

## **Kernels of Hesitant Soft Relations and Functions**

### **نويات الدوال والعلاقات الناعمة المترددة**

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#### **Abstract**

The main purpose of this paper is to study some properties of kernels of hesitant soft relations and functions. We have used measurable simple function as membership value in the soft set theory introduced by Molodtsov in 1999. This paper contains some basic definitions on hesitant soft sets and relations introduced by D. Rout and T.Som[2]. Furthermore, we introduce and discuss kernels of hesitant soft relations and also hesitant soft functions with related some results.

#### **المستخلص**

الغرض الرئيسي من هذا البحث هو دراسة بعض خصائص نويات الدوال والعلاقات الناعمة المترددة. استخدمنا دالة قياسية بسيطة كقيمة لعنصر في نظرية المجموعة الناعمة التي قدمها Molodtsov في سنة 1999. هذا البحث يتضمن بعض التعاريف الأساسية والعلاقات للمجموعات الناعمة المترددة التي عرضت في [2]. علاوة على ذلك، قدمنا وناقشنا نويات العلاقات الناعمة المترددة وكذلك الدوال الناعمة المترددة وبعض النتائج المتعلقة بها.

#### **1. Introduction**

In 1999, Molodtsov introduced the concept of soft sets to solve complicated problems and various types of uncertainties. He introduced the concept that a soft set is an approximate description of an object precisely consisting of two parts, namely predicate and approximate value set. In, Maji et al. [1] introduced several operators for soft set theory: equality of two soft sets, subset and superset of a soft set, complement of a soft set, null soft set, and absolute soft set. Recently, soft set theory has been developed rapidly by some scholars in theory and practice. The latest development to this area would be the introduction of "Hesitant fuzzy sets".

Hesitant fuzzy set, proposed by Torra [3, 4], is characterized by the membership degree of an element to a set presented as several possible values between 0 and 1, and it is a powerful technique in dealing with hesitancy and uncertainty in real a application .Function is a fundamental mathematical concept which is used in many fundamental areas of science and mathematics and has numerous applications.

This paper attempts at creating a theoretical framework in hesitant soft Relations. The rest of this paper is organized as follows. In section two, discusses some preliminaries on hesitant soft relations introduced by D. Rout and T. Som[2]. In section three we gives the concepts of kernels of hesitant soft relations with related results. The last section focuses on hesitant soft functions with theorems.

#### **2. Preliminaries and Basic Definitions**

In this section we will collect the basic definitions and notations as introduced by D. Rout and T. Som [2]

##### **Definition 2.1:** [2]

Let  $X$  be the reference set and  $F[0,1]$  by fuzzy power set then we define **hesitant soft set** by  $h: X \rightarrow F[0,1]$  such that  $h(a) = h_a$  where  $h_a$  is measurable simple function on  $[0,1]$ .

**Empty set:**  $h_a(x) = 0(x) \quad \forall x \in X.$

**Full set:**  $X(x) = 1(x) \quad \forall x \in X.$

Now onwards hesitant soft set will be denoted as HSS.

**Definition 2.2: [2]**

The Score for a HSS is given by  $s(h(a)) = \int h_a(x)dx$ , where  $a$  is a point of  $X$ .

**Definition 2.3: [2]**

Let  $h_1$  and  $h_2$  be two HSS's on  $X$ . Then we say that  $h_1$  is a **subset** of  $h_2$

i.e.,  $h_1 \subseteq h_2$  if and only if  $h_1(x) \subseteq h_2(x) \forall x \in X$

and

$h_1 = h_2$  if and only if  $h_1 \subseteq h_2$  and  $h_2 \subseteq h_1$ .

**Definition 2.4: [2]**

Let  $h_1$  and  $h_2$  be two HSS's on  $X$ . Then we say that  $h_1$  is a **proper subset** of  $h_2$  if  $h_1(x) \subseteq h_2(x) \forall x \in X$  and  $h_2(x) \neq h_1(x)$  for some  $x \in X$

i.e.,  $h_1(x) \subseteq h_2(x) \forall x \in X$  and  $h_2(x) \subset h_1(x)$  for some  $x \in X$ .

**Definition 2.5: [3]**

Let  $h_A$  and  $h_B$  be two HSS's on  $X$ . Then we say that  $h_A$  is a **hesitant equality** of  $h_B$

( $h_A \approx h_B$ ) if  $s(h_A(x)) = s(h_B(x)) \forall x \in X$ .

**Definition 2.6: [22]**

Let  $h_A$  and  $h_B$  be two hesitant soft set on  $X$ , then we say that  $h_A$  is a **hesitant subset** of  $h_B$

(denoted by  $h_A \preceq h_B$ ) if and only if  $s(h_A(x)) \leq s(h_B(x)) \forall x \in X$ .

**Definition 2.7: [3]**

Let  $h_A$  and  $h_B$  be two hesitant soft set on  $X$ , then we say that  $h_A$  is a **hesitant proper subset** of  $h_B$  (denoted by  $h_A < h_B$ ) if  $s(h_A(x)) \leq s(h_B(x)) \forall x \in X$  and  $s(h_A(x)) < s(h_B(x))$  for at least one  $x \in X$ .

**Note: [2]**

The usual or crisp subset notation defined above becomes a special case of the hesitant subset case, however

$h_A \subseteq h_B$ , then  $h_A \preceq h_B$ , but  $h_A \preceq h_B$  is not implies that  $h_A \subseteq h_B$

**Definition 2.8 (Complement): [3]**

Let  $h$  be the HSS in  $X$  and  $x \in X$  be arbitrary point of  $X$ . Then  $h^c(x) = \sum \Psi_{A_i} (1 - \alpha_i)$  where  $h(x) = \sum \Psi_{A_i} \alpha_i$  and  $\Psi_{A_i}$  is a characteristic function

**Proposition 2.9 : [2]**

Let  $h$  be a HSS on  $X$ .  $h_X$  denotes the full set and  $h_0$  denotes the empty set.

- $(h \cup h_X)(x) = h_X(x)$
- $(h \cup h_0)(x) = h(x)$
- $(h \cup h^c)(x) \neq h_X(x)$  and  $(h \cap h^c)(x) \neq h_0(x)$

Similarly,

- $(h \cup h_0)(x) = h_0(x)$  and  $(h \cap h_0)(x) = h_0(x)$
- **Involution:**  $(h^c)^c = h$

Clearly  $1 - (1 - \gamma) = \gamma \quad \forall \gamma \in h(x)$

- **Commutative:**  $h_1 \cup h_2 = h_2 \cup h_1$

$$h_1 \cap h_2 = h_2 \cap h_1$$

- **Associativity:**  $(h_1 \cup h_2) \cup h_3 = h_1 \cup (h_2 \cup h_3)$   
 $(h_1 \cap h_2) \cap h_3 = h_1 \cap (h_2 \cap h_3)$

- **Distributive:**  $h_1 \cap (h_2 \cup h_3) = (h_1 \cap h_2) \cup (h_1 \cap h_3)$

**Definition 2.10: [2]**

A Hesitant soft subset  $\mathcal{R}$  of  $X \times Y$  is called a **Hesitant soft relation**  $\mathcal{R}$  from  $X$  to  $Y$ . i.e.  $\mathcal{R}: X \times Y \rightarrow F[0,1]$

Note:  $HS(X, Y)$  denotes the family of all Hesitant soft relations from  $X$  to  $Y$ .

**Definition 2.11:** [2]

*Identity Relation* on  $X$ , i.e.

$$I: X \times X \rightarrow F[0,1]$$

$$I(x,y) = \begin{cases} 1(x) & \text{if } x = y \\ 0(x) & \text{if } x \neq y \end{cases}$$

**Definition 2.12:** [2]

Let  $\mathcal{R}$  be a Hesitant soft relations on  $X$ . Then the **inverse** of  $\mathcal{R}$  can be defined as:  $\mathcal{R} \in HS(X,Y)$  then  $\mathcal{R} \in HS(Y,X)$  such that  $\mathcal{R}^{-1}(x,y) = \mathcal{R}(y,x)$

**Definition 2.13:** [2]

Let  $\mathcal{R}$  be a Hesitant soft relation on  $X$  then we say  $\mathcal{R}$  is

- (1) **Reflexive** if  $\mathcal{R}(x,x) = 1(x)$
- (2) **Symmetric** if  $\mathcal{R}(x,y) = \mathcal{R}^{-1}(x,y)$ .

### 3. Kernels of Hesitant Soft Relations

In this section, we will introduce the notions of reflexive kernel, symmetric kernel and transitive kernel of a hesitant soft relation, and study their properties.

**Definition 3.1:**

Let  $\mathcal{R}$  and  $\mathcal{P}$  are a Hesitant soft relations on  $X$  to  $Y$ . Then

- a) The union of two Hesitant soft relations  $\mathcal{R}$  and  $\mathcal{P}$  on  $X$  to  $Y$ , denoted  $\mathcal{R} \cup \mathcal{P}$ , is defined by
 
$$\mathcal{R} \cup \mathcal{P} = \{h(a) \times h(b): h(a) \times h(b) \in \mathcal{R} \text{ or } h(a) \times h(b) \in \mathcal{P}\}$$
- b) The intersection of two Hesitant soft relations  $\mathcal{R}$  and  $\mathcal{P}$  on  $X$  to  $Y$ , denoted by  $\mathcal{R} \cap \mathcal{P}$ , is defined

$$\mathcal{R} \cap \mathcal{P} = \{h(a) \times h(b): h(a) \times h(b) \in \mathcal{R} \text{ and } h(a) \times h(b) \in \mathcal{P}\}$$

- c)  $\mathcal{R} \leq \mathcal{P}$  if for any  $(a,b) \in X \times Y$ ,  $h(a) \times h(b) \in \mathcal{R}$ . Then  $h(a) \times h(b) \in \mathcal{P}$ .

**Definition 3.2:**

Let  $\mathcal{R}$  be a Hesitant soft relation from  $X$  to  $Y$  (i.e.  $\mathcal{R}: X \times Y \rightarrow F[0,1]$ ) and  $\mathcal{P}$  be a Hesitant soft relation from  $Y$  to  $Z$  (i.e.  $\mathcal{P}: Y \times Z \rightarrow F[0,1]$ ). Then the composition of  $\mathcal{R}$  and  $\mathcal{P}$ , denoted  $\mathcal{P} \circ \mathcal{R}$ , is a Hesitant soft relation from  $X$  to  $Z$  (i.e.  $\mathcal{P} \circ \mathcal{R}: X \times Z \rightarrow F[0,1]$ ) defined as follows:  $h(a) \times h(c) \in \mathcal{P} \circ \mathcal{R}$  if and only if

$$h(a) \times h(b) \in \mathcal{R} \text{ and } h(b) \times h(c) \in \mathcal{P}$$

**Definition 3.3:**

Let  $\mathcal{R}$  be a Hesitant soft relation on  $X$  then we say  $\mathcal{R}$  is **transitive** if  $\mathcal{R} \circ \mathcal{R} \subseteq \mathcal{R}$ .

$\mathcal{R}$  is an equivalence Hesitant soft relation if it is reflexive, symmetric and transitive.

**Example:**

Consider Hesitant soft relation  $h$  over a universe  $U$  where  $U = \{c_1, c_2, c_3, c_4, c_5\}$ ,

$A = \{w_1, w_2\}$ ,  $h(w_1) = \{c_1, c_2, c_3\}$ ,  $h(w_2) = \{c_4, c_5\}$ .

Let  $\mathcal{R}$  and  $\mathcal{P}$  be two Hesitant soft relations on  $X$  given by  $\mathcal{R} = \{h(w_1) \times h(w_1), h(w_2) \times h(w_1)\}$ ,

$\mathcal{P} = \{h(w_1) \times h(w_1), h(w_2) \times h(w_2)\}$ .

Then  $\mathcal{R} \cup \mathcal{P} = \{h(w_1) \times h(w_1), h(w_2) \times h(w_1), h(w_2) \times h(w_2)\}$

$\mathcal{R} \cap \mathcal{P} = \{h(w_1) \times h(w_1)\}$ .

Let  $\mathcal{Q}$  be Hesitant soft relation  $h$  defined as follows:

$\{h(w_1) \times h(w_1), h(w_2) \times h(w_1), h(w_2) \times h(w_2)\}$ . Then  $\mathcal{R} \leq \mathcal{Q}$ .

$\mathcal{P} \circ \mathcal{R} = \{h(w_1) \times h(w_1), h(w_2) \times h(w_1)\}$ .

Consider a relation  $\mathcal{R}_1$  defined on  $h$  as

$\{h(w_1) \times h(w_1), h(w_2) \times h(w_2), h(w_1) \times h(w_2), h(w_2) \times h(w_1)\}$ . This relation is an equivalence Hesitant soft relation.

**Theorem 3.4:**

Let  $\mathcal{R}, \mathcal{P}$  and  $\mathcal{Q}$  be three Hesitant soft relations on  $X$ . Then

- (1)  $(\mathcal{R}^{-1})^{-1} = \mathcal{R}, (\mathcal{R}^c)^c = \mathcal{R}$ .
- (2)  $\mathcal{R} \cap \mathcal{Q} \subset \mathcal{R}, \mathcal{R} \cap \mathcal{Q} \subset \mathcal{Q}$ .
- (3)  $\mathcal{R} \subset \mathcal{Q} \Rightarrow \mathcal{R}^{-1} \subset \mathcal{Q}^{-1}$ .
- (4) If  $\mathcal{P} \supset \mathcal{Q}$  and  $\mathcal{P} \supset \mathcal{R}$ , then  $\mathcal{P} \supset \mathcal{R} \cup \mathcal{Q}$ .
- (5) If  $\mathcal{P} \subset \mathcal{Q}$  and  $\mathcal{P} \subset \mathcal{R}$ , then  $\mathcal{P} \subset \mathcal{R} \cap \mathcal{Q}$ .
- (6)  $(\mathcal{R} \cup \mathcal{Q})^{-1} = \mathcal{R}^{-1} \cup \mathcal{Q}^{-1}, (\mathcal{R} \cap \mathcal{Q})^{-1} = \mathcal{R}^{-1} \cap \mathcal{Q}^{-1}$ .

**Proof:**

- (1)  $h(a) \times h(b) \in \mathcal{R} \Rightarrow h(b) \times h(a) \in \mathcal{R}^{-1} \Rightarrow h(a) \times h(b) \in (\mathcal{R}^{-1})^{-1}$ .

Then  $(\mathcal{R}^{-1})^{-1} = \mathcal{R}$ .

$h(a) \in \mathcal{R} \Rightarrow h^c(a) \in \mathcal{R}^c \Rightarrow (1 - \gamma)(a) = 1 - h(a) \in \mathcal{R}^c$  for all  $\gamma \in h$   
 $\Rightarrow [1 - (1 - \gamma)](a) = h(a) \in (\mathcal{R}^c)^c = \mathcal{R}$ .

- (2) Let  $h(a) \times h(b) \in \mathcal{R} \cap \mathcal{Q} \Leftrightarrow h(a) \times h(b) \in \mathcal{R}$  and  $h(a) \times h(b) \in \mathcal{Q}$   
 $\Leftrightarrow h(a) \times h(b) \in \mathcal{R}$ .

Hence  $\mathcal{R} \cap \mathcal{Q} \subset \mathcal{R}$ . The proof of  $\mathcal{R} \cap \mathcal{Q} \subset \mathcal{Q}$  is similar.

- (3) Let  $\mathcal{R} \subset \mathcal{Q}$  and let  $h(a) \times h(b) \in \mathcal{R}^{-1} \Rightarrow h(b) \times h(a) \in \mathcal{R}$ , since  $\mathcal{R} \subset \mathcal{Q}$   
 $\Rightarrow h(b) \times h(a) \in \mathcal{Q} \Rightarrow h(a) \times h(b) \in \mathcal{Q}^{-1}$ .

Therefore  $\mathcal{R}^{-1} \subset \mathcal{Q}^{-1}$ .

- (4) and (5) hold.

- (6)  $h(a) \times h(b) \in (\mathcal{R} \cup \mathcal{Q})^{-1} \Leftrightarrow h(b) \times h(a) \in \mathcal{R} \cup \mathcal{Q}$   
 $\Leftrightarrow h(b) \times h(a) \in \mathcal{R}$  or  $h(b) \times h(a) \in \mathcal{Q}$   
 $\Leftrightarrow h(b) \times h(a) \in \mathcal{R}^{-1}$  or  $h(b) \times h(a) \in \mathcal{Q}^{-1}$

$\Leftrightarrow h(a) \times h(b) \in \mathcal{R}^{-1} \cup \mathcal{Q}^{-1}$ .

The proof of  $(\mathcal{R} \cap \mathcal{Q})^{-1} = \mathcal{R}^{-1} \cap \mathcal{Q}^{-1}$  is similar.

**Proposition 3.5:**

Let  $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{P}_1$  and  $\mathcal{P}_2$  be three Hesitant soft relations on  $X$ . Then

- (1) If  $\mathcal{R}_1 \subseteq \mathcal{P}_1$  and  $\mathcal{R}_2 \subseteq \mathcal{P}_2$ , then  $\mathcal{R}_1 \circ \mathcal{R}_2 \subseteq \mathcal{P}_1 \circ \mathcal{P}_2$ .
- (2)  $\mathcal{R}_1 \circ (\mathcal{R}_2 \cup \mathcal{R}_3) = (\mathcal{R}_1 \circ \mathcal{R}_2) \cup (\mathcal{R}_1 \circ \mathcal{R}_3), \mathcal{R}_1 \circ (\mathcal{R}_2 \cap \mathcal{R}_3) \subseteq (\mathcal{R}_1 \circ \mathcal{R}_2) \cap (\mathcal{R}_1 \circ \mathcal{R}_3)$
- (3)  $(\mathcal{R}_1 \circ \mathcal{R}_2)^{-1} = \mathcal{R}_2^{-1} \circ \mathcal{R}_1^{-1}$

**Proof:**

- (1) Let  $h(a) \times h(b) \in \mathcal{R}_1 \circ \mathcal{R}_2 \Leftrightarrow h(a) \times h(c) \in \mathcal{R}_1$  and  $h(c) \times h(b) \in \mathcal{R}_2$  for some  $h(c) \in X$ , since  $\mathcal{R}_1 \subseteq \mathcal{P}_1$  and  $\mathcal{R}_2 \subseteq \mathcal{P}_2$ . Then  $h(a) \times h(c) \in \mathcal{P}_1$  and  $h(c) \times h(b) \in \mathcal{P}_2 \Leftrightarrow h(a) \times h(b) \in \mathcal{P}_1 \circ \mathcal{P}_2$ .

- (2) Clear

- (3)  $h(a) \times h(b) \in (\mathcal{R}_1 \circ \mathcal{R}_2)^{-1} \Leftrightarrow h(b) \times h(a) \in \mathcal{R}_1 \circ \mathcal{R}_2$   
 $\Leftrightarrow h(b) \times h(c) \in \mathcal{R}_1$  and  $h(c) \times h(a) \in \mathcal{R}_2$  for some  $h(c) \in X$   
 $\Leftrightarrow h(c) \times h(b) \in \mathcal{R}_1^{-1}$  and  $h(a) \times h(c) \in \mathcal{R}_2^{-1}$   
 $\Leftrightarrow h(a) \times h(b) \in \mathcal{R}_2^{-1} \circ \mathcal{R}_1^{-1}$ .

**Definition 3.6:**

Let  $\mathcal{R}$  be a hesitant soft relation on  $X$ .

- (1) The maximal reflexive Hesitant soft relation contained in  $\mathcal{R}$  is called **reflexive kernel** of  $\mathcal{R}$ , denoted by  $kr(\mathcal{R})$ .
- (2) The maximal symmetric Hesitant soft relation contained in  $\mathcal{R}$  is called **symmetric kernel** of  $\mathcal{R}$ , denoted by  $ks(\mathcal{R})$ .
- (3) The maximal transitive Hesitant soft relation contained in  $\mathcal{R}$  is called **transitive kernel** of  $\mathcal{R}$ , denoted by  $ts(\mathcal{R})$ .

**Theorem 3.7:**

Let  $\mathcal{R}$  be a Hesitant soft relation on X. Then

- (1)  $kr(\mathcal{R}) = \mathcal{R} \cap I^c$
- (2)  $ks(\mathcal{R}) = \mathcal{R} \cap \mathcal{R}^{-1}$ .
- (3)  $kt(\mathcal{R}) = \mathcal{R} \cap (\mathcal{R} \circ \mathcal{R})$ .

**Proof:**

(1) By Theorem 3.4(2),  $\mathcal{R} \cap I^c \subset \mathcal{R}$  and  $\mathcal{R} \cap I^c \subset I^c$ .

$\forall a \in X$ , by the definition of I,  $h(a) \times h(a) \in I$ , so  $h(a) \times h(a) \notin I^c$ . Hence  $h(a) \times h(a) \notin \mathcal{R} \cap I^c$ , i.e.  $\mathcal{R} \cap I^c$  is an reflexive hesitant soft relation on X

If  $\mathcal{P}$  is reflexive hesitant soft relation on X and  $\mathcal{P} \subset \mathcal{R}$  Then  $\mathcal{P} \subset I^c$ . Hence  $T \subset \mathcal{R} \cap I^c$ . So  $kr(\mathcal{R}) = \mathcal{R} \cap I^c$ .

(2) By Theorem 3.4 (6) and (1),

$$(\mathcal{R} \cap \mathcal{R}^{-1})^{-1} = \mathcal{R} \cap (\mathcal{R})^{-1} = \mathcal{R}^{-1} \cap \mathcal{R} = \mathcal{R} \cap \mathcal{R}^{-1}.$$

i.e.  $\mathcal{R} \cap \mathcal{R}^{-1}$  is a symmetric hesitant soft relation on X. According to Theorem 3.4 (2),  $\mathcal{R} \cap \mathcal{R}^{-1} \subset \mathcal{R}$ . On the other hand, if  $\mathcal{P}$  is a symmetric hesitant soft relation on X and  $\mathcal{P} \subset \mathcal{R}$ .

By Theorem 3.4(3),  $\mathcal{P}^{-1} \subset \mathcal{R}^{-1}$  Then by Theorem 3.4(5),  $\mathcal{P} = \mathcal{P}^{-1} \subset \mathcal{P} \cap \mathcal{P}^{-1}$ .

So  $ks(\mathcal{R}) = \mathcal{R} \cap \mathcal{R}^{-1}$ .

(3) By  $[\mathcal{R} \cap (\mathcal{R} \circ \mathcal{R})] \circ [\mathcal{R} \cap (\mathcal{R} \circ \mathcal{R})] = (\mathcal{R} \cap \mathcal{R}) \circ (\mathcal{R} \cap \mathcal{R}) \subset \mathcal{R} \cap (\mathcal{R} \circ \mathcal{R})$  is transitive hesitant soft relation. i.e.  $\mathcal{R} \cap (\mathcal{R} \circ \mathcal{R})$  is a transitive hesitant soft relation on X and  $\mathcal{R} \cap (\mathcal{R} \circ \mathcal{R}) \subset \mathcal{R}$ .

If  $\mathcal{P}$  is a transitive hesitant soft relation on X and  $\mathcal{R} \subset \mathcal{P}$ . Then  $\mathcal{R} \circ \mathcal{R} \subseteq \mathcal{P}$ . According to Theorem 3.4(4),  $\mathcal{P} \cap (\mathcal{R} \circ \mathcal{R}) = \mathcal{R}$ . So  $kt(\mathcal{R}) = \mathcal{R} \cap (\mathcal{R} \circ \mathcal{R})$ .

Let  $\mathcal{R}$  be Hesitant soft relationh on X given by  $\mathcal{R} = \{h(w_1) \times h(w_1), h(w_2) \times h(w_1)\}$ . By using above theorem we have  $kr(\mathcal{R}) = \{h(w_2) \times h(w_1)\}$ ,  $ks(\mathcal{R}) = \{h(w_1) \times h(w_1)\}$  and  $kt(\mathcal{R}) = \{h(w_1) \times h(w_1), h(w_2) \times h(w_1)\}$ .

**4. Hesitant Soft Functions**

In this section, we will study and debate some results that related hesitant soft functions.

**Definition 4.1:**

Let  $h_A$  and  $h_B$  are two non-empty hesitant soft sets. Then a hesitant soft set relation from X is called a hesitant soft function  $f: (X, h_A) \rightarrow F[0,1]$  if every element in domain has a unique element in the range. Then we write  $f(h(a)) = F_a$ .

**Definition 4.2:**

A function is called injective ( one to one) if  $h(a) \neq h(b)$  implies  $f(h(a)) \neq f(h(b))$ .

**Definition 4.3:**

A function is called surjective (onto) if  $\text{ran } f = F_a$ .

**Definition 4.4:**

A function which is both injective and surjective is called a bijective function.

**Example:**

Let  $U = \{s_1, s_2, s_3, s_4\}$ ,  $A = \{a_1, a_2\}$ ,  $B = \{b_1, b_2\}$ , suppose a hesitant soft set A and B defined as follows  $h_A(a_1) = \{s_1, s_2\}$ ,  $h_A(a_2) = \{s_3, s_4\}$  and  $h_B(b_1) = \{s_1, s_2, s_4\}$ ,  $h_B(b_2) = \{s_2, s_3\}$ . Then a hesitant soft function f from  $(X, h_A)$  to  $(X, h_B)$  defined as follows  $\{h_A(a_1) \times h_B(b_1), h_A(a_2) \times h_B(b_2)\}$  is a bijective function.

**Theorem 4.5:**

Let  $f: (X, h_A) \rightarrow F[0,1]$  be a hesitant soft function and  $h_{A_1}, h_{A_2}$  be hesitant soft subsets of X. Then

- 1)  $h_{A_1} \preceq h_{A_2}$ , then  $f(h_{A_1}) \preceq f(h_{A_2})$
- 2)  $f(h_{A_1} \cup h_{A_2}) = f(h_{A_1}) \cup f(h_{A_2})$
- 3)  $f(h_{A_1} \cap h_{A_2}) \preceq f(h_{A_1}) \cap f(h_{A_2})$ , equality holds if f is one to one.

**Proof:**

1) Suppose  $h_{A_1} \preceq h_{A_2}$  and let  $P(b) \in f(h_{A_1})$ . Then  $P(b) = f(h(a))$  for some  $h(a) \in h_{A_1}$  and  $h_{A_1} \preceq h_{A_2}$ , this is  $P(b) = f(h(a))$  for some  $h(a) \in h_{A_2}$ .

Therefore,  $P(b) \in f(h_{A_2})$ .

2) Let  $P(b) \in f(h_{A_1} \cup h_{A_2})$ . Hence  $P(b) = f(h(a))$  for some  $h(a) \in (h_{A_1} \cup h_{A_2})$ , it follows that  $f(h(a))$  for  $h(a) \in h_{A_1}$  or  $h(a) \in h_{A_2}$ . Thus  $P(b) \in f(h_{A_1})$  or  $P(b) \in f(h_{A_2})$ , i.e.  $P(b) \in f(h_{A_1}) \cup f(h_{A_2})$ . Then  $f(h_{A_1} \cup h_{A_2}) \subseteq f(h_{A_1}) \cup f(h_{A_2})$ .

Now,  $h_{A_1} \preceq h_{A_1} \cup h_{A_2}$  and  $h_{A_2} \preceq h_{A_1} \cup h_{A_2}$ , so  $f(h_{A_1}) \preceq f(h_{A_1} \cup h_{A_2})$  and  $f(h_{A_2}) \preceq f(h_{A_1} \cup h_{A_2})$ . Then  $f(h_{A_1}) \cup f(h_{A_2}) \preceq f(h_{A_1} \cup h_{A_2})$ , i.e.  $f(h_{A_1} \cup h_{A_2}) = f(h_{A_1}) \cup f(h_{A_2})$ .

3) Let  $P(b) \in f(h_{A_1} \cap h_{A_2})$ . Hence  $P(b) = f(h(a))$  for some  $h(a) \in (h_{A_1} \cap h_{A_2})$ , it follows that  $P(b) = f(h(a))$  for  $h(a) \in h_{A_1}$  and  $h(a) \in h_{A_2}$ . Thus  $P(b) \in f(h_{A_1})$  and  $P(b) \in f(h_{A_2})$ , i.e.  $P(b) \in f(h_{A_1}) \cap f(h_{A_2})$ . Then  $f(h_{A_1} \cap h_{A_2}) \preceq f(h_{A_1}) \cap f(h_{A_2})$ . Conversely, Let  $P(b) \in f(h_{A_1}) \cap f(h_{A_2})$ , then  $P(b) \in f(h_{A_1})$  and  $P(b) \in f(h_{A_2})$ .  $P(b) = f(h(a_1))$  for some  $h(a_1) \in f(h_{A_1})$  and  $P(b) = f(h(a_2))$  for some  $h(a_2) \in f(h_{A_2})$ . Now,  $f(h(a_1)) = f(h(a_2)) = P(b)$ . Since  $f$  is one to one, then  $h(a_1) = h(a_2)$ . It follows that  $P(b) \in f(h_{A_1}) \cap f(h_{A_2})$ , i.e.  $f(h_{A_1} \cap h_{A_2}) = f(h_{A_1}) \cap f(h_{A_2})$ , holds if  $f$  is one to one.

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