E is a separated bornological vector space, then x=y implies $x_\gamma \longrightarrow x$ in F, then F is b-closed in E.

(ii) Let (x_γ) is a Cauchy net in F; Since E is complete, then (x_γ) converge bornologically to x in E.

Since F is b-closed, then (x_γ) converges bornologically to x in F.

then F is complete.

Theorem 3.3 :- If E is a complete bornological vector space and F is b-closed subspace of E, then the quotient E/F is complete.

Proof: Let $\mathcal{B}_0$ be a base for the bornology of E. If $\theta: E \to E/F$ is the canonical map, then $\theta(\mathcal{B}_0)$ is a base for the bornology of E/F, Since F is b-closed subspace of E, then by theorem 1.19 the quotient E/F is separated, if E is a complete bornological vector space then every Cauchy net in E converges in E by Definition 3.1. If $\theta: E \to E/F$ is the canonical map, then it is bounded linear map. Thus every bornologically convergent Cauchy net (x_γ) in E, $\theta(x_\gamma)$, $\gamma \in \Gamma$ is converge bornologically Cauchy net in E/F. Whence E/F is complete.

Theorem 3.4 :- Every product of any family of complete bornological spaces is complete.

Proof:- let E_i, $i \in \Gamma \neq \emptyset$ be a family of complete separated bornological vector space and let $E = \prod_{i\in\Gamma} E_i$ be the product of E_i. If (x_γ) is a Cauchy net in E. Then the canonical projection $p_i: E \to E_i$ of a Cauchy net is Cauchy (x^i_γ) in E_i for every $i \in \Gamma$. Since every E_i is complete and it is clear that separated bornological vector space for every $i \in \Gamma$, then (x_γ^i) converges bornologically to a unique point x_i in E_i for every $i \in I$, and let $x=(x_i) \in E$, $i \in \Gamma$, then the net (x_γ) converges bornologically to $x \in E$. Then E is a complete bornological vector space.

Proposition 3.5:- Let $(E_i, u_{ji})_{i\in\Gamma}$ be an inductive system of complete bornological vector space, i.e. E_i is complete for every $i \in \Gamma$ and let $E = \underset{i\in\Gamma}{\overset{\lim}{\rightarrow}} E_i$. Then E is complete if and only if E is separated.

Proof:- Since every complete space is sparated, only the sufficiency needs proving. Assume, then, E to be separated and let u_i be the canonical embedding of E_i in to E. Let $(x_\gamma^i)_{\gamma\in\Gamma}$ be a Cauchy net in E_i, whenever $i \in \Gamma$. Since E_i is a complete bornological vector space then (x_γ^i) converges bornologically to a point $x \in E_i$; then $u_i(x_\gamma^i)$ converges bornologically Cauchy net in E whenever $i \in \Gamma$.

Since E is separated then $u_i(x_\gamma^i)$ has a unique limit then E is complete.

Corollary 3.6 :- Let $(E_i, u_{ji})_{i\in\Gamma}$ be an inductive system of complete bornological space. Let $E = \underset{i\in\Gamma}{\overset{\lim}{\rightarrow}} E_i$. If the maps u_{ji} are injective, then E is complete.

Proof:- Since u_{ji} are injective by proposition 1.13 E is sparated then by proposition 3.5 E is complete.

Corollary 3.7 :- Every bornological direct sum of any family of complete bornological spaces is complete.

Proof:- Let (E_i) be a family of complete bornological space and let

$E = \underset{i \in I}{\oplus} E_i$ be their bornological direct sum. denote I, the set of finite subsets of I, directed under inclusion.

For every J∈I, $E_J = \underset{i \in I}{\oplus} E_i$. the space E_J is bornologically isomorphic to the product $\prod_{i \in I} E_i$. Whence is complete theorem 3.4

If $J \subset J'$, denote by $u_{JJ}{'}$ the canonical embedding of E_J into $E_{J'}$. Then E is the bornological inductive limit of the spaces E_J and the assertion follows from Corollary 3.6 .

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حول فضاء المتجهات البرنولوجي الكامل

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قسم الرياضيات

المستخلص:-

بنية فضاء المتجهات البرنولوجي الكامل تمت دراستها بالاضافة الى بعض العلاقات والمفاهيم المرتبطة بها مثل فضاء القسمة الكامل , فضاء الجداء الكامل و فضاء الجمع المباشر الكامل في فضاء المتجهات البرنولوجي. تضمن البحث ايضا دراسة شبكة كوشي بعد وضع تقارب الشبكة في فضاء المتجهات البرنولوجي المحدب.