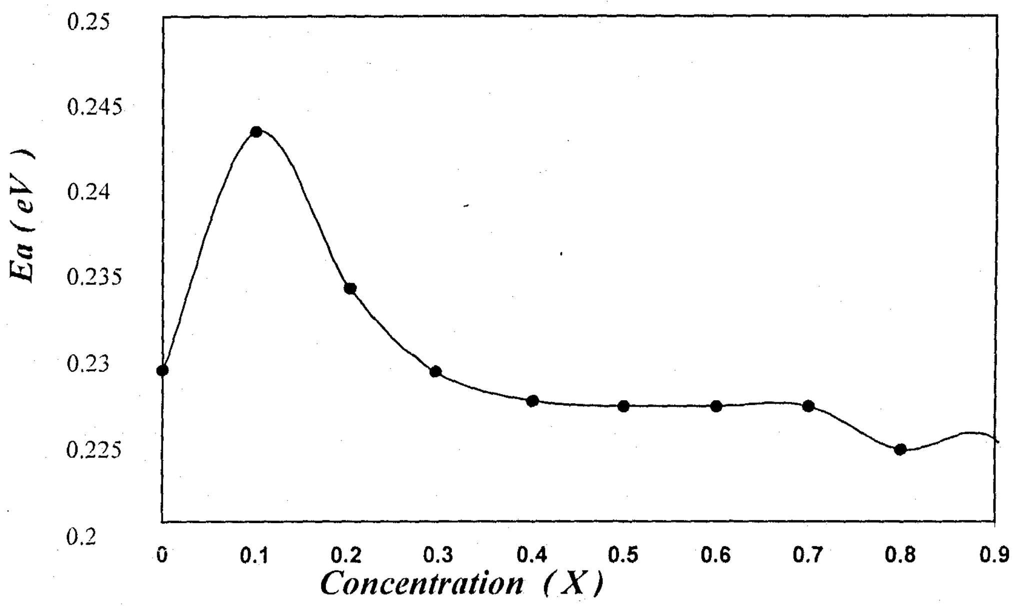

Fig. 5 : plot of activation energy (Ea) vs. concentration (X) from the ferrite system $Mg_{1-x}Mn_xFe_2O_4$

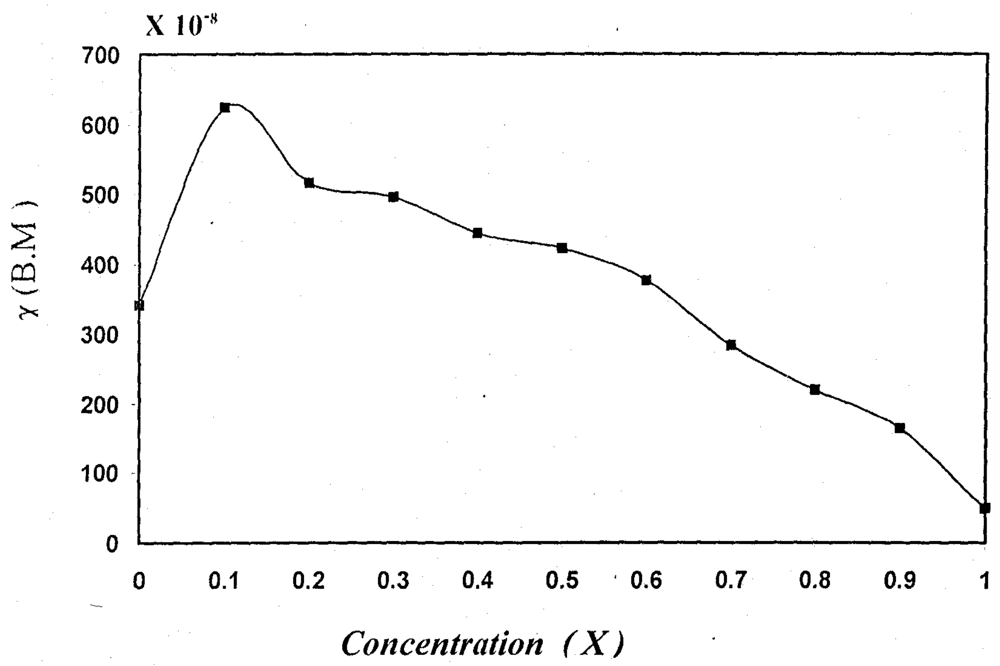

Fig. 6 : plot of magnetic susceptibility (χ) vs. concentration (X) from the ferrite system $Mg_{1-x}Mn_xFe_2O_4$

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ALGEBRA OF QUOTIENTS WITH BOUNDED EVALUATION FOR SOME OPERATOR ALGEBRAS

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الخلاصة

في هذا البحث ، تم ايجاد جبر القسومات ذات القيم المقيدة لجبريات المــؤثرات الخطيــة على فضاء هلبرت المعقد ، تحديداً: مؤثرات هلبرت-شمت ومؤثرات صنف تراس. ايضاً ، تــم دراسة السلوك في احتساب جبر القسومات ذات القيم المقيدة لجب تجميعي اولي كلياً.

ABSTRACT

In this paper, we find the algebra of quotients with bounded evaluation for operator algebras on a complex Hilbert space, namely: Hilbert – Schmidt and trace – class operators. Also, we study the behavior in computing the algebra of quotients with bounded evaluation for totally prime associative algebra.

1. INTRODUCTION AND PRELIMINARIES.

The notion of rings of quotients and the algebras of quotients with bounded evaluation (in which two –sided ideals are used) were introduced by W. S. Martindale for prime rings in [1] and Cabrera-Mohammed for normed (prime) semiprime algebras in [2] respectively and the Martindale rings of quotients extended to semiprime rings by S. A. Amitsur in [3]. It is usual to define these rings of quotients in a concrete form through partially defined centralizers on nonzero (two –sided) ideals, however for our interest we prefer to give an abstract presentation (see for example [4] or [5]) as follows: for a given prime associative algebra A, the right algebra of quotients of A, denoted here by $Q(A)$, is defined as the maximal algebra extension Q of A satisfying the following conditions:(i) if $q \in Q$ then there exists a nonzero ideal I of A with $qI \subseteq A$, (ii) if $q \in Q$, I is a nonzero ideal of A, and $qI = 0$, then $q=0$. Given q in $Q(A)$, and I nonzero ideal of A such that $qI \subseteq A$, we denote by L_q^I the mapping from I into A given by $L_q^I(x) = qx$ for all x in I. Following [6, Chapter 2], when A is additionally a normed algebra, the (right) bounded algebra of quotients of A is defined as the subalgebra of $Q(A)$ given by

$Q_b(A)=\{q \in Q(A) : \exists I$ nonzero ideal of A s.t. $qI \subseteq A$ and L_q^I is continuous$\}$

endowed with the algebra seminorm

$\| q\|_b = inf\{ \| L_q^I \| : I$ nonzero ideal of A s.t. $qI \subseteq A$ and L_q^I is continuous$\}$

It is clear that the inclusion of A into $Q(A)$ become a continuous embedding of A into $Q_b(A)$.

The bounded algebra of quotient has a very good behavior in a nice class of normed algebra called ultraprime and was discussed by Mathieu in [6]. We recall that, for a, b in an associative algebra A, the two –sided multiplication operator $M_{a,b} : A \rightarrow A$ is defined by $M_{a,b}(x) = axb$ for all x in A. It is well-known that the primeness of A is characterized in terms of two –sided multiplication operators as follows: A is prime if and only if $M_{a,b} = 0$ implies either $a=0$ or $b=0$. This fact was used by M. Mathieu to give a characterization of ultraprime associative algebra without any reference to ultrapowers, which become the usual definition now a day. A normed associative algebra A is ultraprime if there exists a positive number K such that

$$K\| a \| \| b \| \leq \| M_{a,b} \| \text{ for all } a, b \text{ in } A.$$

The bounded algebra of quotients provides the appropriate concept of algebra of quotients in the class of ultraprime algebras in the following sense: If A is an ultraprime algebra, then $Q_b(A)$ is an ultraprime algebra and the inclusion of A into $Q_b(A)$is topological. Moreover, if Q is a subalgebra of $Q(A)$ containing A, and if $\| \cdot \|_q$ is an algebra norm on Q for which Q is a topological extension of A, then $(Q, \| \cdot \|_q)$ is continuously embedded in $Q_b(A)$ (see[7, Theorem 4.1]) and [8, Proposition 2.8]).

In section 2 we introduce the notion of algebra of quotients with bounded evaluation for a normed prime associative algebra, and we study this algebra of quotients in the class of totally prime algebra introduced in [9]. Our treatment follows as possible the same objective discussed in the above paragraph for the bounded algebra of quotients. The starting point is a result of purely algebraic nature asserting that if A is a prime associative algebra and if Q is a subalgebra of $Q(A)$ containing A, then $M(A)$ (the multiplication algebra of A) is canonically embedded in $M(Q)$. Using this result, for each q in $Q(A)$ and each nonzero ideal I of A such that $q I \subseteq A$, we can consider the A-valued mapping E_q^I obtained by restriction of the evaluation operator in q to suitable ideal $R_I^{\#}$ of $M(A)$. The algebra of quotients with bounded evaluation $Q_{be}(A)$ for a normed prime associative algebra A originates when the role played in the definition of $Q_b(A)$ by the operators $L_q^I : I \rightarrow A$ (for q in $Q(A)$ and I nonzero ideal of A such that $q I \subseteq A$) is transferred to the evaluation operators E_q^I. Our construction turned out to be the appropriate concept of algebra of quotients in the class of totally prime associative algebras. Concretely, if A is a totally prime associative algebra, then $Q_{be}(A)$ is a totally prime normed algebra, and the inclusion of A into $Q_{be}(A)$ and of $M(A)$ into $M(Q_{be}(A))$ are topological. Moreover, if $(Q, \| \cdot \|)$ is a normed algebra such that Q is a subalgebra of $Q(A)$ containing A and the inclusion of A into Q and of $M(A)$ into $M(Q)$ are topological, then Q is continuously embedded in $Q_{be}(A)$.

Section 3 is devoted to the study of the algebra of quotients with bounded evaluation of some important algebra of operators on a complex Hilbert space H, namely: The algebra $L^1(H)$ of all trace –class operators and the algebra $L^2(H)$ of all Hilbert-Schmidt operators. We will begin by determining the bounded algebra of quotients for norm ideals: If A is a normed algebra which is a norm ideal on H, then $Q_b(A)=BL(H)$ (the Banach algebra of all bounded linear operators on H). Finally, we prove that $Q_{be}(L^1(H))= L^1(H)$ and $Q_{be}(L^2(H))= L^2(H)$.

2. ALGEBRA OF QUOTIENTS WITH BOUNDED EVALUATION OF A NORMED PRIME ALGEBRA.

The multiplication algebra $M(A)$ of an algebra A is defined as the subalgebra of $L(A)$ (the algebra of all linear operators on A) generated by the identity operator Id_A and the set $\{ L_a, R_a : a \in A\}$, where L_a and R_a will mean the operators of left and right (respectively) multiplication by a on A. We begin with a general result on the multiplication algebra of prime associative algebra that relies on the theory of generalized polynomial identities, for which the reader is referred to [2] and [5] for a complete proof.

Proposition 1. *Let A be a prime associative algebra and let Q be a subalgebra of $Q(A)$ containing A. Then for all F in $M(A)$ there exists a unique element $\dot{F}$ in $M(Q)$. Such that $\dot{F}(a)=F(a)$ for all a in A, and the mapping $F \to \dot{F}$ becomes an algebra monomorphism from $M(A)$ into $M(Q)$.*

Proof. See [2, Proposition 1]. ■

For an ideal I of an algebra A we denote by $R_I^\#$ the ideal of $M(A)$ generated by the set $\{R_x : x \in I\}$.

Lemma 1. *Let A be an associative algebra. If I is an ideal of A, then $R_I^\#$ coincides with the left ideal of $M(A)$ generated by the set $\{ R_x : x \in I \}$.*

Proof. Let I be an ideal of A, and let P denotes the left ideal of $M(A)$ generated by the set $\{ R_x : x \in I \}$. To prove that $P= R_I^\#$ it is enough to see that the set $S=\{F \in M(A) : PF \subseteq P\}$ is equal to $M(A)$. It is clear that S is a unital subalgebra of $M(A)$. Note that if F_0 is in $M(A)$ and satisfies that $R_x F_0 \in P$ for all x in I, then the set $V=\{T \in M(A): TF_0 \in P\}$ is a left ideal of $M(A)$ containing $\{ R_x : x \in I \}$, hence P is contained in V. Therefore we can write $S= \{F \in M(A) : R_x F \in P$ for all x in $I \}$. Since A is associative algebra, the equalities $R_x R_a = R_{ax}$, and $R_x\ L_a = L_a\ R_x$ hold for all x in I and a in A. Therefore, L_a, R_a lies in S for all a in A, and so $S=M(A)$. ■

Let A be a prime associative algebra. By Proposition 1, $M(A)$ can be seen inside of $M(Q(A))$, a fact that we will used without mention. Given q in $Q(A)$, it is clear that the set $D=\{F\in M(A) : F(q) \in A\}$ is a left ideal of $M(A)$ and that we can consider the evaluation mapping E_q from D into A defined by $E_q(F)=F(q)$ for all F in D. If I is an ideal of A such that $qI\subseteq A$ then the set $\{R_x : x\in I\}$ is contained in D, hence $R_I^{\#}$ is also contained in D as consequence of lemma 1. Thus, we can consider the restriction of E_q to $R_I^{\#}$, which will be denoted by E_q^I.

Proposition 2. *Let A be a normed prime associative algebra. Then*

$$O_{be}(A) = \{ q\in Q(A) : \exists\ I \text{ nonzero ideal of } A \text{ s.t. } qI\subseteq A \text{ and } E_q^I \text{ is continuous}\}$$

is a subalgebra of Q(A), *and* $|\,.\,| : O_{be}(A)\to R$ *defined by*

$$|q| = \inf\{\| E_q^I \| : I \text{ nonzero ideal of } A \text{ s.t. } qI\subseteq A \text{ and } E_q^I \text{ is continuous}\}$$

is an algebra seminorm. Moreover $O_{be}(A)$ *is subalgebra of* $O_b(A)$ *containing A, and these inclusion are continuous. Precisely* $|a|\le \|a\|$ *for all a in A, and* $\|q\|_b \le |q|$ *for all q in* $O_{be}(A)$.

Proof. See [2, Theorem 1]. ■

For a normed prime associative algebra A, the seminormed algebra $(O_{be}(A), |\,.\,|)$ is called the *algebra of quotients with bounded evaluation of A*. Our next objective is to carry out the study of this algebra of quotient for totally prime associative algebras. Totally prime algebra was introduced in [9] to provides the nonassociative extension of the determination of the extended centroid for ultraprime associative algebras given in [7]. We recall that a normed algebra A is *totally prime* if there exists a positive number K such that $K\ \|a\|\ \|b\| \le \|N_{a,b}\|$ for all a, b in A, where $N_{a,b}$ denotes the bilinear mapping from $M(A)xM(A)$ into A defined by $N_{a,b}(F, G)=F(a)G(b)$ for all F, G in $M(A)$.

Lemma 2. *Let A be a totally prime associative algebra and assume that K is a positive constant such that* $K\ \|a\|\ \|b\| \le \|N_{a,b}\|$ *for all a, b in A. Let q be in* $O_{be}(A)$ *and let I be a nonzero ideal of A such that* $qI\subseteq A$ *and* E_q^I *is continuous. If J a nonzero ideal of A such that* $qJ\subseteq A$, *then* E_q^J *is continuous and*

$$K\ \|E_q^J\| \le \|E_q^I\|.$$

Proof. Let J be a nonzero ideal of A such that $qJ \subseteq A$. For x in I with $\| x \|=1$, F in $R_J^{\#}$ and S, T in $M(A)$ we have

$$N_{F(q),x}(S, T)=SF(q)T(x)= R_{T(x)}SF(q)= E_q^I(R_{T(x)}SF),$$

hence

$$\| N_{F(q),x}(\mathrm{S}, \mathrm{T}) \| = \| E_q^I(R_{T(x)}\mathrm{SF}) \| \leq \| E_q^I \| \| \mathrm{S} \| \| \mathrm{F} \|,$$

and so

$$\| N_{F(q),x} \| \leq \| E_q^I \| \| F \|.$$

Since A is a totally prime algebra, it follow that

$$K \| F(q) \| \leq \| E_q^I \| \| F \|$$

and so E_q^J is continuous and $K \| E_q^J \| \leq \| E_q^I \|$.■

The following result is similar to that obtained in [2].

Theorem 1. *Let A be a totally prime associative algebra, and assume that K is a positive constant such that $K \|a\| \|b\| \leq \|N_{a,b}\|$ for all a, b in A. Then $O_{be}(A)$ is a totally prime normed algebra, and the inclusions of A into $O_{be}(A)$ and of M(A) into $M(O_{be}(A))$ are topological. Precisely, $K \|a\| \leq |a| \leq \|a\|$ for all a in A, and $K \|F\| \leq |F| \leq \|F\|$ for all F in M(A). Moreover, if $(Q, \|\cdot\|_q)$ is a normed algebra such that Q is a subalgebra of Q(A) containing A and the inclusions of A into Q and of M(A) into M(Q) are topological, then Q is continuously embedded in $Q_{be}(A)$.*

Proof. By Proposition 2, $Q_{be}(A)$ is an algebra extension of A and $|\ .\ |$ is an algebra seminorm on $O_{be}(A)$ such that $|a| \leq \|a\|$ for all a in A. Let q be in $Q_{be}(A)$ satisfying $|q| = 0$. By Lemma 2, $E_q^I = 0$ (hence $qI=0$) for all nonzero ideal I of A such that $qI \subseteq A$, and so $q = 0$. Therefore $|\ .\ |$ is a norm on $Q_{be}(A)$. Let a be in A and I be a nonzero ideal of A for x in I with $\| x \| = 1$ and F, G in $M(A)$ we have

$$N_{a.x}(F, G)=F(a)G(x)=R_{G(x)}F(a) = E_a^I(R_{G(x)}F)$$

hence

$$\| Na, x(F, G) \| E_a^I (RG(x)F) \| \leq \| E_a^I \| \| G \| F \|$$

and so

$$\|Na, x \| \leq \| E_a^I \|.$$

Since A is a totally prime algebra it follows that $K \| \mathrm{a} \| \leq \| E_a^I \|$. Now, taking infimum on I we obtain that $K \| \mathrm{a} \| \leq | \mathrm{a} |$.

Our next objective is to prove that the inclusion of $M(A)$ into $Q_{be}(A)$ is topological. Fix F in $M(A)$. If q is in $Q_{be}(A)$ and I is a nonzero ideal of A such that $qI \subseteq A$, then $F(q)I \subseteq R_I^{\#}F(q) \subseteq R_I^{\#}(q) \subseteq A$ and, for G in $R_I^{\#}$ we can write

$$\| E^I_{F(q)}(G) \| = \| GF(q) \| = \| E^I_q \| \, \|G\| \, \|F\|,$$

therefore $\| E^I_{F(q)} \| \leq \| E^I_q \| \, \|F\|$. Consequently, $| F(q) | \leq \| E^I_q \| \, \|F\|$ and, by taking infimum on I, $| F(q) | \leq |q| \, \|F\|$. From this it follows that $|F| \leq \|F\|$. Moreover, for all a in A we have

$$K \|F(a)\| \leq |F(a)| \leq |F| \, |a| \leq |F| \, \|a\|,$$

therefore $K \|F\| \leq |F|$.

Now we prove that $Q_{be}(A)$ is a totally prime algebra. Let q_1, q_2 fixed in $Q_{be}(A)$ and consider I_1, I_2 nonzero ideals of A such that $q_1 I_1 \subseteq A$ and $q_2 I_2 \subseteq A$. For F_1 in $R^{\#}_{I1}$, F_2 in $R^{\#}_{I2}$ and S, T in $M(A)$ we have

$$N_{F_1(q_1),\, F_2(q_2)}(S, T) = SF_1(q_1)\, TF_2(q_2) = N^{I_1\,:\,I_2}_{q_1\,,\,q_2}(SF_1, TF_2),$$

where $N^{I_1\,:\,I_2}_{q_1\,,\,q_2}$ denotes the restriction to $R^{\#}_{I1} \times R^{\#}_{I2}$ of $N_{q_1\, q_2}$. Hence

$$K \| N_{F_1(q_1),\, F_2(q_2)}(S, T) \| = K \| N^{I_1\,:\,I_2}_{q_1\,,\,q_2}(SF_1, TF_2) \| \leq | N^{I_1\,:\,I_2}_{q_1\,,\,q_2}(SF_1, TF_2) |$$

$$\leq | N^{I_1\,:\,I_2}_{q_1\,,\,q_2} | \, |S| \, |F_1| \, |T| \, |F_2| \leq | N^{I_1\,:\,I_2}_{q_1\,,\,q_2} | \, \|S\| \, \|F_1\| \, \|T\| \, \|F_2\|,$$

and so

$$K \| N_{F_1(q_1),\, F_2(q_2)} \| \leq | N^{I_1\,:\,I_2}_{q_1\,,\,q_2} | \, \|F_1\| \, \|F_2\|,$$

therefore

$$K^2 \| F_1(q_1) \| \, \| F_2(q_2) \| \leq | N^{I_1\,:\,I_2}_{q_1\,,\,q_2} | \, \|F_1\| \, \|F_2\|$$

because A is a totally prime algebra. From the last inequality it follows that

$$K^2 \| E^{I_1}_{q_1} \| \, \| E^{I_2}_{q_2} \| \leq | N^{I_1\,.\,I_2}_{q_1\,,\,q_2} |,$$

and so

$$K^2 |q_1| \, |q_2| \leq | N^{I_1\,:\,I_2}_{q_1\,.\,q_2} |$$

Finally, let $(Q, \| \cdot \|_q)$ a normed algebra such that Q is a subalgebra of $Q(A)$ containing A and assume the existence of positive numbers α, β, $\wp$, δ such that $\alpha \|a\| \leq \|a\|_q \leq \beta \|a\|$ for all a in A, and $\wp \|F\| \leq \|F\|_q \leq \delta \|F\|$ for all F in $M(A)$. Let q be in Q, and assume that I is a nonzero ideal of A such that $qI \subseteq A$. For all F in $R_I^{\#}$ we have

$$\| E^I_q(F) \| = \| F(q) \| \leq \alpha^{-1} \| F(q) \|_q \leq \alpha^{-1} \| F \|_q \| q \|_q \leq \alpha^{-1} \delta \| F \| \, \|q\|_q$$

hence E^I_q is continuous and $\| E^I_q \| \leq \alpha^{-1} \delta \|q\|_q$, and so q lies in $Q_{be}(A)$ and $|q| \leq \alpha^{-1} \delta \|q\|_q$. Therefore Q is continuously embedded in $Q_{be}(A)$. ■

Remark 1. Note that under the assumptions of Theorem 1 we have actually prove that

$$K^2 | q_1 | \ | q_2 | \leq | N_{q_1, q_2}^{I_1, I_2} |$$

for all q_1, q_2 in $Q_{be}(A)$ and I_1, I_2 nonzero ideals of A such that $q_1 I_1 \subseteq A$ and $q_2 I_2 \subseteq A$, where $N_{q_1, q_2}^{I_1, I_2}$ denotes the restriction to $R_{I_1}^{\#} \times R_{I_2}^{\#}$ of N_{q_1, q_2}. This condition characterize topologically to the subalgebras of $Q_{be}(A)$ in the following sense :
if $(Q, \| . \|)$ is a normed algebra such that Q is a subalgrbea of $Q(A)$ containing A and the inclusions of A into Q and of $M(A)$ into $M(A)$ are topological, and there exists a positive constant K such that

$$K \| q_1 \|_q \ \| q_2 \|_q \leq \| N_{q_1, q_2}^{I_1, I_2} \|_q$$

for all q_1, q_2 in Q and I_1, I_2 nonzero ideals of A such that $q_1 I_1 \subseteq A$ and $q_2 I_2 \subseteq A$, then Q is topologically embedded in $Q_{be}(A)$.

Indeed , let $(Q, \| . \|)$ be such a normed algebra and consider α, β, $\wp$, δ as in the end of the proof of Theorem 1. We know that $| q | \leq \alpha^{-1}\delta \ \| q \|$ for all q in Q. Fix q in Q, and assume that I is a nonzero ideal of A such that $q I \subseteq A$. Take a in A with $\| a \| = 1$. For each $0 < \varepsilon < 1$ there are F in $R_I^{\#}$ with

$\| F \|_q = 1$ and G in $M(A)$ with $\| G \| = 1$ such that

$$\varepsilon K \| q \| \leq \| N_{q, a}(F, G) \|,$$

therefore

$$\varepsilon K \| q \|_q \leq \| F(q)G(a) \|_q \leq \| F(q) \|_q \ \| G(a) \|_q \leq \beta \ \| F(q) \|$$
$$\leq \beta \ \| E_q^I \| \ \| F \| \leq \beta \wp^{-1} \| E_q^I \|.$$

This shows that $K \| q \|_q \leq \beta \wp^{-1} \ \| E_q^I \|$.

Consequently $K \| q \|_q \leq \beta \wp^{-1} | q |$, as required. ■

3. ALGEBRA OF QUOTIENTS WITH BOUNDED EVALUATION FOR TRACE –CLASS AND HILBERT – SCHMIDT OPERATORS.

For a given complex Hilbert space $(H, <, >)$, we denote by $L(H)$ the algebra of all linear operators on H, and by $BL(H)$ the Banach algebra of all bounded linear operators on H with the operator norm, which will be denoted by $\| . \|_\infty$. As usual, for x, y in H, $x \otimes y$ will denote the rank – one operator on H defined by

$$(x \otimes y)(z) = <z, y> x \text{ for all z in } H.$$

It is well-know that the subset $FL(H)$ of $BL(H)$ consisting of all finite - rank operators can be expressed as follows

$$FL(H) = \{ \sum_{i=1}^{n} x_i \otimes y_i : n \in N, \ x_i, y_i \in H \ (1 \leq i \leq n) \}$$

It is easy to see that if A is a subalgebra of $BL(H)$ containing $FL(H)$, then A is an irreducible algebra of operators on H such that the centralizer

$$\upsilon = \{T \in L(H) : \ TF = FT \text{ for all } F \text{ in } A\}$$

is equal to Cid_H where id_H means the identity operator and C the complex field. Moreover, $FL(H)$ is the sum of all minimal left ideal of A (the socle of A) and it is a minimal ideal of A contained in every ideal of A. Recall also that a norm ideal on H is an ideal A of $BL(H)$ endowed with a norm $\| \, . \, \|$ satisfying the following conditions:

i) $\| x \otimes y \| = \| x \| \, \| y \|$ for all x, y in H (cross-property),

ii) $\| FTG \| \leq \| F \|_\infty \, \| T \| \, \| G \|_\infty$ for all F, G in $BL(H)$ and T in A.

Our first goal is to determine the bounded algebra of quotients of normed algebras that are norm ideals.

Proposition 3. *Let H be a complex Hilbert space and let $(A, \| \, . \, \|)$ be a normed algebra which is a norm ideal on H. Then $(Q_b(A), | \, . \, |) = (BL(H), \| \, . \, \|_\infty)$*

Proof. See [2, Theorem 2]. ■

The following Proposition yield information about properties of the algebra of qoutients with bounded evaluation of a normed algebra which is norm ideal.

Proposition 4. *Let H be a complex Hilbert space and let $(A, \| \, . \, \|)$ be a normed algebra which is a norm ideal on H. Then $Q_{be}(A)$ is a right ideal of $BL(H)$ and $| \, . \, |$ is an algebra norm satisfying the following assertions:*

i) $| T | \leq \| T \|$ for all T in A,

ii) $\| T \|_\infty \leq | T |$ for all T in $Q_{be}(A)$,

iii) $| x \otimes y | = \| x \| \, \| y \|$ for all x, y in H,

iv) $\| TF \| \leq | T | \, \| F \|_\infty$ for all T in $Q_{be}(A)$ and F in $FL(H)$,

v) $| TF | \leq | T | \, \| F \|_\infty$ for all T in $Q_{be}(A)$ and F in $BL(H)$.

Proof. See [3, Proposition 3]. ■

The algebra of trace – class and Hilbert – Schmidt are relevant examples of normed algebras which are norm ideals on a complex Hilbert space . Now, we collect some aspects of these algebras that are interesting for our development and that can be seen for example in [10] or in [11]. Let us fix a complex Hilbert space $(H, \langle \, . \, , \, . \, \rangle)$, and consider the algebra involution $F \rightarrow F^*$ (the adjoint of F relative to the inner product $\langle . , . \rangle$) on $BL(H)$. Suppose that $\{ e_\lambda \}_{\lambda \in \Lambda}$ is an orthonormal basis for H, we define for each F in $BL(H)$

$$\|F\|_2=(\sum_{\lambda\in\Lambda}\|F(e_\lambda)\|^2)^{1/2} \quad \text{and} \quad \|F\|_1=\sum_{\lambda\in\Lambda}<|F|(e_\lambda)e_\lambda>$$

where $|F| = (F^* F)^{1/2}$ (these definitions are independent of the choice of basis) and we say that F is an Hilbert –Schmidt operator if $\|F\|_2 < +\infty$, and we say that F is a trace-class operator if $\|F\|_1 < +\infty$. will denote the set of all Hilbert - Schmidt operators on H and $L^1(H)$ will denote the set of all trace –class operators on H. For $i=1, 2$, $(L^2(H), \|.\|_i)$ is a Banach algebra which is a self-adjoint norm ideal of $BL(H)$ and the involution $*$ is $\|.\|_i$ - isometric. Moreover, the trace of a trace-class operator F is defined by

$$tr(F)=\sum_{\lambda\in\Lambda}<F(e_\lambda), e_\lambda>$$

(again this definition is independent of the chice of basis), and the function

$$tr: L^1(H)\to C$$

is a continuous commutative linear form.

Theorem 2 . *Let H be a complex Hilbert space. Then*

i) $Q_{be}(L^1(H)), |.|_1) = (L^1(H), \|.\|_1)$.

ii) $Q_{be}(L^2(H)), |.|_2) = (L^2(H), \|.\|_2)$.

Proof. *i*)-By Proposition 4, $Q_{be}(L^1(H))$ is a right ideal of $BL(H)$. We claim that if T is a positive operator in $Q_{be}(L^1(H))$, then T lies $L^1(H)$ and $\|T\|_1 \le |T|_1$. Let T be a positive operator in $Q_{be}(L^1(H))$, and suppose that $\{e_\lambda\}_{\lambda\in\Lambda}$ is an orthonormal basis for H. For each finite non - empty subset S of Λ consider the projection

$$P_S=\sum_{\lambda\in S} e_\lambda\otimes e_\lambda ,$$

and note that

$$\sum_{\lambda\in S}<T(e_\lambda), e_\lambda> = \sum_{\lambda\in\Lambda}<TP_S(e_\lambda), e_\lambda> = tr(TP_S) \le \|TP_S\|_1 \le |T|_1 \|P_S\|_\infty$$
$$= |T|_1 ,$$

where the continuity of the trace function and Proposition 4.iv) have been used . As a consequence , the family of non-negative real numbers$\{<T(e_\lambda), e_\lambda>\}_{\lambda\in\Lambda}$ is summable with sum less or equal to $|T|_1$, and so T is a trace - class operator and

$$\|T\|_1 \le |T|_1 .$$

Now , let F be in $Q_{be}(L^1(H))$. By the polar decomposition of F^* there exist a unique partial isometry W such that $F^* = W|F^*|$, and moreover $|F^*| = W^* F^*$ (see [9; Theorem 2.3.4]) . By Proposition 4.v) , $|F^*| = FW$ belongs to $Q_{be}(L^1(H))$ and

$$\||F^*|\|_1 = |FW|_1 \le |F|_1 \|W\|_\infty = |F|_1.$$

Since $|F^*|$ is positive, from the first part of the proof we have that $|F^*|$ is a trace - class operator and $\| |F^*| \|_1 \leq |F|_1$. Now, taking into account that $\| F \|_1 = \| F^* \|_1 = \| |F^*| \|_1$, It follows that F is a trace-class operator and $\| F \|_1 \leq |F|_1$. The converse inequality holds by Proposition 4.i).

ii) – Again by Proposition 4, $Q_{be}(L^2(H))$ is a right ideal of $BL(H)$. Let T be in $Q_{be}(L^2(H))$, and suppose that $\{e_\lambda\}_{\lambda\in\wedge}$ is an orthonormal basis for H. If S is a finite non-empty subset of $\wedge$ and if we consider the projection $P_S = \sum_{\lambda\in S} e_\lambda \otimes e_\lambda$, then

$$\sum_{\lambda\in S} \| T(e_\lambda) \|^2 = \sum_{\lambda\in\wedge} \| TP_S(e_\lambda) \|^2 = \| TP_S \|_2^2 \leq |T|_2^2 \, \|P_S\|_\infty^2 = |T|_2^2,$$

where we have used Proposition 4. Iv). As a consequence, the family of non-negative numbers $\{\| T(e_\lambda) \|^2\}_{\lambda\in\wedge}$ is summable with sum less or equal to $|T|_2^2$ and so T is a Hilbert- Schmidt operator and $\| T \|_2 \leq |T|_2$. The converse inequality holds by Proposition 4.i.). ■

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