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Some properties of two − fuzzy pre − Hilbert space

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ABSTRACT: We introduce the definition of a two-fuzzy pre Hilbert space (two-fuzzy inner product space) and discuss some properties of this spaces ,and we use the definition of two-fuzzy pre Hilbert space to introduce the definitions of (level complete in two-fuzzy Hilbert , fuzzy orthogonality) . Also, introduce some important theorems with their proofs. Moreover, crisp inner product and orthogonality are established .

Keywords: fuzzy set, Two-fuzzy pre-Hilbert space, α-norm,

Mathematics Subject Classification : 30C45

1. INTRODUCTION

The concept of fuzzy set was introduced by Zadeh [2] in 1965 as an extension of the classical notion of set. A satisfactory theory of 2-norm on a linear space has been introduced and developed by Gahler in [4]. The concept of fuzzy norm and α-norm were introduced by Bag and Samanta and the notions of convergent and Cauchy sequences were also discussed in [6]. Zhang [1] has defined fuzzy linear space in a different way. RM. Somasundaram and ThangarajBeaula defined the notion of fuzzy 2-normed linear space $(F(X)\,,N)$ or 2- fuzzy ,2-normed linear space. Some standard results in fuzzy 2-normed linear spaces were extended . The famous closed graph theorem and Riesz Theorem were also established in 2-fuzzy 2-normed linear space. In [5] , we have introduced the new concept of 2-fuzzy inner product space on $F(X)$, the set of all fuzzy sets of X. In this paper studysome properties of two-pere-fuzzy Hilbert spaces and results are discussed about the two concepts.

2. Preliminaries

Definition 2.1.[3]. Let $F(X)$ be a vector space over the complex field C. The fuzzy subset η defined as a mapping from $F(X) \times F(X) \times C \longrightarrow [0,1]$ such that for all $f,g,h \in F(X),\ \alpha \in C$

(1) $For\ s,t \in C, \eta(f+g,h,|t|+|s|) \geq min\{\eta(f,h,|t|),\eta(g,h,|s|)\}$

(2) $For\ s,t \in C, \eta(f,g,|st|) \geq min\{\eta(f,f,|s|^2),\eta(g,g,|t|^2)\}$

(3) $For\ t \in C,\ \eta(f,g,t) = \eta(g,f,t)$

(4) $\eta(\alpha f, g, t) = \eta(f, g, \frac{1}{|\alpha|}), \alpha(\neq 0) \in C, t \in C.$

(5) $\eta(f, f, t) = 0, for\ all\ t \in C/\mathbb{R}^+$

(6) $\eta(f, f, t) = 1\ for\ all\ t > 0\ if\ and\ only\ if\ f = 0$

(7) $\eta(f, f,): \mathbb{R} \rightarrow I(= [0,1])$ is a monotonic non- decreasing function of R and $lim\, \eta(f, f, t) = 1\ as\ t \rightarrow \infty$

Then η is said to be two - fuzzy pre-Hilbert (2 - FPHS) on $F(X)$ and the pair $(F(X), \eta)$is called a two-fuzzy pre-Hilbert space (2-FPHS).

Definition 2.2. [5]. Let$(F(X), \eta)$ be a 2-FPHS satisfying the condition $\{\eta(f, f, t^2) > 0, when\ t > 0\}$implies that $= 0$. Then for all$\alpha \in (0,1)$, define $\|f\|_\alpha = inf\{t : \eta(f, f, t^2) \geq \alpha\}$ a crisp norm on $F(X)$, called the a-norm on $F\ (X)$ generated by η. Now using these definitions let us define fuzzy norm on F(X) and verify the conditions as follows:

Theorem 2.3.[5]. Let η be a two-fuzzy inner product on F(X). Then $N : F(X) \times \mathbb{R} \rightarrow [0,1]$ defined by

$$N(f, t) = \begin{cases} \eta(f, f, t^2) & when\ t \in \mathbb{R}, t > 0 \\ 0 & when\ t \in \mathbb{R}\ , t < 0 \end{cases}$$

is a fuzzy norm on $F(X)$.

(N1) By define$N(f\ , t) = 0\ , for\ all\ t \in \mathbb{R}\ and\ t \leq 0$.

Proof: (N2) Again from (6), for all $t > 0$, $\eta(f, f, t^2) = 1\ if\ and\ only\ if\ f = 0$ therefore it follows that $N(f, t) = 1\ if\ and\ only\ if\ f = 0$.

(N3) $For\ all\ t > 0\ and\ c \neq 0$,

$$N(cf, t) = \eta(cf, cf, t^2) = \eta(f, cf, \frac{t^2}{|c|}) = \eta(f, f, \frac{t^2}{|c|^2}) = N(f, \frac{t}{|c|}) .$$

(N4) To prove that $N(f + g, s + t) \geq min\{N(f, s), N(g, t)\}$ for every $s, t \in \mathbb{R}\ ,\ f, g \in F(X)$, let us consider the following cases:

(a) $s + t < 0$,

(b) $s = t = 0\ ; s > 0,\ t < 0\ or\ s < 0\ ,\ t > 0$,

(c) $s + t > 0\ ; s, t \geq 0$.

Let us prove (c).

$N(f + g\ , s + t) = \eta(f + g, f + g, (s + t)^2)$

$= \eta(f + g, f + g\ , s^2 + st + st + t^2)$

$\geq \eta(f, f, s^2) \wedge \eta(g, g, t^2) \wedge \eta(f, g, st)$

$\geq \eta(f, f, s^2) \wedge \eta(g, g, t^2)$

$= N(f, s) + N(g\ , t)$

(a) and (b) follows immediately.

(N5) From (7) $\eta(f\ , f, .)$ is a monotonic non-decreasing function and tends to 1 as$t \rightarrow \infty$ to. Thus $N(f, .)$ is a monotonic non-decreasing function and tends to 1 as $t \rightarrow \infty$to.

Theorem 2.4.[5]. (Parallelogram Law) Let n be a two- fuzzy inner product on $F(X)$, $\alpha \in (0,1)\ and\ \|\ \ \|_\alpha$ be the α-norm generated from 2-FIP η on $F(X)$. Then

$\|f - g\|_\alpha^2 + \|f + g\|_\alpha^2 = 2(\|f\|_\alpha^2 + \|\ g\|_\alpha^2\ )$

Theorem 2.5.[5] If a two - fuzzy pre-Hilbert space $(F(X),\eta)$ is strictly convex, and if $\eta(f,g,t) = \|f\|_\alpha\|g\|_\alpha$ then f and g are linearly dependent

Theorem 2.6.[5]. Let $(F(X),\eta)$ be a two-fuzzy pre-Hilbert space. If $\eta(f,h,t) = n(g,h,t)$ for all $h \in F(X)$ then f and g are dependent.

Definition 2.7.[5].A sequence $\{f_n\}$ in a fuzzy two-normed linear space $(F(X),N)$ is called a Cauchy sequence with respect to a-norm if $lim\,\|f_n - f_m\| = 0$as $n,m \to \infty$.

Definition 2.8. [5]. A sequence $\{f_n\}$ in a fuzzy two-normed linear called a convergent sequence with respect to α-norm if there exists $f \in F(X)$ such that if $lim\,\|f_n - f_m\| = 0$as $n,m \to \infty$

Definition 2.9.[5]. A fuzzy two-normed linear space $(F(X),N)$ is said to be complete if every Cauchy sequence converges.

Definition 2.10.[5] A complete fuzzy 2-normed linear space $(F(X),N)$ is called two-fuzzy Banach space.

Definition 2.11.[5] A complex 2-fuzzy Banach space $(F(X),N)$ is said to be 2- fuzzy Hilbert space if its norm is induced by the two-fuzzy inner product.

Theorem 2.12. [5] A closed convex two-fuzzy subset C of a two-fuzzy Hilbert space $F(X)$ contains a unique element in $F(X)$ with smallest a-norm.

Definition 2.13.[5] Let $F(X)$ be a vector space over the complex field C. Let η be a two-fuzzy inner product on $F\,(X)$. Let

$$N(f,t) = \begin{cases} \eta(f,f,t^2) & when\, t \in \mathbb{R}, t > 0 \\ 0 & when\, t \le 0 \end{cases}$$

be the fuzzy norm induced by the two-fuzzy inner product. Let

$\|f\|_\alpha = inf\,\{t > 0 : N\,(f,t) \ge \alpha\}$.

If $\|\quad\|_\alpha$ a satisfies parallelogram law then defineα-two-inner product as

$\langle f,g\rangle_\alpha = F_\alpha + iG_\alpha,$

$$where\, F_\alpha = \frac{1}{4}(\|f + g\|_\alpha^2 - \|f - g\|_\alpha^2) and$$

$$G_\alpha = \frac{1}{4}(\|f + ig\|_\alpha^2 - \|f - ig\|_\alpha^2), \alpha \in (0,1) \ldots\ldots\ldots\ldots (\#).$$

3.Main results:

In this section, some of the basic results related to this work are given .

Theorem 3.1. Every two-fuzzy pre-Hilbert space is a two-fuzzy normed space.

Proof: let $(F(X),\eta,*)$ be a fuzzy pre-Hilbert space. Define

$$N\,(f,t) = \begin{cases} \eta\,(f,f,t^2) & ,t > 0 \\ 0 & ,t \le 0 \end{cases} \quad For\ all\ f,g \in F\,(X), t \in R.$$

The axioms (1, 3, 4 ,5,6) in Definition (3.3.1) satisfied Now to prove (2)

$$N(f+g,t+s) = \eta(f+g,f+g,(t+s)^2)$$
$$= \eta(f+g\,,f+g,t^2+ts+ts+s^2)$$
$$\ge \eta(f+g,f+g,t^2) \wedge \eta(f+g\,,f+g,s^2) \wedge \eta(f+g\,,f+g,ts)$$
$$\ge \eta(f,f,t^2) \wedge \eta(g,g,s^2) = N(f,t) * N(g,s).$$

Therefore $(F(X),N,*)$ is a two- fuzzy normed space.

The relation between two-fuzzy metric space and two-fuzzy pre-Hilbert space is given in next theorem .

Theorem 3.2. Every two-fuzzy pre-Hilbert space is a two-fuzzy metric space.

Proof: Let $(X, F, *)$be a fuzzy pre-Hilbert space.

Defined $M(f,g,t) = \begin{cases} \eta(f-g, f-g, t^2) & ,t > 0 \\ 0 & ,t \le 0 \end{cases}$,

for all $f,g,h \in F(X)$ and $t,s \in R$.

(1) $M(f,g,t) = \eta(f-g, f-g, t^2) > 0$ for all $t > 0$

(2) $M(f,g,t) = 1 \Leftrightarrow M(f,g,t) = \eta(f-g, f-g, t^2) = 1$
$\Leftrightarrow f - g = 0 \Leftrightarrow f = g$ for all $t > 0$;

(3) $M(f,g,t) = \eta(f-g, f-g, t^2) = \eta(g-f, g-f, t^2) = M(g,f,t)$;

(4) $M(f,g,t) * M(g,h,s) = \eta(f-g, f-g, t^2) \wedge \eta(g-h, g-h, s^2)$
$\le \eta(f-g, f-g, t^2) \wedge \eta(g-h, g-h, s^2) \wedge \eta(f-g, f-g, ts)$
$= \eta(f-g, f-g, t^2 + s^2 + st) = \eta(f-g, f-g, (t+s)^2)$
$= M(f,h,t+s)$;

(5) $M(f,g,\bullet) = \eta(f-g, f-g, t^2): (0,\infty) \to [0,1]$ is continuous.

Therefore $(F(X), M, *)$is a two-fuzzy metric space.

The proof of the next theorem is straightforward it is omitted .

Theorem 3.3. Let $F(X)$ be a linear space over the complex field C. Let η be a two-fuzzy inner product on $F(X)$

(i) For $f,g,h \in F(X)$ and $t,s \in C, \eta(f, g+h, t+s) \ge \eta(f,g,t) \wedge \eta(f,h,s)$.

(ii) For $\lambda \in C$ and $\lambda \ne 0, \eta(f, \lambda g, t) = \eta(\lambda f, g, t)$.

(iii) $\forall t \in \mathbb{R}$ and $t > 0$, $\eta(0,0,t) = 1 \ge \eta(f,g,t)$, $\forall f,g \in F(X)$.

Remark 3.4 .

(i) let $f \in F(X)$, $\eta(f, f . t^2) > 0 \Rightarrow f = 0$

(ii) $\forall f, g \in F(X)$ and $p, q \in \mathbb{R}$,
$\eta(f+g, f+g, 2q^2) \wedge \eta(f-g, f-g, 2p^2) \ge \eta(f, f, p^2) \wedge \eta(g, g, q^2)$

3.5.Minimizing vector

Definition 3.5.1. Let $(F(X), \eta)$be a two-fuzzy pre-Hilbert satisfying (Remark3.4). F(X) is said to be level complete (l-complete) if for any $\alpha \in (0,1)$, every Cauchy sequence converges in F(X) w.r.t the$\alpha - norm, \| \quad \|_\alpha$, generated by the fuzzy norm N which is induced by two $-$ fuzzy inner product η .

Theorem 3.5.2. Let $(F(X), \eta)$ be a two- FPH space satisfying (Remark3.4) and $M(\ne \emptyset)$ be a convex subset of $F(X)$ which is level complete. Let $f \in F(X)$. Then for each $\alpha \in (0,1)$, $\exists$ a unique $g_0^\alpha \in M$such that $m_{g_0^\alpha}^{(\alpha)} = inf_{g \in M}\left\{m_g^{(\alpha)}\right\}$, where $m_g^{(\alpha)} = \wedge\{t \in \mathbb{R}^+, N(f,t) \ge \alpha\}$, N being the fuzzy norm induced by the two- FIP function η .

Proof :

Observe that for each$\alpha \in (0,1)$ and $g \in M$,

$m_g^{(\alpha)} = inf\{t \in \mathbb{R}^+ : N(f-g,t) \geq \alpha\} \|f-g\|_\alpha$ where $\| \ \|_\alpha$ the $\alpha - norm$ induced from the fuzzy norm N which is obtained from the two- fuzzy inner product η. By Definition 2-3-14, $(F(X), \langle , \rangle_\alpha)$ is an two-fuzzy pre-Hilbert space for each $\alpha \in (0,1)$, where $\langle , \rangle_\alpha$ is given by (#). Also M is level complete and convex. So for each $\alpha \in (0,1)$, M is a convex complete subset of $(F(X), \langle , \rangle_\alpha)$.

Hence by the minimizing vector theorem in crisp two- pre-Hilbert space we get the result.

3.6. Tow-Fuzzy Orthogonality

This section deals with the concept of two-fuzzy pre-Hilbert space and some of its properties.

Definition 3.6.1.[5] Let $(F(X), \eta)$ be two-fuzzy inner product space. If $f, g \in F(X)$ be such that $\langle f, g\rangle_\alpha = 0$, for all $\alpha \in (0,1)$ then f and g are two-fuzzy orthogonal to each other and is denoted by $f \perp_\alpha g$. With the help of this many more results can be established.

Definition 3.6.2. Let $\alpha \in (0,1)$ and $(F(X), \eta)$ be a $2-\mathrm{FPH}$ space satisfying (Remark 3.3.16). Now if $f, g \in F(X)$ be such that $\langle f, g\rangle_\alpha = 0$, then we say that *f, g* are $\alpha - \mathrm{fuzzy}$ orthogonal to each other and is denoted by $f \perp_\alpha g$. Let M be a subset of *V* and $x \in V$. Now if $\langle f, g\rangle_\alpha = 0 \ \forall g \in M$, then we say that f is $\alpha - \mathrm{fuzzy}$ orthogonal to M and is denoted by $f \perp_\alpha M$

Lemma 3.6.3.(orthogonality) Let $(F(X), \eta)$ be a $\mathrm{two} - \mathrm{FPH}$ space satisfying(Remark3.4).. Let $Y (\neq \emptyset)$ be a subspace of *F(X*) which is level complete and $f \in F(X)$ be fixed. Then for each $\alpha \in (0,1)$, $h^\alpha (= f - g_0^\alpha)$ is a-fuzzy orthogonal to Y, where $g_0^\alpha \in Y$ is such that $m_{g_0^\alpha}^{(\alpha)} = inf_{g \in M} \left\{m_g^{(\alpha)}\right\}$

Proof :

As $(F(X), \langle , \rangle_\alpha)$ is a crisp two- pre-Hilbert space and Y is closed w.r.t the $\alpha - norm, \| \ \|_\alpha$,, the result follows.

Definition 3.6.4. Let $(\mathrm{F(X)}, \eta)$ be a $2-\mathrm{FPH}$ space satisfying (Remark 3.4). Now if $f, g \in F(X)$ be such that $\langle f, g\rangle_\alpha = 0, \forall \alpha \in (0,1)$, then we say that f, g are fuzzy orthogonal to each other and is denoted by $f \perp g$.

Thus $f \perp g \ iff \ f \perp_\alpha g, \forall \alpha \in (0,1)$.

Theorem 3.6.5. Let $(F(X), \eta)$ be a two-FPH space satisfying(Remark3.4). such that $\eta(f, f, \bullet)$ is strictly increasing and lower semi continuous for any $f \in F(X)$. Then for $f, g \in F(X)$, $f \perp g \ iff \ \eta(f+g, f+g, t^2) = \eta(f-g, f-g, t^2)$
And $\eta(f+ig, f+ig, t^2) = \eta(f-ig, f-ig, t^2), \forall t > 0$.

Proof: The condition is necessary.

$Let\ f \perp g, then < f, g >_\alpha = F_\alpha + iG_\alpha = 0, \forall \alpha \in (0,1)$
$where F_\alpha = 4(\|f+g\|_\alpha^2 - \|f-g\|_\alpha^2) \ \forall \alpha \in (0,1) \ and$
$G_\alpha = 4(\|f+ig\|_\alpha^2 - \|f-ig\|_\alpha^2), \alpha \in (0,1)$.
$Then \|f+g\|_\alpha^2 - \|f-g\|_\alpha^2 = 0, \forall \alpha \in (0,1) \ldots\ldots\ldots\ldots (i)$
$And \ \|f+ig\|_\alpha^2 - \|f-ig\|_\alpha^2 = 0, \forall \alpha \in (0,1) \ldots\ldots\ldots\ldots (ii)$

From (*i*)*we get* $\|f+g\|_\alpha = \|f-g\|_\alpha \ \forall \alpha \in (0,1) \ldots\ldots\ldots$ (*iii*)
$\wedge\{t > 0; N(f + g,t) \geq \alpha\} = \wedge\{s > 0; N(x-y,s) > \alpha\}, \forall \alpha \in (0,1)$
$\Rightarrow \wedge\{t > 0; \eta(f+g,f+g,t^2) \geq \alpha\} =$
$\wedge\{s > 0; \eta(f-g,f-g,s^2) \geq \alpha\}, \forall \alpha \in (0,1).$
Now if possible let $\eta(f+g,f+g,s^2) \neq \eta(f-g,f-g,s^2)$ *for some* $s > 0$.
Without loss of generality let
$\eta(f+g,f+g,s^2) > \eta(f-g,f-g,s^2)$*for some* $s > 0 = \alpha_0$ (*say*)
Then by our assumption that $\eta(f,f,\bullet)$is strictly increasing and lower semi continuous $\forall f \in F(X)$, we get:-
$\wedge\{t > 0; \eta(f+g,f+g,t^2) \geq \alpha_0\} < \wedge\{r > 0; \eta(f-g,f-g,r^2) \geq \alpha_0\} = s$
$\Rightarrow \|f+g\|_{\alpha_0} < \|f-g\|_{\alpha_0}$, which is a contradiction of (iii).
So,$\eta(f+g,f+g,t^2) = \eta(f-g,f-g,t^2), t > 0$
Similarly from (ii) we can prove that
$\eta(f+ig,f+ig,t^2) = \eta(f-ig,f-ig,t^2) \ \forall t > 0$ The sufficiency of the conditions readily follows.

Theorem 3.6.6. Let $(F(X),\eta)$ *be a two* $-$ *FPH* space satisfying(Remark3.4). and $\alpha \in (0,1)$. *If* $\eta(f,f,\bullet)$ is strictly increasing and *continuous* $\forall f \in F(X)$, then
$f \perp_\alpha g$ *iff*
$\{(\eta(f+g,f+g,t^2) \geq \alpha$ *iff* $\eta(f-g,f-g,t^2) \geq \alpha, \forall t > 0)$ *and*
$(\eta(f+ig,f+ig,t^2) \geq \alpha$ *iff* $\eta(f-ig,f-ig,t^2) \geq \alpha, \forall t > 0)\}$.

Proof: *Let* $f \perp_\alpha g$
Then $< f,g >_\alpha = 0 = F_\alpha + iG_\alpha$
$\Rightarrow F_\alpha = 0$
$\Rightarrow \|f+g\|_\alpha = \|f-g\|_\alpha \ldots\ldots\ldots\ldots\ldots\ldots$ (*i*)
and $G_\alpha = 0$
$\Rightarrow \|f+i\,g\|_\alpha = \|f-i\,g\|_\alpha \ldots\ldots\ldots\ldots\ldots\ldots$ (*ii*)
Now from (i) we have,
$L = \Lambda\{t > 0; \eta(f+g,f+g,t^2) \geq \alpha\} = \Lambda\{s > 0; \eta(f-g,f-g,s^2) \geq \alpha\}$
$= M$
If possible let $\eta(f+g,f+g,p^2) \geq \alpha$ *and* $\eta(f-g,f-g,p^2) < \alpha$ *for some* $p \in R^+$*and* $p > 0$.
Then $\exists\, r,s$ *such that* $\eta(f-g,f-g,p^2) < r < s < \alpha \leq \eta(f+g,f+g,p^2)$
We consider the following cases.
If $\eta(f+g,f+g,p^2) > \alpha > \eta(f-g,f-y,p^2)$,
Then $M > p > L$ (Since η is strictly increasing and continuous and hence upper semi continuous $\forall f \in F(X)$), which can not hold.
If $\eta(f+g,f+g,p^2) = \alpha > \eta(f-g,f-g,p^2)$,
Then$L = p < M$ (Since η is strictly increasing and continuous and hence lower semi continuous $\forall f \in F(X)$,which also can not hold.
Thus $\eta(f+g,f+g,t^2) \geq \alpha$ *iff* $\eta(f-g,f-g,t^2) \geq \alpha, Vt > 0$
Similarly we can prove that

$\eta(f+g, f+ig, t^2) \geq \alpha \; iff \; \eta(f-g, f-ig, t^2) \geq \alpha, \forall t > 0$ The converse part readily follows.

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