On Extension the Stability Region of Implicit-Explicit Linear Multistep Methods for Ordinary Differential Equation

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الملخص

إن دراسة خاصية استقرارية طرق متعددة الخطوات (الصريحة – الضمنية) لمعادلات التفاضلية الاعتيادية تم تحليلها على مناطق الاستقرارية المعرفة من خلال المسائل الاختيارية وان هذه الدراسة أجريت على الطرق متعددة الخطوات الاعتيادية بالنسبة للمعادلات التفاضلية الاعتيادية.

في هذا البحث تم إعطاء فكرتين أساسيتين لتوسيع مناطق الاستقرارية للطرق (الصريحة – الضمنية) ومن الرتبة الثانية وأيضا تم استخدام طريقة اويلر (الصريحة – الضمنية) ذات الرتبة الأولى لتوسيع هذه المناطق. وكانت النتائج التي تم الحصول عليها تبين مدى فاعلية استخدام الطرق المعدلة عن مثيلاتها.

Abstract

Stability properties of implicit – explicit (IMEX) of linear multistep methods for ordinary differential equations are analyzed on the basis of stability regions defined by using scalar test equations. This study is closely related to the stability analysis of the standard linear multistep methods for ordinary differential equation.

In this paper we give two idea to extend the stability region for the second – order IMEX method, also have extended the region IMEX Euler method, the simplest IMEX method of first order, numerical results are presented which show superiority of improving method.

1- Introduction

In this paper we look at the general ODE problem

$$\frac{du}{dt} = f(t,u(t)) + g(t,u(t)) \qquad (1.1)$$

Where f represent the stiff part and g represents the non stiff part. In many applications, f is linear and g is non linear. In order to solve such equation efficiently, a special type of numerical method called implicit-explicit (IMEX) method is often used, which is obtained by applying an implicit formula with a good stability property to the f term and an explicit formula to the g term [7,10].

One of the simplest examples is the IMEX Euler method:

$$u_{n+1} = u_n + \Delta t f(t_{n+1}, u_{n+1}) + \Delta t g(t_n, u_n) \qquad (1.2)$$

Which is obtained by applying the implicit Euler formula to the f term and explicit Euler formula to the g term.

Here, Δt is the step-size, $t_n = t_0 + n\Delta t$, and u_n denotes an approximate value of $u(t_n)$.

This method is of order one in accuracy.

In this paper, we discuss IMEX methods of linear multi-step type.

For numerical solution of (1.1) we consider IMEX linear multi-step method

$$\sum_{j=0}^{k} \alpha_j u_{n+j} = \Delta t \sum_{j=0}^{k} \beta_j f(t_{n+j}, u_{n+j}) + \Delta t \sum_{j=0}^{k-1} \beta_j^* g(t_{n+j} u_{n+j}) \qquad (1.3)$$

Here, α_j, β_j denote coefficients of k-step linear multi-step method. Coefficients β_j^* are determined by $\beta_j^* = \beta_j + \beta_k \gamma_j$ with coefficients γ_j of a suitable extrapolation to incorporate β_k into the other coefficients [10].

For the study of stability of such IMEX methods, Frank et al.[5] proposed

$$\frac{du}{dt} = \lambda u(t) + \mu u(t) , \lambda, \mu \in C \qquad (1.4)$$

As a test equation (see Refs[1,2,5,7] for related studies), where $\lambda u(t)$ and $\mu u(t)$ correspond to f term and g term in (1.1).respectively.

Application of method (1.3) to this test equation yields the difference equation

$$\sum_{j=0}^{k} \alpha_j u_{n+j} = z \sum_{j=0}^{k} \beta_j u_{n+j} + \omega \sum_{j=0}^{k-1} \beta_j^* u_{n+j}, \; z = \Delta t \lambda, \; \omega = \Delta t \mu \qquad (1.5)$$

The stability region S of the IMEX method is defined as a region of the parameters $(z, \omega) = (\Delta t \lambda, \Delta t \mu)$ such that the zero solution of (1.5) is asymptotically stable. we can study the stability of the method on the basis of S in the similar manner as in the case of the standard linear multi-step methods [8,9].

However, S is a region in $\mathbb{C}^{2.}$ it is not easy to construct a new scheme by adjusting the parameters α_j, β_j and γ_j so as to enlarge the stability region S.

To overcome this difficulty, we consider, in analogy to (1.4), the test equation

$$\frac{du}{dt} = \lambda u(t) + \mu u(t-\tau), \lambda, \mu \in C \tag{1.6}$$

originally proposed by Barwell in 1975 [10] for the study of the stability of numerical method for delay differential equation (DDE$_s$) with constant delay. Here, $\tau > 0$ is a constant delay. when the step-size Δt is given in the form $\Delta t = \frac{\tau}{m}$, (m≥1 ; integer), (1.7)

an IMEX method can be applied to DDE$_s$ of form [2]

$$\frac{du}{dt} = f(t, u(t)) + g(t, u(t-\tau)) \tag{1.8}$$

2- Stability regions

We assume that the linear multi-step method determined from α_j, β_j is of order $p \geq 1$, and γ_j satisfy for any sufficiently smooth function $\varphi(t)$ [10] and $\sum_{j=0}^{k-1} j^q \gamma_j = k^q$,q=0,1,…,p-1 (2.1)

For example in the case $k = p = 2$, from the condition $\gamma_0 + \gamma_1 = 1$, $\gamma_1 = 2$ coefficients γ_1, γ_0 are uniquely determined as $\gamma_1 = 2, \gamma_0 = -1$ which gives a linear extrapolation determined from (2.1) and satisfies [10] $\xi^k - (\xi - 1)^k - \sum_{j=0}^{k-1} \gamma_j \xi^j$

The pair gives a polynomial extrapolation. Condition (2.1) assures that the local error of method (1.3) is $O(\Delta t^{p+1})$.

Here, we assume that $u(t_n + \Delta t) \approx u(t_n), as \Delta t \to 0$.

So, we can say that $\delta u(t_n + \Delta t) \approx \delta u(t_n), 0 \leq \delta \leq 1, as \Delta t \to 0$ (2.2)

In this paper we shall discuss two cases:

- First:suppose that $\Phi = \delta u_{n+k} + \delta u_{n+k-1}$

Then, the IMEX linear multi-step method (1.3) for equation (2.2) is represented as

$$\Phi + \sum_{j=0}^{k} \alpha_j u_{n+j} = \Delta t \sum_{j=0}^{k} \beta_j f(t_{n+j}, u_{n+j}) + \Delta t \sum_{j=0}^{k-1} \beta_j^* g(t_{n+j}, u_{n+j}), \tag{2.3a}$$

Or

$$(1+\delta)(\alpha_k u_{n+k} + \alpha_{k-1} u_{n+k-1}) + \sum_{j=0}^{k-2} \alpha_j u_{n+j} = \Delta t \sum_{j=0}^{k} \beta_j f(t_{n+j}, u_{n+j}) + \Delta t \sum_{j=0}^{k-1} \beta_j^* g(t_{n+j}, u_{n+j}) \tag{2.3b}$$

- Second: if we assume that $\Phi = \delta u_{n+k} + \delta u_{n+k-1} + \ldots + \delta u_n$

This idea we can not applied except when we have a method such that $\sum_{j=0}^{k} \alpha_j u_{n+k}$, $k > 2$.

So, the IMEX linear multi-step method(1.3) for equation(2.2) is represented as

$$\Phi + \sum_{j=0}^{k}\alpha_j u_{n+j} = \Delta t\sum_{j=0}^{k}\beta_j f(t_{n+j},u_{n+j}) + \Delta t\sum_{j=0}^{k-1}\beta_j^* g(t_{n+j},u_{n+j}), \tag{2.4a}$$

Or

$$(1+\delta)\sum_{j=0}^{k}\alpha_j u_{n+j} = \Delta t\sum_{j=0}^{k}\beta_j f(t_{n+j},u_{n+j}) + \Delta t\sum_{j=0}^{k-1}\beta_j^* g(t_{n+j},u_{n+j}) \tag{2.4b}$$

where α_j, β_j denote coefficient of a k-step linear multi-step method.

Application of method (2.3) and (2.4) to the test equation yields the difference equations

$$\Phi + \sum_{j=0}^{k}\alpha_j u_{n+j} = z\sum_{j=0}^{k}\beta_j u_{n+j} + \omega\sum_{j=0}^{k-1}\beta_j^* u_{n+j}\ ,$$

$$z = \Delta t\lambda\ ,\omega = \Delta t\mu\ \ ,\ \Phi = \delta\, u_{n+k} + \delta\, u_{n+k-1} \tag{2.5a}$$

Or

$$\alpha_k u_{n+k} + \alpha_{k-1} u_{n+k-1} + \frac{1}{(1+\delta)}\sum_{j=0}^{k-2}\alpha_j u_{n+j} = \frac{z}{(1+\delta)}\sum_{j=0}^{k}\beta_j u_{n+j} + \frac{\omega}{(1+\delta)}\sum_{j=0}^{k-1}\beta_j^* u_{n+j} \tag{2.5b}$$

And

$$\Phi + \sum_{j=0}^{k}\alpha_j u_{n+j} = z\sum_{j=0}^{k}\beta_j u_{n+j} + \omega\sum_{j=0}^{k-1}\beta_j^* u_{n+j}\ , \tag{2.6a}$$

$$z = \Delta t\lambda\ ,\omega = \Delta t\mu\ \ ,\ \Phi = \delta\, u_{n+k} + \delta\, u_{n+k-1} + \dots + \delta\, u_n$$

Or

$$\sum_{j=0}^{k}\alpha_j u_{n+j} = \frac{z}{(1+\delta)}\sum_{j=0}^{k}\beta_j u_{n+j} + \frac{\omega}{(1+\delta)}\sum_{j=0}^{k-1}\beta_j^* u_{n+j} \tag{2.6b}$$

Introducing polynomials (for more related see [3,4,9]) (from (2.3))

$$p(\xi) = \alpha_k \xi^k + \alpha_{k-1}\xi^{k-1} + \frac{1}{(1+\delta)}\sum_{j=0}^{k-2}\alpha_j \xi^j\ ,$$

$$\text{(when } \Phi = \delta u_{n+k} + \delta u_{n+k-1}\text{)} \tag{2.7}$$

$$\sigma(\xi) = \frac{1}{(1+\delta)}\sum_{j=0}^{k}\beta_j \xi^j\ ,\ \sigma^*(\xi) = \frac{1}{(1+\delta)}\sum_{j=0}^{k-1}\beta_j^* \xi^j$$

Also. we introducing polynomials (from (2.4))

$$p(\xi) = \sum_{j=0}^{k}\alpha_j \xi^j\ ,$$

$$\text{(when } \Phi = \delta u_{n+k} + \delta u_{n+k-1} + \dots + \delta u_n\text{)} \tag{2.8}$$

$$\sigma(\xi) = \frac{1}{(1+\delta)}\sum_{j=0}^{k}\beta_j \xi^j\ ,\ \sigma^*(\xi) = \frac{1}{(1+\delta)}\sum_{j=0}^{k-1}\beta_j^* \xi^j$$

And putting $\eta(\xi;z) = \rho(\xi) - z\sigma(\xi)$ (2.9)

We can write the characteristic equations of the difference equation (2.5) and (2.6) in the form

$$\eta(\xi;z) - \omega\sigma^*(\xi) = 0 \tag{2.10}$$

And represent the stability region S of method (2.3) and (2.4) as[5,6]

$$S=\{(z,\omega)\in c^2 :(2.10)\Rightarrow |\xi|<1\} \qquad (2.11)$$

For example, in the case of the IMEX Euler method (1.2), it follows form $u_{n+1}-u_n=\frac{1}{(1+\delta)}u_{n+1}+\frac{1}{(1+\delta)}u_n$ IMEX Improving Euler(IMEX IEuler) method

$$\rho(\xi)=\xi-1,\ \sigma(\xi)=\frac{\xi}{(1+\delta)},\ \sigma^*(\xi)=\frac{1}{(1+\delta)}$$

that the characteristic equation is $\xi-1-\frac{z}{(1+\delta)}\xi-\frac{\omega}{(1+\delta)}=0$

which is rewritten as $(1+\frac{\omega}{(1+\delta)})\Big/(1-\frac{z}{(1+\delta)})$

hence, the stability region is represented as

$$S=\{(z,\omega)\in C^2 :\left|1+\frac{\omega}{(1+\delta)}\right|<\left|1-\frac{z}{(1+\delta)}\right|\} \qquad (2.12)$$

the intersection of the stability region S and the $z-$plane $\{(z,0):z\in C\}$ is identified with the region $S_A=\{z\in C:\eta(\xi;z)=0\Rightarrow|\xi|<1\}$

in the complex, which corresponds to the standard stability region of the implicit formulas. For each $z\in S_A$, we denote by Γ_z the set of all ω such that (2.7) has a root $|\xi|=1$. This set is a curve in the complex plane represented in the form

$$\Gamma_z:\frac{\eta(\xi;z)}{\sigma^*(\xi)}\ ,\ \xi=e^{i\theta}\ ,\ 0\le\theta\le 2\pi \qquad (2.13)$$

Which gives the boundary of the z-section $S\cap\{(\omega,z):\omega\in C\}$

For the IMEX IEuler method, we have

$$\frac{\eta(\xi;z)}{\sigma^*(\xi)}=(-1+(1-\frac{z}{(1+\delta)})\xi)\Big/\frac{1}{(1+\delta)}$$

Where $\eta(\xi;z)=\xi-1-\frac{z}{(1+\delta)}\xi\ ,\ \sigma^*(\xi)=\frac{1}{(1+\xi)}$

Hence, Γ_z is circle centered at -2 with radius $\left|1-\frac{z}{(1+\delta)}\right|$ (Fig 1)

Restricting the variable z on to real line

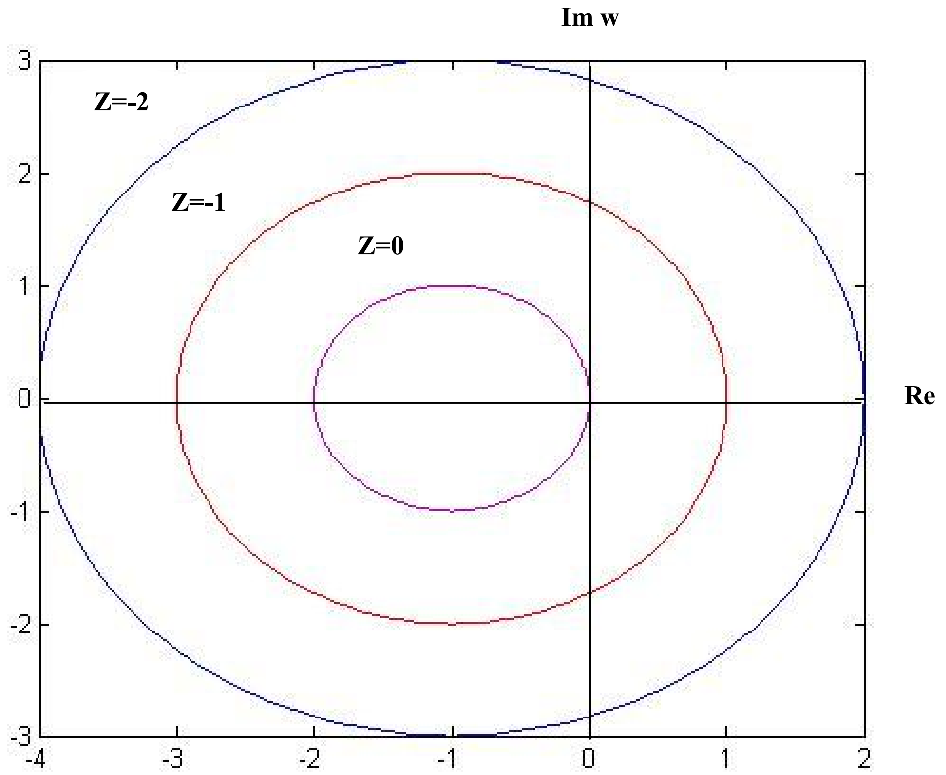

Fig. 1 (a): stability region of the IMEX Euler method

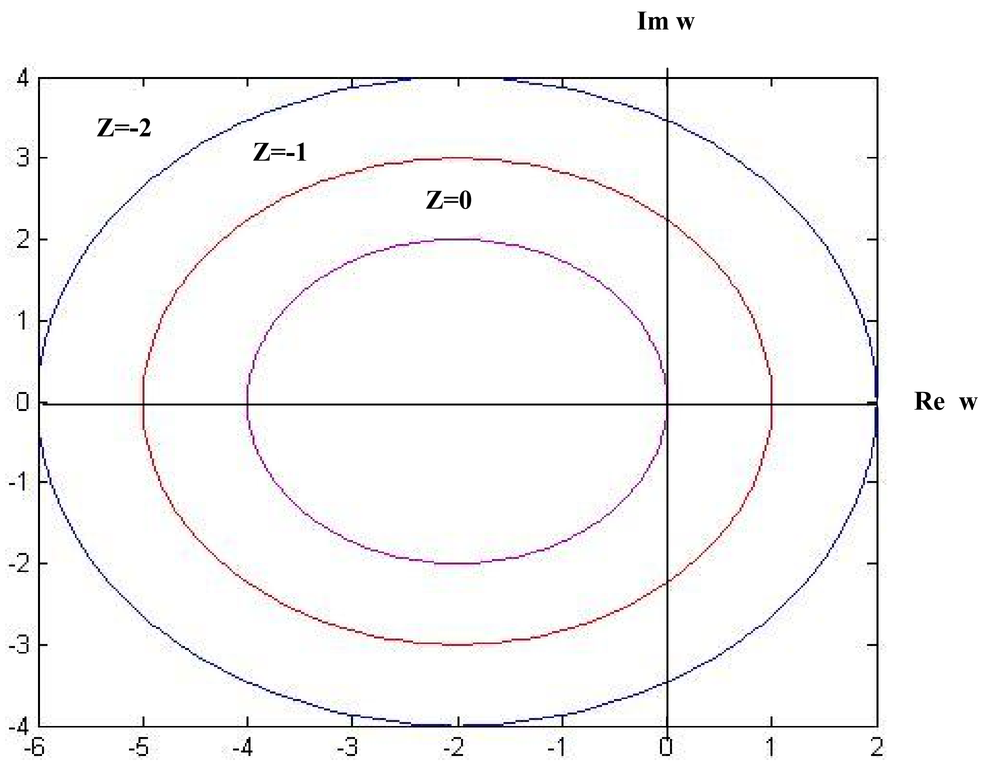

Fig. 1 (b): stability region of the IMEX IEuler method

A second-order two-step IMEX method defined with BDF2 (two-step backward differentiation formula)
$\alpha_2 = 3/2, \alpha_1 = -2, \alpha_0 = 1/2, \beta_2 = 1, \beta_1 = \beta_0 = 0$) and
$\beta_1^* = 2, \beta_0^* = -1$ $(\gamma_1 = 2, \gamma_0 = -1)$

$$\frac{3}{2}u_{n+2} - 2u_{n+1} + \frac{1}{2}u_n = \Delta t f(t_{n+2}, u_{n+2}) + \Delta t(2g(t_{n+1}, u_{n+1}) - g(t_n, u_n))$$

is called (BDF2) method see [5].
This methods have three terms of u_n, so we discuss the first and second idea.
First: By (2.3) and ($k = 2$) we have

$$(1+\delta)(\alpha_2 u_{n+2} + \alpha_1 u_{n+1}) + \alpha_0 u_n = \Delta t(\beta_2 f(t_{n+2}, u_{n+2}) + \beta_1 f(t_{n+1}, u_{n+1}) + \beta_0 f(t_n, u_n) + \beta_1^* g(t_{n+1}, u_{n+1}) + \beta_0^* g(t_n, u_n))$$

So, from the above condition

$$(1+\delta)(\frac{3}{2}u_{n+2} - 2u_{n+1}) + \frac{1}{2}u_n = \Delta t f(t_{n+2}, u_{n+2}) + \Delta t(2g(t_{n+1}, u_{n+1}) - g(t_n, u_n))$$

we have IMEX Improving BDF2(IBDF2) method

$$\frac{3}{2}u_{n+2} - 2u_{n+1} + \frac{1}{2(1+\delta)}u_n = \frac{\Delta t}{(1+\delta)} f(t_{n+2}, u_{n+2}) + \frac{2\Delta t}{(1+\delta)} g(t_{n+1}, u_{n+1}) - \frac{\Delta t}{(1+\delta)} g(t_n, u_n)$$

From (2.5)

$$\frac{3}{2}u_{n+2} - 2u_{n+1} + \frac{1}{2(1+\delta)}u_n = \frac{z}{(1+\delta)}u_{n+2} + \frac{2\omega}{(1+\delta)}u_{n+1} - \frac{\omega}{(1+\delta)}u_n$$

, $z = \Delta t \lambda$, $\omega = \Delta t \mu$
In the case of this method

$$\rho(\xi) = \frac{3}{2}\xi^2 - 2\xi + \frac{1}{2(1+\delta)}, \sigma(\xi) = \frac{1}{(1+\delta)}\xi^2, \sigma^*(\xi) = \frac{1}{(1+\delta)}(2\xi - 1)$$

From (2.9) we have $\eta(\xi, z) = \frac{3}{2}\xi^2 - 2\xi + \frac{1}{2(1+\delta)} - \frac{z}{(1+\delta)}\xi^2$ and we denote by Γ_z the set of all ω such that (2.10) has a real with $|\xi| = 1$

$$\Gamma_z : \frac{\eta(\xi; z)}{\sigma^*(\xi)} = \frac{\frac{3}{2}\xi^2 - 2\xi + \frac{1}{2(1+\delta)} - \frac{z}{(1+\delta)}\xi^2}{\frac{1}{(1+\delta)}(2\xi - 1)} = \frac{3(1+\delta)\xi^2 - 4(1+\delta)\xi + 1 - 2z\xi^2}{2(2\xi - 1)}$$

For negative z is simple closed curve as show in Fig(2).
In this idea, the parameters δ appears in left (assume $\delta_1 = \delta, 0 \geq |\delta_1| < 1$) and right hands of IBDF2.

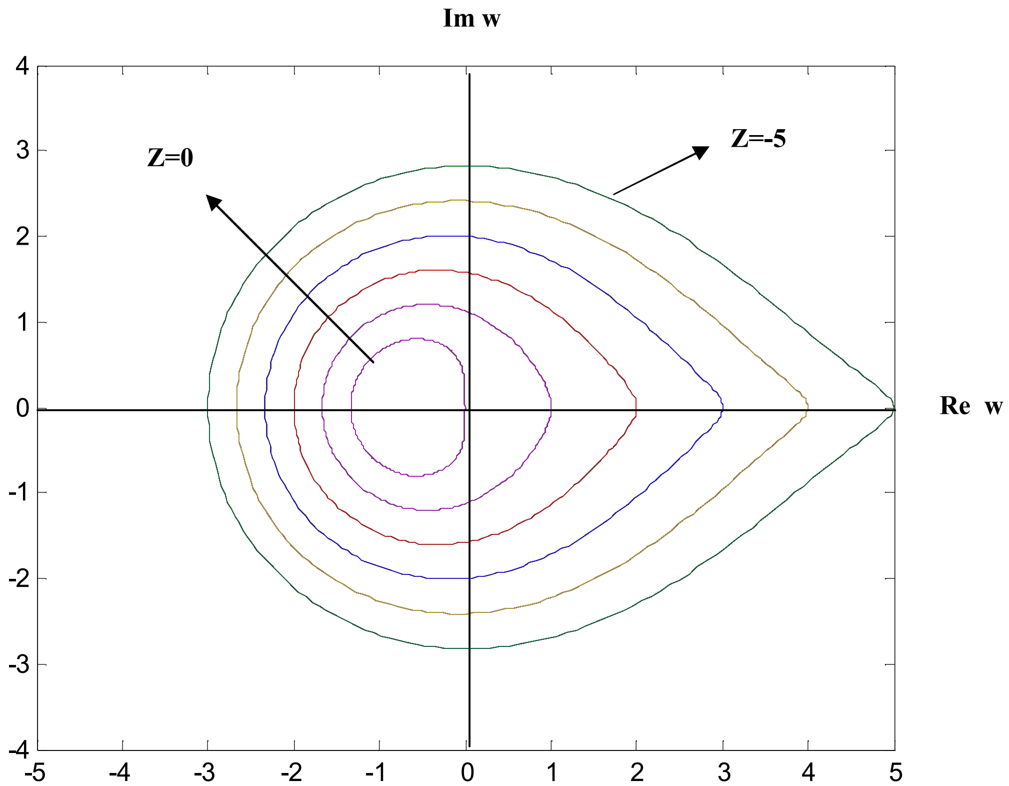

Fig. 2 (a): stability region of the IMEX BDF2 method

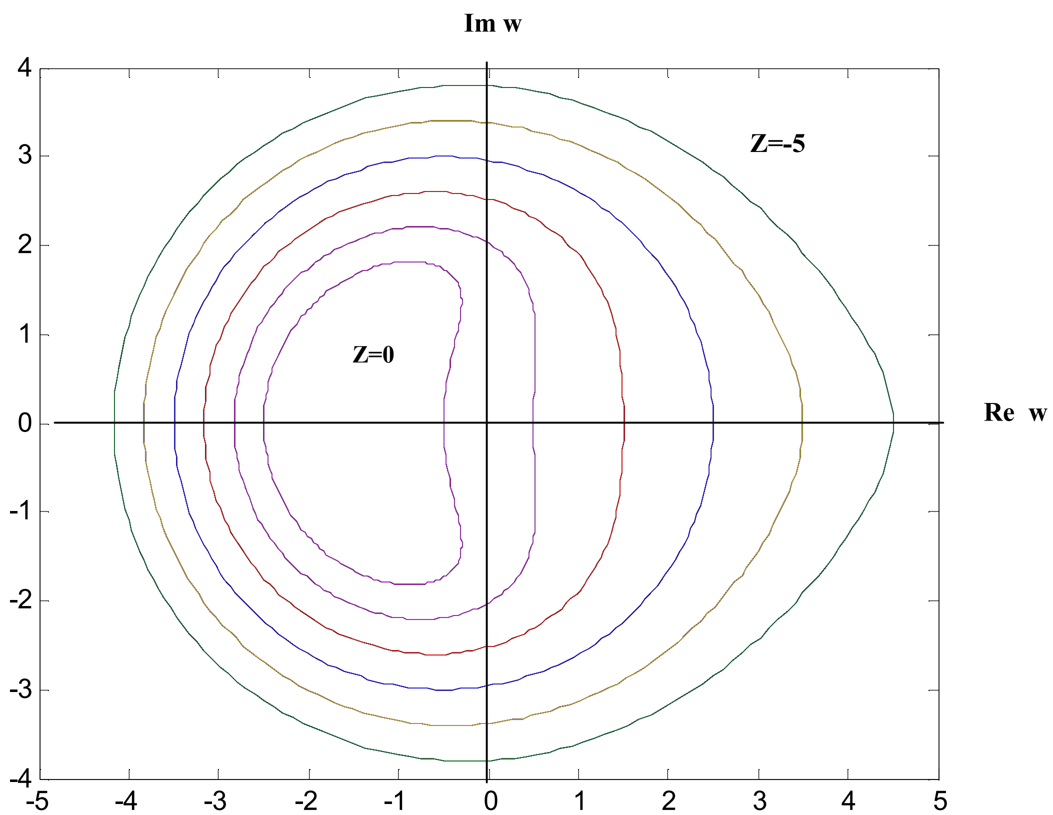

Fig. 2 (b): stability region of the IMEX IBDF2 method

Second: same as first idea, by (2.4) we have (IMEX IBDF2 method)

$$\frac{3}{2}u_{n+2}-2u_{n+1}+\frac{1}{2}u_n=\frac{1}{(1+\delta)}f(t_{n+2},u_{n+2})+\frac{2}{(1+\delta)}g(t_{n+1},u_{n+1})-\frac{1}{(1+\delta)}g(t_n,u_n)$$

From (2.6)

$$\frac{3}{2}u_{n+2}-2u_{n+1}+\frac{1}{2}u_n=\frac{z}{(1+\delta)}u_{n+2}+\frac{2\omega}{(1+\delta)}u_{n+1}-\frac{\omega}{(1+\delta)}u_n$$

$z=\Delta t\lambda\ ,\omega=\Delta t\mu$

In the case of this method

$$\rho(\xi)=\frac{3}{2}\xi^2-2\xi+\frac{1}{2},\sigma(\xi)=\frac{1}{(1+\delta)}\xi^2,\sigma^*(\xi)=\frac{1}{(1+\delta)}(2\xi-1)$$

From (2.9) we have $\eta(\xi,z)=\frac{3}{2}\xi^2-2\xi+\frac{1}{2}-\frac{z}{(1+\delta)}\xi^2$ and we denote by Γ_z the set of all ω such that (2.10) has a real with $|\xi|=1$

$$\Gamma_z:\frac{\eta(\xi;z)}{\sigma^*(\xi)}=\frac{\frac{3}{2}\xi^2-2\xi+\frac{1}{2}-\frac{z}{(1+\delta)}\xi^2}{\frac{1}{(1+\delta)}(2\xi-1)}=\frac{3(1+\delta)\xi^2-4(1+\delta)\xi+(1+\delta)-2z\xi^2}{2(2\xi-1)}$$

For negative z is simple closed curve as show in Fig 2(c)

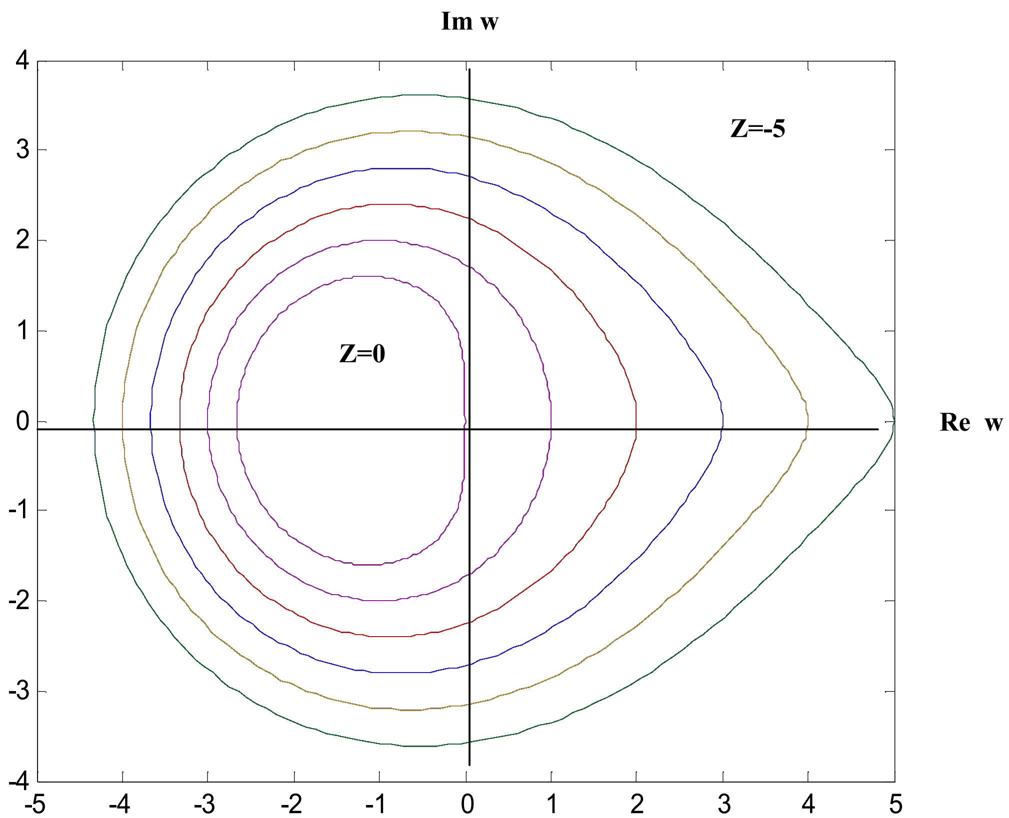

Fig. 2(c) stability region of the IMEX IBDF2 method

Under condition (1.7) for the step-size, an IMEX linear multi-step method for the DDF (1.8) is defined by

$$\Phi+\sum_{j=0}^{k}\alpha_j u_{n+j}=\Delta t\sum_{j=0}^{k}\beta_j f(t_{n+j},u_{n+j})+\Delta t\sum_{j=0}^{k-1}\beta_j^* g(t_{n+j},u_{n-m+j}) \qquad (2.14)$$

Application of the method to the test equation (1.6) yields

$$\alpha_k u_{n+k}+\alpha_{k-1}u_{n+k-1}+\frac{1}{(1+\delta)}\sum_{j=0}^{k-2}\alpha_j u_{n+j}=\frac{z}{(1+\delta)}\sum_{j=0}^{k}\beta_j u_{n+j}+\frac{\omega}{(1+\delta)}\sum_{j=0}^{k-1}\beta_j^* u_{n-m+j} \qquad (2.15)$$

$z=\Delta t\lambda\ ,\ \Delta t\mu\ ,\Phi=\delta\,u_{n+k}+\delta\,u_{n+k-1}$

Or

$$\sum_{j=0}^{k}\alpha_j u_{n+j}=\frac{z}{(1+\delta)}\sum_{j=0}^{k}\beta_j u_{n+j}+\frac{\omega}{(1+\delta)}\sum_{j=0}^{k-1}\beta_j^* u_{n-m+j} \qquad (2.16)$$

$z=\Delta t\lambda\ ,\ \Delta t\mu\ ,\Phi=\delta\,u_{n+k}+\delta\,u_{n+k-1}+\ldots+\delta\,u_n$

And the characteristic equation of the difference equation (2.15) and (2.16) is written in the form $\xi^m\eta(\xi;z)-\omega\sigma^*(\xi)=0$ (2.17)

Using this equation, we define the p-stability region S_P of the IMEX method as [10] $S_P=\bigcap_{m\geq 0}S_P^{(m)}\ ,\ S_P^{(m)}=\{(z,\omega)\in C^2:(2.13)\Rightarrow|\xi|<1\}$ (2.18)

3- Analysis of p-stability regions

Toshiyuki [10] try to find a second-order IMEX method with larger S by the construction of a scheme with larger S_P than that of the IMEX BDF2 method.

Furthermore, define the curve Γ^* in the complex plane by

$$\Gamma^*:\frac{\sigma^*(\xi)}{\sigma(\xi)}\ ,\ \xi=e^{i\theta}\ ,\ 0\leq\theta\leq 2\pi \qquad (2.19)$$

and put $r=\sup\{|\omega|:\omega\in\Gamma^*\}$ (2.20)

Then, have $r\geq 1$.

Using the form of Φ will not effect for the results of equations (2.19) and (2.20), but reusing Γ_z will extend the stability region.

For any second order two-step linear multi-step method is represented in

$$\alpha_2=a\ ,\ \alpha_1=1-2a\ ,\ \alpha_0=a-1$$

the form [10]

$$\beta_2=b\ ,\ \beta_1=\frac{1}{2}=a-2b\ ,\ \beta_0\frac{1}{2}-a+b$$

With the real parameters a,b and conditions $a>\frac{1}{2}\ ,\ b>\frac{a}{2}$, since $\rho(\xi)$ is written as $\rho(\xi)=(a\xi+1-a)(\xi-1)$

Moreover, it follows form $\gamma_1=2\ ,\ \gamma_0=-1$ that $\beta_1^*=\frac{1}{2}+a\ ,\ \beta_0^*=\frac{1}{2}-a$

And [10] shows that the supremum r (when $a=b$) is written as $r=\dfrac{2a+1}{2a-1}$, moreover, when taking $a=b=20$ [10] obtain $r=1.051$ and Γ_z for much larger stability region (stabilized second-order method) than of the IMEX BDF2 method.
Her, we show that under (2.3) and (2.4) method we obtain the method more larger than that (stabilized second-order method).
By (2.3) we have

$$au_{n+2}+(1-2a)u_{n+1}+\frac{(a-1)}{(1+\delta)}u_n=\frac{b}{(1+\delta)}f(t_{n+2},u_{n+2})+\frac{(\frac{1}{2}+a-2b)}{(1+\delta)}f(t_{n+1},u_{n+1})+$$
$$+\frac{(\frac{1}{2}-a+b)}{(1+\delta)}f(t_n,u_n)+\frac{(\frac{1}{2}+a)}{(1+\delta)}g(t_{n+1},u_{n+1})+\frac{(\frac{1}{2}-a)}{(1+\delta)}g(t_n,u_n)$$

From (2.5)

$$au_{n+2}+(1-2a)u_{n+1}+\frac{(a-1)}{(1+\delta)}u_n=\frac{zb}{(1+\delta)}u_{n+2}+\frac{z(\frac{1}{2}+a-2b)}{(1+\delta)}u_{n+1}+\frac{z(\frac{1}{2}-a+b)}{(1+\delta)}u_n$$
$$+\frac{\omega(\frac{1}{2}+a)}{(1+\delta)}u_{n+1}+\frac{\omega(\frac{1}{2}-a)}{(1+\delta)}u_n\ ,\ z=\Delta t\,\lambda\ ,\ \omega=\Delta t\,\mu$$

In the case of this method

$$\rho(\xi)=a\xi^2+(1-2a)\xi+\frac{(a-1)}{(1+\delta)},$$
$$\sigma(\xi)=\frac{b}{(1+\delta)}\xi^2+\frac{(0.5+a-2b)}{(1+\delta)}\xi+\frac{(0.5-a+b)}{(1+\delta)},$$
$$\sigma^*(\xi)=\frac{(0.5+a)}{(1+\delta)}\xi+\frac{(0.5-a)}{(1+\delta)}$$

So,

$$\Gamma_z:\frac{\eta(\xi;z)}{\sigma^*(\xi)}=\frac{a\xi^2+(1-2a)\xi+\dfrac{(a-1)}{(1+\delta)}-\dfrac{z}{(1+\delta)}(b\xi^2+(0.5+a-2b)\xi+(0.5-a+b))}{\dfrac{1}{(1+\delta)}((0.5+a)\xi+(0.5-a))}$$
$$=\frac{a(1+\delta)\xi^2+(1-2a)(1+\delta)\xi+(a-1)-z(b\xi^2+(0.5+a-2b)\xi+(0.5-a+b))}{(0.5+a)\xi+(0.5-a)}$$

Now, by (2.4) we have

$$au_{n+2}+(1-2a)u_{n+1}+(a-1)u_n=\frac{b}{(1+\delta)}f(t_{n+2},u_{n+2})+\frac{(\frac{1}{2}+a-2b)}{(1+\delta)}f(t_{n+1},u_{n+1})+$$

$$+\frac{(\frac{1}{2}-a+b)}{(1+\delta)}f(t_n,u_n)+\frac{(\frac{1}{2}+a)}{(1+\delta)}g(t_{n+1},u_{n+1})+\frac{(\frac{1}{2}-a)}{(1+\delta)}g(t_n,u_n)$$

From (2.6)

$$au_{n+2}+(1-2a)u_{n+1}+(a-1)u_n=\frac{zb}{(1+\delta)}u_{n+2}+\frac{z(\frac{1}{2}+a-2b)}{(1+\delta)}u_{n+1}+\frac{z(\frac{1}{2}-a+b)}{(1+\delta)}u_n$$

$$+\frac{\omega(\frac{1}{2}+a)}{(1+\delta)}u_{n+1}+\frac{\omega(\frac{1}{2}-a)}{(1+\delta)}u_n\ ,\ z=\Delta t\,\lambda\ ,\ \omega=\Delta t\,\mu$$

In the case of this method

$$\rho(\xi)=a\xi^2+(1-2a)\xi+(a-1),\ \sigma(\xi)=\frac{b}{(1+\delta)}\xi^2+\frac{(0.5+a-2b)}{(1+\delta)}\xi+\frac{(0.5-a+b)}{(1+\delta)},$$

$$\sigma^*(\xi)=\frac{(0.5+a)}{(1+\delta)}\xi+\frac{(0.5-a)}{(1+\delta)}$$

$$\Gamma_z:\frac{\eta(\xi;z)}{\sigma^*(\xi)}=\frac{a\xi^2+(1-2a)\xi+(a-1)-\frac{z}{(1+\delta)}(b\xi^2+(0.5+a-2b)\xi+(0.5-a+b))}{\frac{1}{(1+\delta)}((0.5+a)\xi+(0.5-a))}$$

$$=\frac{a(1+\delta)\xi^2+(1-2a)(1+\delta)\xi+(a-1)(1+\delta)-z(b\xi^2+(0.5+a-2b)\xi+(0.5-a+b))}{(0.5+a)\xi+(0.5-a)}$$

<u>3- Numerical results</u>

Here we show by numerical examples [9] the performance of the improved methods. From the coming tables we show the stability behavior of the normal methods and improved methods by using different values for the step size h.

These values, of h showed the stability behavior of the newly developed methods.

We include test results of the following problems for the IMEX Euler method and IMEX second two-step method also IMEX second two-step method with a=b=20.

<u>Example-1</u>: $y''+1001y'+1000y\ ,\ y(0)=1, y'(0)=-1$

$$y(x)=e^{-x}+e^{-1000}$$

<u>Example-2</u>: $\varepsilon\, y''+y'=0\ ,\ y(0)=1, y(1)=2$

$$y(x)=\frac{2-\exp(-1/\varepsilon)-\exp(-x/\varepsilon)}{1-\exp(-1/\varepsilon}$$

Example-3: $y'' + 21y' + 20y \quad , y(0) = 1, y'0) = -1$

$y(x) = e^{-x} + e^{-20}$

Table (1) and (2) displays the results with the IMEX Euler and IMEX IEuler methods. We can see that the method (IMEX IEuler) produce stable solutions with much larger h than that IMEX Euler method.

Also, with the IMEX IBDF2 and IMEX BDF2 (see tables 3-10) indicate the effect of step size h which lies outside of the stability region of IMEX BDF2 method and inside for the IMEX IBDF2 method.

On the other hand, the IMEX IBDF2 method with second case is more stable solution than the IMEX IBDF2 method with first case. But, when taken δ_1 differences from δ we have stability results.

4- conclusions

In this paper we discuss two idea and we see that these methods(IMEX Improving Euler method and IMEX Improving BDF2method) its more stability than the methods developed by Toshiyuki KOTO(IMEX Euler method and IMEX BDF2method).

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Step size =0.05

$$y'' + 21y' + 20y = 0$$

X	Theoretical solution	Numerical solution of IEU method	Error of IEU method	Numerical solution of EU method	Error of EU method
0.05	1.319108865672156	1.812222222222222	0.493113356550066	2.052500000000000	0.733391134327844
1	0.367879443232596	0.585178006474125	0.217298563241529	0.100556402773683	0.468435846006279
2	0.135335283236613	0.154523051570657	0.019187768334045	2.901709408161799	3.037044691398412
3	0.049787068367864	0.040803607301091	0.008983461066773	21.663314171561830	21.713101239929692
4	0.018315638888734	0.010774666639432	0.007540972249302	1.595809859695625e+02	1.595993016084512e+02
5	0.006737946999085	0.002845175926095	0.003892771072991	1.175251826606274e+03	1.175258564553273e+03
6	0.002478752176666	7.513017637876931e-04	0.001727450412879	8.655232990428390e+03	8.655235469180567e+03
7	9.118819655545162e-04	1.983899607379422e-04	7.134920048165740e-04	6.374212758587310e+04	6.374212849775506e+04
8	3.354626279025119e-04	5.238717439338315e-05	2.830754535091287e-04	4.694337896421047e+05	4.694337899775673e+05
9	1.234098040866796e-04	1.383344212940241e-05	1.095763619572772e-04	3.457181164130893e+06	3.457181164254303e+06
10	4.539992976248485e-05	3.652881136717617e-06	4.174704862576723e-05	2.546067595757721e+07	2.546067595762261e+07

Table (1) : Numerical results of IEU (δ =0.5) method and EU method .

Step size =0.015

$\varepsilon\ y'' + y' = 0, \varepsilon = 0.01$

X	Theoretical solution	Numerical solution of IEU method	Error of IEU method	Numerical solution of EU method	Error of EU method
0.015	1.776869839851570	1.996093750000000	0.223130160148430	1.750000000000000	0.026869839851570
0.15	1.999999694097679	2.001174444536446	0.001174750438766	130	1.280000003059023e+02
0.30	1.999999999999907	2.000308967673098	3.089676731915514e-04	131074	131072
0.45	2	2.000081281848617	8.128184861666554e-05	134217730	134217728
0.60	2	2.000021383269157	2.138326915712696e-05	1.374389534740000e+11	1.374389534720000e+11
0.75	2	2.000005625415855	5.625415854471072e-06	1.407374883553300e+14	1.407374883553280e+14
0.90	2	2.000001479909518	1.479909518042888e-06	1.441151880758559e+17	1.441151880758559e+17
1.05	2	2.000000389328050	3.893280497990759e-07	1.475739525896764e+20	1.475739525896764e+20
1.20	2	2.000000102422702	1.024227018753265e-07	1.511157274518287e+23	1.511157274518287e+23
1.35	2	2.000000026944912	2.694491163524049e-08	1.547425049106725e+26	1.547425049106725e+26
1.50	2	1.999999991898803	8.101197268572946e-09	1.584563250285287e+29	1.584563250285287e+29
1.65	2	2.000000001864825	1.864824739072901e-09	1.622592768292134e+32	1.622592768292134e+32
1.80	2	2.000000000490590	4.905902351026725e-10	1.661534994731145e+35	1.661534994731145e+35
1.95	2	2.000000000129063	1.290625384342548e-10	1.701411834604692e+38	1.701411834604692e+38

Table (2) : Numerical results of IEU (δ =0.5) method and EU method .

Step size =0.1 (First case)

$$y'' + 21y' + 20y = 0$$

X	Theoretical solution	Numerical solution of IBDF2 method	Error of IBDF2 method	Numerical solution of BDF2 method	Error of BDF2 method
0.1	1.040172701272572	0.864864864864865	0.175307836407707	0.100000000000000	1.140172701272572
4	0.018315638888734	0.014309507522831	0.004006131365903	3.400074047538720e+05	3.400073864382331e+05
8	3.354626279025119e-04	2.081560443943902e-05	3.562782323419509e-04	1.359807673864999e+14	1.359807673864999e+14
12	6.144212353328210e-06	1.409269523871745e-05	7.948482885389243e-06	5.438359010233444e+22	5.438359010233444e+22
16	1.125351747192591e-07	3.979789108786082e-07	5.105140855978673e-07	2.174994985880914e+31	2.174994985880914e+31
20	2.061153622438558e-09	1.332680337402335e-08	1.126564975158479e-08	8.698585694150505e+39	8.698585694150505e+39
24	3.775134544279098e-11	2.875186713547607e-10	3.252700167975517e-10	3.478876667287276e+48	3.478876667287276e+48
28	6.914400106940203e-13	3.956601835756914e-12	3.265161825062894e-12	1.391327658510566e+57	1.391327658510566e+57
32	1.266416554909418e-14	7.035453149140475e-14	5.769036594231058e-14	5.564418743381209e+65	5.564418743381209e+65
36	2.319522830243570e-16	7.542522073258293e-15	7.774474356282650e-15	2.225410798261433e+74	2.225410798261433e+74
40	4.248354255291589e-18	3.517011080835734e-16	3.474527538282818e-16	8.900216625338383e+82	8.900216625338383e+82

Table (3) : Numerical results of IBDF2 (δ =0.4, δ_1 = -0.45) method and BDF2 method .

Step size = 0.1(Second case)

$$y'' + 21y' + 20y = 0$$

X	Theoretical solution	Numerical solution of IBDF2 method	Error of IBDF2 method	Numerical solution of BDF2 method	Error of BDF2 method
0.1	1.040172701272572	0.950000000000000	0.090172701272572	0.100000000000000	1.140172701272572
4	0.018315638888734	0.018194231095980	1.214077927545534e-04	3.400074047538720e+05	3.400073864382331e+05
8	3.354626279025119e-04	3.484707698997363e-04	1.300814199722448e-05	1.359807673864999e+14	1.359807673864999e+14
12	6.144212353328210e-06	6.700293521008944e-06	5.560811676807337e-07	5.438359010233444e+22	5.438359010233444e+22
16	1.125351747192591e-07	1.259208698742138e-07	1.338569515495468e-08	2.174994985880914e+31	2.174994985880914e+31
20	2.061153622438558e-09	2.347232716461780e-09	2.860790940232218e-10	8.698585694150505e+39	8.698585694150505e+39
24	3.775134544279098e-11	4.372019349018589e-11	5.968848047394910e-12	3.478876667287276e+48	3.478876667287276e+48
28	6.914400106940203e-13	8.147436755353200e-13	1.233036648412997e-13	1.391327658510566e+57	1.391327658510566e+57
32	1.266416554909418e-14	1.518675543002552e-14	2.522589880931347e-15	5.564418743381209e+65	5.564418743381209e+65
36	2.319522830243570e-16	2.830907683035232e-16	5.113848527916624e-17	2.225410798261433e+74	2.225410798261433e+74
40	4.248354255291589e-18	5.276943514714216e-18	1.028589259422627e-18	8.900216625338383e+82	8.900216625338383e+82

Table (4) : Numerical results of IBDF2 (δ =1) method and BDF2 method .

Step size =0.002 (First case)

$$y'' + 1001y' + 1000y = 0$$

X	Theoretical solution	Numerical solution of IBDF2 method	Error of IBDF2 method	Numerical solution of BDF2 method	Error of BDF2 method
0.004	0.996007989343992	0.997334332658333	0.001326343314342	0.994677328000000	0.001330661343991
0.080	0.923116346386636	0.924411804988129	0.001295458601493	3.967458686904738	3.044342340518103
0.160	0.852143788966211	0.853254089901039	0.001110300934828	1.192809433611261e+07	1.192809348396882e+07
0.240	0.786627861066553	0.787573825508518	9.459644419644864e-04	4.570847161995962e+13	4.570847161995883e+13
0.320	0.726149037073691	0.726949378816755	8.003417430635373e-04	1.751549280657117e+20	1.751549280657117e+20
0.400	0.670320046035639	0.670991572149895	6.715261142551388e-04	6.711939327305943e+26	6.711939327305943e+26
0.480	0.618783391806141	0.619341185254219	5.577934480777147e-04	2.572016102026799e+33	2.572016102026799e+33
0.560	0.571209063848815	0.571666649289019	4.575854402040225e-04	9.855969350277760e+39	9.855969350277760e+39
0.640	0.527292424043049	0.527661918325668	3.694942826194536e-04	3.776808852676557e+46	3.776808852676557e+46
0.720	0.486752255959972	0.487044504690633	2.922487306610266e-04	1.447273687925382e+53	1.447273687925382e+53
0.800	0.449328964117222	0.449553665540325	2.247014231034350e-04	5.545954824472313e+59	5.545954824472313e+59
0.880	0.414782911681581	0.414948729026548	1.658173449664013e-04	2.125210675195637e+66	2.125210675195637e+66
0.960	0.382892885975112	0.383007549307375	1.146633322627877e-04	8.143810321057266e+72	8.143810321057266e+72
1.040	0.353454681958780	0.353525080485440	7.039852666018698e-05	3.120709269882304e+79	3.120709269882304e+79

Table (5) : Numerical results of IBDF2 (δ =1, δ_1 =-0.00005) method and BDF2 method .

Step size =0.002 (Second case)

$$y'' + 1001\,y' + 1000\,y = 0$$

X	Theoretical solution	Numerical solution of IBDF2 method	Error of IBDF2 method	Numerical solution of BDF2 method	Error of BDF2 method
0.004	0.996007989343992	0.997335999333333	0.001328009989342	0.994677328000000	0.001330661343991
0.080	0.923116346386636	0.924501136035672	0.001384789649037	3.967458686904738	3.044342340518103
0.160	0.852143788966211	0.853422225242536	0.001278436276324	1.192809433611261e+07	1.192809348396882e+07
0.240	0.786627861066553	0.787808111657584	0.001180250591031	4.570847161995962e+13	4.570847161995883e+13
0.320	0.726149037073691	0.727238642768418	0.001089605694727	1.751549280657117e+20	1.751549280657117e+20
0.400	0.670320046035639	0.671325968480920	0.001005922445280	6.711939327305943e+26	6.711939327305943e+26
0.480	0.618783391806141	0.619712057985838	9.286661796972240e-04	2.572016102026799e+33	2.572016102026799e+33
0.560	0.571209063848815	0.572066407146528	8.573432977131423e-04	9.855969350277760e+39	9.855969350277760e+39
0.640	0.527292424043049	0.528083922151174	7.914981081257322e-04	3.776808852676557e+46	3.776808852676557e+46
0.720	0.486752255959972	0.487482965877312	7.307099173398668e-04	1.447273687925382e+53	1.447273687925382e+53
0.800	0.449328964117222	0.450003554458738	6.745903415167342e-04	5.545954824472313e+59	5.545954824472313e+59
0.880	0.414782911681581	0.415405692506728	6.227808251464406e-04	2.125210675195637e+66	2.125210675195637e+66
0.960	0.382892885975112	0.383467836325317	5.749503502051057e-04	8.143810321057266e+72	8.143810321057266e+72
1.040	0.353454681958780	0.353985475280021	5.307933212408966e-04	3.120709269882304e+79	3.120709269882304e+79

Table (6) : Numerical results of IBDF2 (δ =1) method and BDF2 method .

Step size = 0.015 (First case)

$$\varepsilon\, y'' + y' = 0\,,\ \varepsilon = 0.01$$

X	Theoretical solution	Numerical solution of IBDF2 method	Error of IBDF2 method	Numerical solution of BDF2 method	Error of BDF2 method
0.015	1.776869839851570	1.750000000000000	0.026869839851570	2.500000000000000	0.723130160148430
0.150	1.999999694097679	1.992386888096835	0.007612806000845	1.000584210914877	0.999415483182802
0.300	1.999999999999907	1.999668276101549	3.317238983577830e-04	0.431255406963110	1.568744593036797
0.450	2	2.000013827894327	1.382789432691212e-05	0.011202289933945	1.988797710066055
0.60	2	2.000037966559722	3.796655972232443e-05	0.155941088089072	1.844058911910928
0.750	2	2.000048484022029	4.848402202917157e-05	1.488210270869652	0.511789729130348
0.900	2	2.000058500354282	5.850035428212053e-05	4.755666607510483	2.755666607510483
1.050	2	2.000068501034364	6.850103436395472e-05	10.577561356069847	8.577561356069847
1.200	2	2.000078501385237	7.850138523712857e-05	18.946793694824862	16.946793694824862
1.350	2	2.000088501782404	8.850178240438211e-05	28.455464395393047	26.455464395393047
1.500	2	2.000098502229882	9.850222988161406e-05	35.323532419514116	33.323532419514116
1.650	2	2.000108502727388	1.085027273881600e-04	32.516687157414310	30.516687157414310
1.800	2	2.000118503274898	1.185032748982628e-04	9.546331217538949	7.546331217538949
1.950	2	2.000128503872412	1.285038724123666e-04	46.091414331622040	48.091414331622040

Table (7) : Numerical results of IBDF2 ($\delta = 1, \delta_1 = 0.000001$) method and BDF2 method

Step size =0.015 (Second case)

$$\varepsilon\, y'' + y' = 0, \;\; \varepsilon = 0.01$$

X	Theoretical solution	Numerical solution of IBDF2 method	Error of IBDF2 method	Numerical solution of BDF2 method	Error of BDF2 method
0.015	1.776869839851570	1.750000000000000	0.026869839851570	2.500000000000000	0.723130160148430
0.150	1.999999694097679	1.992378390175124	0.007621303922556	1.000584210914877	0.999415483182802
0.300	1.999999999999907	1.999649776111770	3.502238881363251e-04	0.431255406963110	1.568744593036797
0.450	2	1.999985327786715	1.467221328521617e-05	0.011202289933945	1.988797710066055
0.60	2	1.999999466268900	5.337311004272038e-07	0.155941088089072	1.844058911910928
0.750	2	1.999999983529931	1.647006886429381e-08	1.488210270869652	0.511789729130348
0.900	2	1.999999999614421	3.855786800954775e-10	4.755666607510483	2.755666607510483
1.050	2	1.999999999996986	3.014033467252375e-12	10.577561356069847	8.577561356069847
1.200	2	2.000000000000356	3.557154570899002e-13	18.946793694824862	16.946793694824862
1.350	2	2.000000000000020	2.042810365310288e-14	28.455464395393047	26.455464395393047
1.500	2	1.999999999999990	1.043609643147647e-14	35.323532419514116	33.323532419514116
1.650	2	1.999999999999987	1.310063169057685e-14	32.516687157414310	30.516687157414310
1.800	2	1.999999999999986	1.376676550535194e-14	9.546331217538949	7.546331217538949
1.950	2	1.999999999999986	1.376676550535194e-14	46.091414331622040	48.091414331622040

Table (8) : Numerical results of IBDF2 (δ =1) method and BDF2 method

Step size =0. 03(First case)

$\varepsilon y'' + y' = 0, \ \varepsilon = 0.02$

X	Theoretical solution	Numerical solution of IBDF2(a=b=20) method	Error of IBDF2(a=b=20) method	Numerical solution of BDF2(a=b=20) method	Error of BDF2(a=b=20) method
0.03	1.776869839851570	1.937500000000000	0.160630160148430	2.500000000000000	0.723130160148430
0.30	1.999999694097679	1.962370370282104	0.037629323815576	65.307824844192770	63.307825150095084
0.60	1.999999999999907	1.977255316393915	0.022744683605992	6.442745538550658e+04	6.442545538550658e+04
0.90	2	1.986293499429314	0.013706500570686	6.597142776784954e+07	6.597142576784954e+07
1.20	2	1.991726905575670	0.008273094424330	6.755473984425363e+10	6.755473984225363e+10
1.50	2	1.994974612986215	0.005025387013785	6.917605359838356e+13	6.917605359838156e+13
1.80	2	1.996906458329869	0.003093541670131	7.083627888474251e+16	7.083627888474251e+16
2.10	2	1.998048554068537	0.001951445931463	7.253634957797640e+19	7.253634957797640e+19
2.40	2	1.998717299677139	0.001282700322861	7.427722196784783e+22	7.427722196784783e+22
2.70	2	1.999102502483158	8.974975168420940e-04	7.605987529507615e+25	7.605987529507615e+25
3.00	2	1.999317903697922	6.820963020777171e-04	7.788531230215801e+28	7.788531230215801e+28
3.30	2	1.999431629746284	5.683702537164770e-04	7.975455979740980e+31	7.975455979740980e+31
3.60	2	1.999484477082621	5.155229173794407e-04	8.166866923254760e+34	8.166866923254760e+34
3.90	2	1.999500873785333	4.991262146674114e-04	8.362871729412878e+37	8.362871729412878e+37

Table (9) : Numerical results of IBDF2 ($\delta = 1, \delta_1 = 0.0000001$) method and BDF2 method .

Step size =0.03 (Second case)
$\varepsilon\, y'' + y' = 0,\ \varepsilon = 0.02$

X	Theoretical solution	Numerical solution of IBDF2(a=b=20) method	Error of IBDF2(a=b=20) method	Numerical solution of BDF2(a=b=20) method	Error of BDF2(a=b=20) method
0.03	1.776869839851570	1.967741935483871	0.190872095632301	2.500000000000000	0.723130160148430
0.30	1.999999694097679	1.980266499505787	0.019733194591892	65.307824844192770	63.307825150095084
0.60	1.999999999999907	1.988161078000563	0.011838921999344	6.442745538550658e+04	6.442545538550658e+04
0.90	2	1.992899411326480	0.007100588673520	6.597142776784954e+07	6.597142576784954e+07
1.20	2	1.995742358903052	0.004257641096948	6.755473984425363e+10	6.755473984225363e+10
1.50	2	1.997447581668470	0.002552418331530	6.917605359838356e+13	6.917605359838156e+13
1.80	2	1.998470124514628	0.001529875485372	7.083627888474251e+16	7.083627888474251e+16
2.10	2	1.999083160954544	9.168390454559194e-04	7.253634957797640e+19	7.253634957797640e+19
2.40	2	1.999450620311540	5.493796884599167e-04	7.427722196784783e+22	7.427722196784783e+22
2.70	2	1.999670843172778	3.291568272223433e-04	7.605987529507615e+25	7.605987529507615e+25
3.00	2	1.999802807229115	1.971927708850352e-04	7.788531230215801e+28	7.788531230215801e+28
3.30	2	1.999881874673444	1.181253265556403e-04	7.975455979740980e+31	7.975455979740980e+31
3.60	2	1.999929243857753	7.075614224705262e-05	8.166866923254760e+34	8.166866923254760e+34
3.90	2	1.999957620210428	4.237978957233324e-05	8.362871729412878e+37	8.362871729412878e+37

Table (10) : Numerical results of IBDF2 (δ =0.55) method and BDF2 method .