Periodic Solutions for Nonlinear System of Differential Equations of Third Order with Boundary Conditions

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الملخص

يتضمن البحث دراسة وجود وتقارب الحلول الدورية لنظام من المعادلات التفاضلية اللاخطية من الرتبة الثالثة ذات شروط حدودية، وذلك بالاعتماد على الطريقة التحليلية–العددية لدراسة الحلول الدورية للمعادلات التفاضلية الاعتيادية اللاخطية لـ Samolilenko. كذلك تقودنا هذه الدراسة إلى توسيع الطريقة السابقة، وكذلك توسيع النتائج التي توصل إليها Butris.

ABSTRACT

In this study we investigate the existence and approximation of the periodic solutions for nonlinear system of differential equations of third order with boundary conditions.

The numerical-analytic method has been used to study the periodic solutions of the ordinary differential equations that were introduced by Samoilenko. Also these investigation lead us to the improving and extending the above method and the results of Butris.

1. Introduction

There are many subject in physics and technology using mathematical methods that depends of nonlinear differential equations and boundary value problems and it became clear that the existence and uniqueness of periodic solutions and its algorithm structure from more important problems in the present time $[1,4,6]$.

The periodic solutions for some nonlinear systems of differential equations and boundary value problems have been used to study many problems for example $[2,3,5,7]$.

In this paper we use the above method for investigating the periodic solution for nonlinear system of differential equations of third order with boundary conditions.

Our work is to extend the results of Butris $[3]$. And also this study lead us to the improving and extending the above method.

Consider the following system of nonlinear differential equation, which has the form:

$$\frac{d^3x(t)}{dt^3} = f\left(t, x, \frac{dx}{dt}, \frac{d^2x}{dt^2}\right) , \qquad \dots\dots (1.1)$$

with boundary conditions

$$\left.\begin{aligned} A_1x(0) + C_1x(T) &= d_1 \\ A_2\dot{x}(0) + C_2\dot{x}(T) &= d_2 \\ A_3\ddot{x}(0) + C_3\ddot{x}(T) &= d_3 \end{aligned}\right\} \qquad \dots\dots (1.2)$$

Here $x \in G_1 \subseteq R^n$, $\dot{x} \in G_2 \subseteq R^n$ and $\ddot{x} \in G_3 \subseteq R^n$ where G_1, G_2, G_3 are closed domain subset of Euclidean spaces R^n.

Let the vectors function:

$$f(t,x,\dot{x},\ddot{x}) = \left(f_1(t,x,\dot{x},\ddot{x}), f_2(t,x,\dot{x},\ddot{x}), \dots\dots\dots, f_n(t,x,\dot{x},\ddot{x})\right)$$

where the function $f(t, x, \dot{x}, \ddot{x})$, is defined and continuous on the domain:

$$(t,x,\dot{x},\ddot{x}) \in [0,T] \times G_1 \times G_2 \times G_3 \qquad \dots\dots (1.3)$$

and periodic in t of period T.

when $A_1 = (A_{1ij})$, $A_2 = (A_{2ij})$, $A_3 = (A_{3ij})$, $C_1 = (C_{1ij})$, $C_2 = (C_{2ij})$, $C_3 = (C_{3ij})$ are $(n \times n)$ matrices.

Suppose that the function $f(t,x,\dot{x},\ddot{x})$ satisfies the following inequalities:

$$\left\| f(t,x,\dot{x},\ddot{x}) \right\| \le M \quad , \qquad \dots\dots (1.4)$$

$$\left\| f(t,x_1,\dot{x}_1,\ddot{x}_1) - f(t,x_2,\dot{x}_2,\ddot{x}_2) \right\| \le K_1\|x_1 - x_2\| + K_2\|\dot{x}_1 - \dot{x}_2\| + K_3\|\ddot{x}_1 - \ddot{x}_2\| \quad \dots\dots (1.5)$$

for all $t \in [0,T]$ and $x, x_1, x_2 \in G_1$, $\dot{x}, \dot{x}_1, \dot{x}_2 \in G_2$, $x, \ddot{x}_1, \ddot{x}_2 \in G_3$ where M, K_1, K_2 & K_3 are positive constants.

We define the non-empty sets as follows:

$$\left.\begin{aligned} G_{3f} &= G_3 - M\frac{T}{2} + \beta_3 \\ G_{2f} &= G_2 - M\left(\frac{T}{2}\right)^2 + \beta_2 \\ G_{1f} &= G_1 - M\left(\frac{T}{2}\right)^3 + \beta_1 \end{aligned}\right\} \qquad \dots\dots (1.6)$$

where $\beta_1 = \frac{t}{T}\left\| c_1^{-1} d_1 - \left(c_1^{-1} A_1 + E\right)\ddot{x}_0 \right\|$, $\beta_2 = \frac{t}{T}\left\| c_2^{-1} d_2 - \left(c_2^{-1} A_2 + E\right)\ddot{x}_0 \right\|$,
$\beta_3 = \frac{t}{T}\left\| c_3^{-1} d_3 - \left(c_3^{-1} A_3 + E\right)\ddot{x}_0 \right\|$, E is identity matrixes, and $\|\cdot\| = \max_{t\in[0,T]} |\cdot|$.

Furthermore, we suppose that the greatest eigen-value of the following matrix:

$$\Lambda_0 = \begin{pmatrix} K_1\left(\frac{T}{2}\right)^3 & K_2\left(\frac{T}{2}\right)^3 & K_3\left(\frac{T}{2}\right)^3 \\ K_1\left(\frac{T}{2}\right)^2 & K_2\left(\frac{T}{2}\right)^2 & K_3\left(\frac{T}{2}\right)^2 \\ K_1\left(\frac{T}{2}\right) & K_2\left(\frac{T}{2}\right) & K_3\left(\frac{T}{2}\right) \end{pmatrix}$$

is less than unity. i.e.

$$\lambda_{\max}\left(\Lambda_0\right) < 1 \qquad \ldots\ldots (1.7)$$

Lemma 1:

Let $f(t,x,\dot{x},\ddot{x})$ be continuous vector function in the interval $[0,T]$ then:

$$\left\| \int_0^t \left[f(s,x,\dot{x},\ddot{x}) - \frac{1}{T}\left(\left(c_3^{-1} A_3 + E\right)\ddot{x}_0 - c_3^{-1} d_3 + \int_0^T f(t,x,\dot{x},\ddot{x})dt \right) \right] ds \right\| \le M\alpha(t) + \beta_3$$

Satisfying for $0 \le t \le T$ and $\alpha(t) \le \frac{T}{2}$ where $\alpha(t) = 2t(1-\frac{t}{T})$,
$\beta_3 = \frac{t}{T}\left\| c_3^{-1} d_3 - \left(c_3^{-1} A_3 + E\right)\ddot{x}_0 \right\|$.

proof:

$$\le \left\| \int_0^t f(s,x,\dot{x},\ddot{x}) - \frac{t}{T}\int_0^t f(s,x,\dot{x},\ddot{x})ds + \frac{t}{T}\int_t^T f(t,x,\dot{x},\ddot{x})dt \right\| + \left\| \frac{t}{T}\left[c_3^{-1} d_3 - \left(c_3^{-1} A_3 + E\right)\ddot{x}_0 \right] \right\|$$

$$\le \left(1 - \frac{t}{T}\right)tM + \frac{t}{T}\left(T - t\right)M + \beta_3$$

$$= M\,\frac{T}{2} + \beta_3$$

From lemma 1 we obtain:

$$\left\| Lf\left(t,x,\dot{x},\ddot{x}\right) \right\| \le M\alpha(t) + \beta_3$$

$$\le M\frac{T}{2} + \beta_3 \qquad \ldots\ldots (1.8)$$

$$\left\| L^2 f(t,x,\dot{x},\ddot{x}) \right\| \le \alpha(t) \left\| Lf(t,x,\dot{x},\ddot{x}) \right\|$$
$$\le M\left(\frac{T}{2}\right)^2 + \beta_3 \qquad \text{… … (1.9)}$$

and

$$\left\| L^3 f(t,x,\dot{x},\ddot{x}) \right\| \le \alpha(t) \left\| L^2 f(t,x,\dot{x},\ddot{x}) \right\|$$
$$\le M\left(\frac{T}{2}\right)^3 + \beta_3 \qquad \text{… … (1.10)}$$

2. Approximate Solution

The investigation of approximate solution of the problem (1.1), (1.2) will be introduced by the following theorem:

Theorem 1:

If the system (1.1) with boundary conditions (1.2) defined in the domain (1.3), continuous in t, x and satisfy the inequalities (1.4) and (1.5), then the sequence of functions:

$$x_{m+1}(t,x,\dot{x},\ddot{x}) = x_0 + \dot{x}_0 t + \ddot{x}_0 t^2 + L^3 f(t, x_m(t,x_0), \dot{x}_m(t,x_0), \ddot{x}_m(t,x_0)) \qquad \text{… … (2.1)}$$

with

$$x_0(t,x_0) = x_0 \quad , \quad \frac{dx_m(t,x_0)}{dt} = \dot{x}_m(t,x_0) \quad , \quad \frac{d^2 x_m(t,x_0)}{dt^2} = \ddot{x}_m(t,x_0) \quad , \quad m = 0,1,2,...$$

periodic in t with period T, converges uniformly when $m \to \infty$ on the domain:

$$(t, x_0, \dot{x}_0, \ddot{x}_0) \in [0,T] \times G_1 \times G_2 \times G_3 \qquad \text{… … (2.2)}$$

to the limit function $x^0(t, x_0, \dot{x}_0, \ddot{x}_0)$ which is satisfying the equation:

$$x(t,x,\dot{x},\ddot{x}) = x_0 + \dot{x}_0 t + \ddot{x}_0 t^2 + L^3 f(t,x,\dot{x},\ddot{x}) \ , \qquad \text{… … (2.3)}$$

And it's a unique solution of (1.1) provided that:

$$\left\| x_\infty(t, x_0, \dot{x}_0, \ddot{x}_0) - x_0 \right\| \le M\left(\frac{T}{2}\right)^3 + \beta_3 \qquad \text{… … (2.4)}$$

$$\left\| \dot{x}_\infty(t, x_0, \dot{x}_0, \ddot{x}_0) - \dot{x}_0 \right\| \le M\left(\frac{T}{2}\right)^2 + \beta_3 \qquad \text{… … (2.5)}$$

$$\left\| \ddot{x}_\infty(t, x_0, \dot{x}_0, \ddot{x}_0) - \ddot{x}_0 \right\| \le M\frac{T}{2} + \beta_3 \qquad \text{… … (2.6)}$$

$$\left.\begin{array}{l} \left\| x_\infty(t, x_0, \dot{x}_0, \ddot{x}_0) - x_m(t, x_0, \dot{x}_0, \ddot{x}_0) \right\| \\ \left\| \dot{x}_\infty(t, x_0, \dot{x}_0, \ddot{x}_0) - \dot{x}_m(t, x_0, \dot{x}_0, \ddot{x}_0) \right\| \\ \left\| \ddot{x}_\infty(t, x_0, \dot{x}_0, \ddot{x}_0) - \ddot{x}_m(t, x_0, \dot{x}_0, \ddot{x}_0) \right\| \end{array}\right\} \le \Lambda_0^m (E - \Lambda)^{-1} V_0 \qquad \text{… … (2.7)}$$

for all $t \in [0,T]$, $x_0 \in G_{1f}$, $\dot{x}_0 \in G_{2f}$, $\ddot{x}_0 \in G_{3f}$ where

$$V_0(t,x_0,y_0,z_0)=\begin{pmatrix}\|x_1(t,x_0,\dot{x}_0,\ddot{x}_0)-x_0\|\\ \|\dot{x}_1(t,x_0,\dot{x}_0,\ddot{x}_0)-\dot{x}_0\|\\ \|\ddot{x}_1(t,x_0,\dot{x}_0,\ddot{x}_0)-\ddot{x}_0\|\end{pmatrix}$$

Proof:

Setting m=0 in (2.1) and using lemma 1 and the sequence of the functions (2.1) we get:

$$\|x_1(t,x_0,\dot{x}_0,\ddot{x}_0)-x_0\|=\|L^3 f(t,x_0(t,x_0),\dot{x}_0(t,x_0),\ddot{x}_0(t,x_0))\|$$

$$\leq M\left(\frac{T}{2}\right)^3+\beta_3 \qquad \dots\dots (2.8)$$

That is $x_1(t,x_0,\dot{x}_0,\ddot{x}_0)\in G_1$, for all $t\in[0,T]$, $x_0\in G_{1f}$.

By induction we have:

$$\|x_m(t,x_0,\dot{x}_0,\ddot{x}_0)-x_0\|=\|L^3 f(t,x_{m-1}(t,x_0),\dot{x}_{m-1}(t,x_0),\ddot{x}_{m-1}(t,x_0))\|$$

$$\leq M\left(\frac{T}{2}\right)^3+\beta_3 \qquad \dots\dots (2.9)$$

where $x_m(t,x_0,\dot{x}_0,\ddot{x}_0)\in G_1$, for all $t\in[0,T]$, $x_0\in G_{1f}$.

By derivative the equation (2.1), and setting m=0, we have:

$$\|\dot{x}_1(t,x_0,\dot{x}_0,\ddot{x}_0)-\dot{x}_0\|=\|L^2 f(t,x_0(t,x_0),\dot{x}_0(t,x_0),\ddot{x}_0(t,x_0))\|$$

$$\leq M\left(\frac{T}{2}\right)^2+\beta_3 \qquad \dots\dots (2.10)$$

That is $\dot{x}_1(t,x_0,\dot{x}_0,\ddot{x}_0)\in G_2$, for all $t\in[0,T]$, $\dot{x}_0\in G_{2f}$.

By mathematical induction we have:

$$\|\dot{x}_m(t,x_0,\dot{x}_0,\ddot{x}_0)-\dot{x}_0\|=\|L^2 f(t,x_{m-1}(t,x_0),\dot{x}_{m-1}(t,x_0),\ddot{x}_{m-1}(t,x_0))\|$$

$$\leq M\left(\frac{T}{2}\right)^2+\beta_3 \qquad \dots\dots (2.11)$$

that is $\dot{x}_m(t,x_0,\dot{x}_0,\ddot{x}_0)\in G_2$, for all $t\in[0,T]$, $\dot{x}_0\in G_{2f}$.

By using the second derivative of (2.1) and setting m=0, we get:

$$\|\ddot{x}_1(t,x_0,\dot{x}_0,\ddot{x}_0)-\ddot{x}_0\|=\|Lf(t,x_0(t,x_0),\dot{x}_0(t,x_0),\ddot{x}_0(t,x_0))\|$$

$$\leq M\frac{T}{2}+\beta_3 \qquad \dots\dots (2.12)$$

That is $\ddot{x}_1(t,x_0,\dot{x}_0,\ddot{x}_0)\in G_3$, for all $t\in[0,T]$, $\ddot{x}_0\in G_{3f}$.

Also by induction we have:

$$\left\|\ddot{x}_m\left(t,x_0,\dot{x}_0,\ddot{x}_0\right)-\ddot{x}_0\right\|=\left\|Lf\left(t,x_{m-1}(t,x_0),\dot{x}_{m-1}(t,x_0),\ddot{x}_{m-1}(t,x_0)\right)\right\|$$

$$\leq M\frac{T}{2}+\beta_3 \qquad \ldots\ldots (2.13)$$

That is $\ddot{x}_m(t,x_0,\dot{x}_0,\ddot{x}_0)\in G_3$, for all $t\in[0,T]$, $\ddot{x}_0\in G_{3f}$.

We prove that the sequence of functions (2.1) is uniformly convergent in (2.2). From (2.1), when m=1 we get:

$$\begin{pmatrix}\|x_2(t,x_0,\dot{x}_0,\ddot{x}_0)-x_1(t,x_0,\dot{x}_0,\ddot{x}_0)\|\\ \|\dot{x}_2(t,x_0,\dot{x}_0,\ddot{x}_0)-\dot{x}_1(t,x_0,\dot{x}_0,\ddot{x}_0)\|\\ \|\ddot{x}_2(t,x_0,\dot{x}_0,\ddot{x}_0)-\ddot{x}_1(t,x_0,\dot{x}_0,\ddot{x}_0)\|\end{pmatrix}\leq\Lambda_0(t)\begin{pmatrix}\|x_1(t,x_0,\dot{x}_0,\ddot{x}_0)-x_0\|\\ \|\dot{x}_1(t,x_0,\dot{x}_0,\ddot{x}_0)-\dot{x}_0\|\\ \|\ddot{x}_1(t,x_0,\dot{x}_0,\ddot{x}_0)-\ddot{x}_0\|\end{pmatrix}$$

Now when m=2 we get the following:

$$\begin{pmatrix}\|x_3(t,x_0,\dot{x}_0,\ddot{x}_0)-x_2(t,x_0,\dot{x}_0,\ddot{x}_0)\|\\ \|\dot{x}_3(t,x_0,\dot{x}_0,\ddot{x}_0)-\dot{x}_2(t,x_0,\dot{x}_0,\ddot{x}_0)\|\\ \|\ddot{x}_3(t,x_0,\dot{x}_0,\ddot{x}_0)-\ddot{x}_2(t,x_0,\dot{x}_0,\ddot{x}_0)\|\end{pmatrix}\leq\Lambda_0^2(t)\begin{pmatrix}\|x_2(t,x_0,\dot{x}_0,\ddot{x}_0)-x_1(t,x_0,\dot{x}_0,\ddot{x}_0)\|\\ \|\dot{x}_2(t,x_0,\dot{x}_0,\ddot{x}_0)-\dot{x}_1(t,x_0,\dot{x}_0,\ddot{x}_0)\|\\ \|\ddot{x}_2(t,x_0,\dot{x}_0,\ddot{x}_0)-\ddot{x}_1(t,x_0,\dot{x}_0,\ddot{x}_0)\|\end{pmatrix}$$

By mathematical induction we have:

$$\begin{pmatrix}\|x_{m+1}(t,x_0,\dot{x}_0,\ddot{x}_0)-x_m(t,x_0,\dot{x}_0,\ddot{x}_0)\|\\ \|\dot{x}_{m+1}(t,x_0,\dot{x}_0,\ddot{x}_0)-\dot{x}_m(t,x_0,\dot{x}_0,\ddot{x}_0)\|\\ \|\ddot{x}_{m+1}(t,x_0,\dot{x}_0,\ddot{x}_0)-\ddot{x}_m(t,x_0,\dot{x}_0,\ddot{x}_0)\|\end{pmatrix}\leq\Lambda_0^m(t)\begin{pmatrix}\|x_m(t,x_0,\dot{x}_0,\ddot{x}_0)-x_{m-1}(t,x_0,\dot{x}_0,\ddot{x}_0)\|\\ \|\dot{x}_m(t,x_0,\dot{x}_0,\ddot{x}_0)-\dot{x}_{m-1}(t,x_0,\dot{x}_0,\ddot{x}_0)\|\\ \|\ddot{x}_m(t,x_0,\dot{x}_0,\ddot{x}_0)-\ddot{x}_{m-1}(t,x_0,\dot{x}_0,\ddot{x}_0)\|\end{pmatrix}$$

$$\ldots\ldots (2.14)$$

for m = 1,2,... .

Rewrite the inequalities (2.14), in vector from:

$$V_{m+1}(t)\leq\Lambda_0^m(t)V_m(t) \qquad \ldots\ldots (2.15)$$

where

$$V_{m+1}(t,x_0,\dot{x}_0,\ddot{x}_0)=\begin{pmatrix}\|x_{m+1}(t,x_0,\dot{x}_0,\ddot{x}_0)-x_m(t,x_0,\dot{x}_0,\ddot{x}_0)\|\\ \|\dot{x}_{m+1}(t,x_0,\dot{x}_0,\ddot{x}_0)-\dot{x}_m(t,x_0,\dot{x}_0,\ddot{x}_0)\|\\ \|\ddot{x}_{m+1}(t,x_0,\dot{x}_0,\ddot{x}_0)-\ddot{x}_m(t,x_0,\dot{x}_0,\ddot{x}_0)\|\end{pmatrix}$$

$$\Lambda_0(t)=\begin{pmatrix}K_1(\alpha(t))^3 & K_2(\alpha(t))^3 & K_3(\alpha(t))^3\\ K_1(\alpha(t))^2 & K_2(\alpha(t))^2 & K_3(\alpha(t))^2\\ K_1(\alpha(t)) & K_2(\alpha(t)) & K_3(\alpha(t))\end{pmatrix}$$

$$V_m(t,x_0,\dot{x}_0,\ddot{x}_0)=\begin{pmatrix}\|x_m(t,x_0,\dot{x}_0,\ddot{x}_0)-x_{m-1}(t,x_0,\dot{x}_0,\ddot{x}_0)\|\\ \|\dot{x}_m(t,x_0,\dot{x}_0,\ddot{x}_0)-\dot{x}_{m-1}(t,x_0,\dot{x}_0,\ddot{x}_0)\|\\ \|\ddot{x}_m(t,x_0,\dot{x}_0,\ddot{x}_0)-\ddot{x}_{m-1}(t,x_0,\dot{x}_0,\ddot{x}_0)\|\end{pmatrix}$$

It follows from inequality (2.15) that:

$$V_{m+1} \le \Lambda_0^m V_m \qquad \dots \dots (2.16)$$

where

$$\Lambda_0 = \max_{t\in[0,T]} \Lambda_0(t)$$

this leads to the estimation:

$$\sum_{i=1}^{m} V_i \le \sum_{i=1}^{m} \Lambda_0^{i-1} V_0 \qquad \dots \dots (2.18)$$

Since the matrix Λ_0 has eigen-values $\lambda_1 = 0$, $\lambda_2 = 0$ and $\lambda_3 = \lambda_{\max}(\Lambda_0) = \left(\frac{T}{2}\right)^3 K_1 - \left(\frac{T}{2}\right)^2 K_2 - \frac{T}{2} K_3$

then the series (2.18) is uniformly convergent, i.e.

$$\lim_{m\to\infty} \sum_{i=1}^{m} \Lambda_0^{i-1} V_0 = \sum_{i=1}^{\infty} \Lambda_0^{i} V_0 = (E - \Lambda_0)^{-1} V_0 \qquad \dots \dots (2.19)$$

The limiting relation (2.19) signifies a uniform convergent of the sequence $x_m(t,x_0,\dot{x}_0,\ddot{x}_0)$, $\dot{x}_m(t,x_0,\dot{x}_0,\ddot{x}_0)$, $\ddot{x}_m(t,x_0,\dot{x}_0,\ddot{x}_0)$

$$\left.\begin{aligned} &\lim_{m\to\infty} x_m(t,x_0,\dot{x}_0,\ddot{x}_0) = x_\infty(t,x_0,\dot{x}_0,\ddot{x}_0) \\ &\lim_{m\to\infty} \dot{x}_m(t,x_0,\dot{x}_0,\ddot{x}_0) = \dot{x}_\infty(t,x_0,\dot{x}_0,\ddot{x}_0) \\ &\lim_{m\to\infty} \ddot{x}_m(t,x_0,\dot{x}_0,\ddot{x}_0) = \ddot{x}_\infty(t,x_0,\dot{x}_0,\ddot{x}_0) \end{aligned}\right\} \qquad \dots \dots (2.20)$$

By inequality (2.20), the estimation:

$$\begin{pmatrix} \|x_\infty(t,x_0,\dot{x}_0,\ddot{x}_0) - x_m(t,x_0,\dot{x}_0,\ddot{x}_0)\| \\ \|\dot{x}_\infty(t,x_0,\dot{x}_0,\ddot{x}_0) - \dot{x}_m(t,x_0,\dot{x}_0,\ddot{x}_0)\| \\ \|\ddot{x}_\infty(t,x_0,\dot{x}_0,\ddot{x}_0) - \ddot{x}_m(t,x_0,\dot{x}_0,\ddot{x}_0)\| \end{pmatrix} \le \Lambda_0^m (E - \Lambda_0)^{-1} V_0 \qquad \dots \dots (2.21)$$

is true for m=1,2,3,…

Thus $x_\infty(t,x_0,\dot{x}_0,\ddot{x}_0)$ is a solution of differential equations (1.1).

3. Uniqueness solution

The study of the uniqueness solution of the problem (1.1), (1.2) will be introduced by the following:

Theorem 2:

Let all assumptions and conditions of theorem 1 be given then the problem (1.1), (1.2) has a unique solution $x = x_\infty(t,x_0,\dot{x}_0,\ddot{x}_0)$ on the domain (2.2).

Proof:

We have to show to that $x(t,x_0,\dot{x}_0,\ddot{x}_0)$ is a unique solution of problem (1.1), (1.2). On the contrary, we suppose that there is at least two different solutions $x(t,x_0,\dot{x}_0,\ddot{x}_0)$ and $\hat{x}(t,x_0,\dot{x}_0,\ddot{x}_0)$ of the problem (1.1), (1.2).

From (2.3) the following inequalities are holds:

$$\left\|x(t,x_0,\dot{x}_0,\ddot{x}_0)-\hat{x}(t,x_0,\dot{x}_0,\ddot{x}_0)\right\| \le \left(\frac{T}{2}\right)^3 \left(K_1\left\|x(t)-\hat{x}(t)\right\| + K_2\left\|\dot{x}(t)-\dot{\hat{x}}(t)\right\| + K_3\left\|\ddot{x}(t)-\ddot{\hat{x}}(t)\right\|\right) \qquad \ldots\ldots (3.1)$$

on differentiating (3.1) we should also obtain:

$$\left\|\dot{x}(t,x_0,\dot{x}_0,\ddot{x}_0)-\dot{\hat{x}}(t,x_0,\dot{x}_0,\ddot{x}_0)\right\| \le \left(\frac{T}{2}\right)^2 \left(K_1\left\|x(t)-\hat{x}(t)\right\| + K_2\left\|\dot{x}(t)-\dot{\hat{x}}(t)\right\| + K_3\left\|\ddot{x}(t)-\ddot{\hat{x}}(t)\right\|\right) \qquad \ldots\ldots (3.2)$$

on differentiating (3.2) we should also obtain:

$$\left\|\ddot{x}(t,x_0,\dot{x}_0,\ddot{x}_0)-\ddot{\hat{x}}(t,x_0,\dot{x}_0,\ddot{x}_0)\right\| \le \frac{T}{2} \left(K_1\left\|x(t)-\hat{x}(t)\right\| + K_2\left\|\dot{x}(t)-\dot{\hat{x}}(t)\right\| + K_3\left\|\ddot{x}(t)-\ddot{\hat{x}}(t)\right\|\right) \qquad \ldots\ldots (3.3)$$

Inequalities (3.1), (3.2) and (3.3) would lead to the estimation:

$$\begin{pmatrix} \left\|x(t,x_0,\dot{x}_0,\ddot{x}_0)-\hat{x}(t,x_0,\dot{x}_0,\ddot{x}_0)\right\| \\ \left\|\dot{x}(t,x_0,\dot{x}_0,\ddot{x}_0)-\dot{\hat{x}}(t,x_0,\dot{x}_0,\ddot{x}_0)\right\| \\ \left\|\ddot{x}(t,x_0,\dot{x}_0,\ddot{x}_0)-\ddot{\hat{x}}(t,x_0,\dot{x}_0,\ddot{x}_0)\right\| \end{pmatrix} \le \Lambda_0 \begin{pmatrix} \left\|x(t,x_0,\dot{x}_0,\ddot{x}_0)-\hat{x}(t,x_0,\dot{x}_0,\ddot{x}_0)\right\| \\ \left\|\dot{x}(t,x_0,\dot{x}_0,\ddot{x}_0)-\dot{\hat{x}}(t,x_0,\dot{x}_0,\ddot{x}_0)\right\| \\ \left\|\ddot{x}(t,x_0,\dot{x}_0,\ddot{x}_0)-\ddot{\hat{x}}(t,x_0,\dot{x}_0,\ddot{x}_0)\right\| \end{pmatrix}$$

By iterating we should find that:

$$\begin{pmatrix} \left\|x(t,x_0,\dot{x}_0,\ddot{x}_0)-\hat{x}(t,x_0,\dot{x}_0,\ddot{x}_0)\right\| \\ \left\|\dot{x}(t,x_0,\dot{x}_0,\ddot{x}_0)-\dot{\hat{x}}(t,x_0,\dot{x}_0,\ddot{x}_0)\right\| \\ \left\|\ddot{x}(t,x_0,\dot{x}_0,\ddot{x}_0)-\ddot{\hat{x}}(t,x_0,\dot{x}_0,\ddot{x}_0)\right\| \end{pmatrix} \le \Lambda_0^m \begin{pmatrix} \left\|x(t,x_0,\dot{x}_0,\ddot{x}_0)-\hat{x}(t,x_0,\dot{x}_0,\ddot{x}_0)\right\| \\ \left\|\dot{x}(t,x_0,\dot{x}_0,\ddot{x}_0)-\dot{\hat{x}}(t,x_0,\dot{x}_0,\ddot{x}_0)\right\| \\ \left\|\ddot{x}(t,x_0,\dot{x}_0,\ddot{x}_0)-\ddot{\hat{x}}(t,x_0,\dot{x}_0,\ddot{x}_0)\right\| \end{pmatrix}$$

But $\Lambda_0^m \to 0$ as $m \to \infty$, hence proceeding in the last inequality to the limit we should obtain the equalities $x(t,x_0,\dot{x}_0,\ddot{x}_0)=\hat{x}(t,x_0,\dot{x}_0,\ddot{x}_0)$, $\dot{x}(t,x_0,\dot{x}_0,\ddot{x}_0)=\dot{\hat{x}}(t,x_0,\dot{x}_0,\ddot{x}_0)$ and $\ddot{x}(t,x_0,\dot{x}_0,\ddot{x}_0)=\ddot{\hat{x}}(t,x_0,\dot{x}_0,\ddot{x}_0)$ which proves the solution is a unique and this completes the proof of the theorem.

4. Existence of Solution

The problem of existence solution of the system (1.1) is uniquely connected with the existence of zeros of the function $\Delta(x_0)$, which has the form:-

$$\Delta(x_0)=\frac{1}{T}\left[\left(c_1^{-1}A_1+E\right)x_0+\left(c_1^{-1}A_1+E\right)\dot{x}_0t+\left(c_1^{-1}A_1+E\right)\ddot{x}_0t^2-c_1^{-1}d_1+\int_0^T L^2 f\left(t,x_\infty,\dot{x}_\infty,\ddot{x}_\infty\right)dt\right]$$

… … (4.1)

Since this function is approximately determined from the sequence of functions:

$$\Delta_m(x_0)=\frac{1}{T}\left[\left(c_1^{-1}A_1+E\right)x_0+\left(c_1^{-1}A_1+E\right)\dot{x}_0t+\left(c_1^{-1}A_1+E\right)\ddot{x}_0t^2-c_1^{-1}d_1+\int_0^T L^2 f\left(t,x_m,\dot{x}_m,\ddot{x}_m\right)dt\right]$$

… … (4.2)

m=0,1,2,… .

Theorem 3:

Let all assumptions and conditions of theorem 1 were given, then the following inequality:

$$\left\|\Delta(x_0)-\Delta_m(x_0)\right\|\le\left\langle\begin{pmatrix}K_1\left(\frac{T}{2}\right)^2\\ K_2\left(\frac{T}{2}\right)^2\\ K_3\left(\frac{T}{2}\right)^2\end{pmatrix},\ \Lambda_0^m(E-\Lambda_0)^{-1}V_0\right\rangle=\delta_m \qquad \text{… … (4.3)}$$

is holds for all $m\ge 1$ and $x_0\in G_N$.

Proof:

According to (4.1) and (4.2) we have:

$$\left\|\Delta_1(x_0)-\Delta_m(x_0)\right\|=\left\|\frac{1}{T}\left[\left(c_1^{-1}A_1+E\right)x_0+\left(c_1^{-1}A_1+E\right)\dot{x}_0t+\left(c_1^{-1}A_1+E\right)\ddot{x}_0t^2-c_1^{-1}d_1+\int_0^T L^2 f\left(t,x_\infty,\dot{x}_\infty,\ddot{x}_\infty\right)dt\right]-\right.$$

$$\left.-\frac{1}{T}\left[\left(c_1^{-1}A_1+E\right)x_0+\left(c_1^{-1}A_1+E\right)\dot{x}_0t+\left(c_1^{-1}A_1+E\right)\ddot{x}_0t^2-c_1^{-1}d_1+\int_0^T L^2 f\left(t,x_m,\dot{x}_m,\ddot{x}_m\right)dt\right]\right\|$$

$$=\frac{1}{T}\left\|\left[\left(c_1^{-1}A_1+E\right)x_0+\left(c_1^{-1}A_1+E\right)\dot{x}_0t+\left(c_1^{-1}A_1+E\right)\ddot{x}_0t^2-c_1^{-1}d_1+\int_0^T L^2 f\left(t,x_\infty,\dot{x}_\infty,\ddot{x}_\infty\right)dt-\right.\right.$$

$$\left.\left.-\left(c_1^{-1}A_1+E\right)x_0-\left(c_1^{-1}A_1+E\right)\dot{x}_0t-\left(c_1^{-1}A_1+E\right)\ddot{x}_0t^2+c_1^{-1}d_1-\int_0^T L^2 f\left(t,x_m,\dot{x}_m,\ddot{x}_m\right)dt\right]\right\|$$

$$=\frac{1}{T}\int_0^T\left\|L^2 f\left(t,x_\infty,\dot{x}_\infty,\ddot{x}_\infty\right)-L^2 f\left(t,x_m,\dot{x}_m,\ddot{x}_m\right)\right\|dt$$

$$\le\left(\alpha(t)\right)^2\left[K_1\left\|x_\infty(t,x_0)-x_m(t,x_0)\right\|+K_2\left\|\dot{x}_\infty(t,x_0)-\dot{x}_m(t,x_0)\right\|+\right.$$

$$\left.+K_3\left\|\ddot{x}_\infty(t,x_0)-\ddot{x}_m(t,x_0)\right\|\right]$$

$$\leq \left(\frac{T}{2}\right)^2 \left[K_1 \left\|x_\infty(t,x_0) - x_m(t,x_0)\right\| + K_2 \left\|\dot{x}_\infty(t,x_0) - \dot{x}_m(t,x_0)\right\| + \right.$$
$$\left. + K_3 \left\|\ddot{x}_\infty(t,x_0) - \ddot{x}_m(t,x_0)\right\|\right]$$
$$\leq \left\langle \begin{pmatrix} K_1\left(\frac{T}{2}\right)^2 \\ K_2\left(\frac{T}{2}\right)^2 \\ K_3\left(\frac{T}{2}\right)^2 \end{pmatrix}, \ \Lambda_0^m (E-\Lambda_0)^{-1} V_0 \right\rangle = \delta_m$$

where $\langle \ , \ \rangle$ denotes the scalar product in the space R^2 and K_1, K_2, K_3 , δ_m are a positive constants.

Remark 1: [7]

When $R^n = R^1$, i.e. when x is a scalar, theorem 4 can be strengthens by giving up the requirement that the singular point should be isolated, thus we have.

Theorem 4:

Let the system of equations (1.1) be defined on the domain (1.3). Suppose that for $m \geq 0$, the function $\Delta_m(x_0)$ defined according to formula (4.2) satisfies the inequalities:

$$\left.\begin{array}{l} \min\limits_{a+M\left(\frac{T}{2}\right)^3+\beta_3 \leq x_0 \leq b-M\left(\frac{T}{2}\right)^3+\beta_3} \Delta_m(x_0) \leq -\delta_m \\ \max\limits_{a+M\left(\frac{T}{2}\right)^3+\beta_3 \leq x_0 \leq b-M\left(\frac{T}{2}\right)^3+\beta_3} \Delta_m(x_0) \geq \delta_m \end{array}\right\} \quad \ldots\ldots (4.5)$$

for $m \geq 0$, when $\beta_3 = \frac{t}{T}\left\|c_3^{-1} d_3 - \left(c_3^{-1} A_3 + E\right)\ddot{x}_0\right\|$, M, β_3, δ_m are positive constant. Then (1.1), has a periodic solution $x = x(t)$ such that $a + M\left(\frac{T}{2}\right)^3 + \beta_3 \leq x_0 \leq b - M\left(\frac{T}{2}\right)^3 + \beta_3$.

Proof:

Let x_1, x_2 be any point in the interval $a+M\left(\frac{T}{2}\right)^3 + \beta_3 \leq x_0 \leq b - M\left(\frac{T}{2}\right)^3 + \beta_3$ such that:

$$\left.\begin{aligned}\Delta_m(x_1) &= \min_{a+M\left(\frac{T}{2}\right)^3+\beta_3\le x_0\le b-M\left(\frac{T}{2}\right)^3+\beta_3} \Delta_m(x_0)\\ \Delta_m(x_2) &= \max_{a+M\left(\frac{T}{2}\right)^3+\beta_3\le x_0\le b-M\left(\frac{T}{2}\right)^3+\beta_3} \Delta_m(x_0)\end{aligned}\right\} \quad \ldots\ldots (4.6)$$

By using the inequalities (4.4) and (4.5) we have:

$$\left.\begin{aligned}\Delta(x_1) &= \Delta_m(x_1) + \left(\Delta(x_1) - \Delta_m(x_1)\right) < 0\\ \Delta(x_2) &= \Delta_m(x_2) + \left(\Delta(x_2) - \Delta_m(x_2)\right) > 0\end{aligned}\right\} \quad \ldots\ldots (4.7)$$

From the continuity of (4.1) and (4.2) there exists a point $x^0 = x_0$, $x^0 \in [x_1, x_2]$ such that $\Delta^0(x_0) \equiv 0$.

thus $x = x(t, x_0)$ is a periodic solution for $x_0 \in \left[a + M\left(\frac{T}{2}\right)^3 + \beta_3, b - M\left(\frac{T}{2}\right)^3 + \beta_3\right]$.

Remark 2: [2]

If the set G_{3f} dose not degenerate to a point, then the $\Delta-$constant of the system (3.2) may be considered as the function $\Delta = \Delta(0, x_0)$ given on the set $R^1 \times G_{3f}$. The properties are defined by:

Theorem 5:

Let

$\Delta : D_{3f} \to R^n$,

$$\Delta(x_0) = \frac{1}{T}\left[\left(c_1^{-1}A_1 + E\right)x_0 + \left(c_1^{-1}A_1 + E\right)\dot{x}_0 t + \left(c_1^{-1}A_1 + E\right)\ddot{x}_0 t^2 - c_1^{-1}d_1 + \int_0^T L^2 f\left(t, x_\infty, \dot{x}_\infty, \ddot{x}_\infty\right)dt\right] \quad \ldots\ldots (4.8)$$

where $x_\infty(t, x_0)$ is the limit of a sequence of periodic functions (2.1), then the following inequalities are holds:

$$\|\Delta(x_0)\| \le \xi + \left(\frac{T}{2}\right)^2 M \quad \ldots\ldots (4.9)$$

where $\xi = \frac{1}{T}\left\|\left(c_1^{-1}A_1 + E\right)x_0 + \left(c_1^{-1}A_1 + E\right)\dot{x}_0 t + \left(c_1^{-1}A_1 + E\right)\ddot{x}_0 t^2 - c_1^{-1}d_1\right\|$, and

$$\left\|\Delta(x_0^1) - \Delta(x_0^2)\right\| \le \left(\frac{T}{2}\right)^2\left[K_1\left(W_1 + W_2\frac{U_5 Q_3}{W_5}\right) + K_2\left(W_3 + W_4 B_2\frac{U_5 Q_3}{W_5}\right) + K_3\left(\frac{U_5 Q_3}{W_5}\right)\right] \quad \ldots\ldots (4.10)$$

for all $x_0, x_0^1, x_0^2 \in G_{3f}$ and $U_1 = \dfrac{1}{\left(1-\left(\frac{T}{2}\right)^3 K_1\right)}$, $U_2 = \dfrac{\left(\frac{T}{2}\right)^3}{\left(1-\left(\frac{T}{2}\right)^3 K_1\right)}$,

$U_3 = \dfrac{1}{\left(1-\left(\frac{T}{2}\right)^2 K_2\right)}$, $U_4 = \dfrac{\left(\frac{T}{2}\right)^2}{\left(1-\left(\frac{T}{2}\right)^2 K_2\right)}$, $U_5 = \dfrac{1}{\left(1-\frac{T}{2}K_1\right)}$, $U_6 = \dfrac{\frac{T}{2}}{\left(1-\frac{T}{2}K_1\right)}$

$Q_1 = \|x_0^1 - x_0^2\| + \|\dot{x}_0^1 - \dot{x}_0^2\| t + \|\ddot{x}_0^1 - \ddot{x}_0^2\| t^2$, $Q_2 = \|\dot{x}_0^1 - \dot{x}_0^2\| + \|\ddot{x}_0^1 - \ddot{x}_0^2\| t$, $Q_3 = \|\ddot{x}_0^1 - \ddot{x}_0^2\|$,

$W_1 = \dfrac{U_1 Q_1}{1-U_2K_2U_4K_1} + \dfrac{U_2K_2U_4Q_2}{1-U_2K_2U_4K_1}$, $W_2 = \dfrac{1+K_2U_4}{1-U_2K_2U_4K_1}$, $W_3 = U_3Q_2 + U_4K_1W_1$,

$W_4 = (K_1W_2 + K_3)$, and $W_5 = 1 - [U_6K_1(W_1+W_2) + U_6K_2(W_3+W_4U_4)]$.

Proof:

From the properties to the function $x_\infty(t, x_0)$ established by theorem 1, it follows that the function $\Delta(x_0)$ is continuous and bounded in the domain $R^1 \times G_{3f}$.

$$\begin{aligned}\|\Delta(x_0)\| &= \left\| \frac{1}{T}\left[(c_1^{-1}A_1 + E)x_0 + (c_1^{-1}A_1 + E)\dot{x}_0 t + (c_1^{-1}A_1 + E)\ddot{x}_0 t^2 - c_1^{-1}d_1 + \int_0^T L^2 f(t, x_\infty, \dot{x}_\infty, \ddot{x}_\infty)dt \right]\right\| \\ &\le \frac{1}{T}\left\|(c_1^{-1}A_1 + E)x_0 + (c_1^{-1}A_1 + E)\dot{x}_0 t + (c_1^{-1}A_1 + E)\ddot{x}_0 t^2 - c_1^{-1}d_1\right\| + \frac{1}{T}\int_0^T \left\|L^2 f(t, x_\infty, \dot{x}_\infty, \ddot{x}_\infty)\right\|dt \\ &\le \xi + (\alpha(t))^2 M \\ &\le \xi + \left(\frac{T}{2}\right)^2 M\end{aligned}$$

By using (4.8), we have:

$$\begin{aligned}\|\Delta(x_0^1) - \Delta(x_0^2)\| \le \left(\frac{T}{2}\right)^2 \Big[K_1 \|x_\infty(t, x_0^1) - x_\infty(t, x_0^2)\| + K_2 \|\dot{x}_\infty(t, x_0^1) - \dot{x}_\infty(t, x_0^2)\| + \\ + K_3 \|\ddot{x}_\infty(t, x_0^1) - \ddot{x}_\infty(t, x_0^2)\| \Big] \quad \ldots \ldots (4.11)\end{aligned}$$

where $x_\infty(t, x_0^1)$ and $x_\infty(t, x_0^2)$ are the solutions of the integral equation:

$$x(t, x_0^k, \dot{x}_0^k, \ddot{x}_0^k) = x_0^k + \dot{x}_0^k t + \ddot{x}_0^k t^2 + L^3 f(t, x(t, x_0^k), \dot{x}(t, x_0^k), \ddot{x}(t, x_0^k)) \quad \ldots \ldots (4.12)$$

where k=1,2.

From (4.12), we find that:

$$\begin{aligned}\|x(t, x_0^1, \dot{x}_0^1, \ddot{x}_0^1) - x(t, x_0^2, \dot{x}_0^2, \ddot{x}_0^2)\| \le \|x_0^1 - x_0^2\| + \|\dot{x}_0^1 - \dot{x}_0^2\| t + \|\ddot{x}_0^1 - \ddot{x}_0^2\| t^2 + \left(\frac{T}{2}\right)^3 \Big[K_1 \|x_\infty(t, x_0^1) - x_\infty(t, x_0^2)\| + \\ + K_2 \|\dot{x}_\infty(t, x_0^1) - \dot{x}_\infty(t, x_0^2)\| + K_3 \|\ddot{x}_\infty(t, x_0^1) - \ddot{x}_\infty(t, x_0^2)\| \Big]\end{aligned}$$

$$\left(1-\left(\frac{T}{2}\right)^3 K_1\right)\left\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\right\| \le \left\|x_0^1-x_0^2\right\|+\left\|\dot{x}_0^1-\dot{x}_0^2\right\|t+\left\|\ddot{x}_0^1-\ddot{x}_0^2\right\|t^2+$$

$$+\left(\frac{T}{2}\right)^3\left[K_2\left\|\dot{x}_\infty(t,x_0^1)-\dot{x}_\infty(t,x_0^2)\right\|+K_3\left\|\ddot{x}_\infty(t,x_0^1)-\ddot{x}_\infty(t,x_0^2)\right\|\right]$$

and

$$\left\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\right\| \le \frac{1}{\left(1-\left(\frac{T}{2}\right)^3 K_1\right)}\left[\left\|x_0^1-x_0^2\right\|+\left\|\dot{x}_0^1-\dot{x}_0^2\right\|t+\left\|\ddot{x}_0^1-\ddot{x}_0^2\right\|t^2\right]+$$

$$+\frac{\left(\frac{T}{2}\right)^3}{\left(1-\left(\frac{T}{2}\right)^3 K_1\right)}\left[K_2\left\|\dot{x}_\infty(t,x_0^1)-\dot{x}_\infty(t,x_0^2)\right\|+K_3\left\|\ddot{x}_\infty(t,x_0^1)-\ddot{x}_\infty(t,x_0^2)\right\|\right]$$

$$\left\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\right\| \le U_1Q_1+U_2\left[K_2\left\|\dot{x}_\infty(t,x_0^1)-\dot{x}_\infty(t,x_0^2)\right\|+K_3\left\|\ddot{x}_\infty(t,x_0^1)-\ddot{x}_\infty(t,x_0^2)\right\|\right] \quad \ldots\ldots (4.13)$$

On differentiating $x_\infty(t,x_0^1)$ and $x_\infty(t,x_0^2)$, we get:

$$\left\|\dot{x}_\infty(t,x_0^1)-\dot{x}_\infty(t,x_0^2)\right\| \le \frac{1}{\left(1-\left(\frac{T}{2}\right)^2 K_2\right)}\left[\left\|\dot{x}_0^1-\dot{x}_0^2\right\|+\left\|\ddot{x}_0^1-\ddot{x}_0^2\right\|t\right]+$$

$$+\frac{\left(\frac{T}{2}\right)^2}{\left(1-\left(\frac{T}{2}\right)^2 K_2\right)}\left[K_1\left\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\right\|+K_3\left\|\ddot{x}_\infty(t,x_0^1)-\ddot{x}_\infty(t,x_0^2)\right\|\right]$$

$$\left\|\dot{x}_\infty(t,x_0^1)-\dot{x}_\infty(t,x_0^2)\right\| \le U_3Q_2+U_4\left[K_1\left\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\right\|+K_3\left\|\ddot{x}_\infty(t,x_0^1)-\ddot{x}_\infty(t,x_0^2)\right\|\right] \quad \ldots\ldots (4.14)$$

Again on differentiating $x_\infty(t,x_0^1)$ and $x_\infty(t,x_0^2)$, we get:

$$\left\|\ddot{x}_\infty(t,x_0^1)-\ddot{x}_\infty(t,x_0^2)\right\| \le \frac{1}{\left(1-\frac{T}{2}K_3\right)}\left\|\ddot{x}_0^1-\ddot{x}_0^2\right\|+$$

$$+\frac{\frac{T}{2}}{\left(1-\frac{T}{2}K_3\right)}\left[K_1\left\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\right\|+K_2\left\|\dot{x}_\infty(t,x_0^1)-\dot{x}_\infty(t,x_0^2)\right\|\right]$$

$$\left\|\ddot{x}_\infty(t,x_0^1)-\ddot{x}_\infty(t,x_0^2)\right\| \le U_5 Q_3 + U_6\left[K_1\left\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\right\| + K_2\left\|\dot{x}_\infty(t,x_0^1)-\dot{x}_\infty(t,x_0^2)\right\|\right] \quad \ldots\ldots (4.15)$$

Using the inequalities (4.13), (4.14) and (4.15) in (4.11) we have the inequality (4.10), and this proves the theorem.

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