Some Nonstandard Results of Continuous, Monotonic Functions

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Received 13/7/2004 Accepted 16/2/2005

الخلاصة

الهدف من هذا البحث هو استخدام بعض مفاهيم التحليل الغير القياسي الذي أوجده Robinson, A. (8) ووضعه Nelson, E. (6) بأسلوب منطقي للحصول على بعض النتائج الجديدة للدوا ل المستمرة أو الرتيبة عندما يعرف على منطلق غير قياسي.

ABSTRACT

The aim of this paper is to use some concepts of nonstandard analysis given by Robinson, [8], and axiomatized by Nelson, E. [6] to present a new result of continuous and monotonic function when they are defined on nonstandard sets.

Keywords: nonstandard analysis, continuous functions, and monotonic functions

1- INTRODUCTION

In conventional analysis (classical analysis) the set of real numbers does not contain infinitely small and infinitely large elements, while with nonstandard analysis one can easily tangible such elements. In this paper the problem of continuity and monotonic of functions with respect to such elements is considered to be study. Ismail T.H gave some other results on continuous and monotonic functions in 1991 [4] and 1994 [5].

Through this paper we need the following definitions, notions and theorems of nonstandard analysis.

Every set or elements defined in a classical mathematics is called **standard** [6], [10] A real number x is called **unlimited** if and only if $|x| > r$ for all real $r > 0$ otherwise called **limited** [2], [6].

The set of all unlimited real numbers denoted by $\overline{\mathbf{R}}$, and the set of all limited real numbers denoted by $\underline{\mathbf{R}}$

A real number x is called **infinitesimal** if and only if $|x| < r$ for all positive standard real number r [2], [10].

Two real numbers x and y are said to be **infinitely close** if an only if $x - y$ is infinitesimal and denoted by $x \cong y$ [2]

If x is a limited number in $\mathbf{R}$, then it is infinitely close to a unique standard real number, this unique number is called the **standard part** of x or **shadow** of x denoted by $st(x)$ $^o x$ [2][6].

If f is a real valued function then: -

1- f is called continuous at x_o if f and x_o are standard and for all x, $x \cong x_o$ [6].

2- f is called s-continuous at x_0 if for all x, $x \cong x_0$ then $f(x) \cong (x_0)$ [6].

$mond(x) = \{y \in \mathbf{R} \ s.t \ x \cong y\}$ for limited x

$\alpha - mond(x) = \{y \in \mathbf{R} : \frac{y-z}{\alpha} \cong 0 \}$

$\alpha - micromond(x) = \{y : y - x < \varepsilon^n \ \forall \text{standadd } n\}$

$\alpha - galaxy(x) = \{y \in \mathbf{R} : \frac{y-x}{\alpha} \text{ limited } \}$ [2]

Cauchy Principle:-

If P is an internal property such that $p(n)$ holds for all standard $n \in \mathbf{N}$ then there exist $\omega \in \mathbf{N}$ such that $p(n)$ holds for all $n \leq \omega$ [9]

2- MAIN RESULTS

Lemma 2.1

If f is a positive increasing function defined on $[n, m]$ such that $f(x) \cong 0$ for all $x \in [n, m]$ where $n, m \in \mathbf{N}$, and $n < m$, then

$R_{n-1}(x) \cong \int_n^\infty f(x)$ for unlimited n, where R_{n-1} is the remainder of the series expansion of the sequence $\langle f(n) \rangle_{n \in \mathbf{N}}$

Proof:

Since $n, m \in \mathbf{N}$, so it is enough to prove the result for $x \in [n, n+1]$, and the result for $x \in [n, m]$ is obtain consequentially.

Now, since f is an increasing function then for all x, $n \leq x \leq n+1$ we have

$$f(n) \le f(x) \le f(n=1)$$

$$\Rightarrow \int_n^{n+1} f(n)\,dx \le \int_n^{n+1} f(x)\,dx \le \int_n^{n+1} f(n+1)\,dx$$

$$\Rightarrow f(n) \le \int_n^{n-1} f(x)\,dx \le f(n+1)$$

$$\Rightarrow \sum_{n=0}^{k} f(n) \le \int_0^{k+1} f(x)\,dx \le \sum_1^{k+1} f(n)$$

$$\sum_0^{k} f(n) - \sum_1^{k+1} f(n) \le \int_0^{k+1} f(x)\,dx - \sum_1^{k+1} f(n) \le 0$$

$$f(0) - f(k+1) \le \int_0^{k+1} f(x)\,dx - \sum_1^{k+1} f(n) \le 0$$

but $f(0) - f(k-1) \cong \varepsilon$, where ε is a negative number, then $\int_0^{k+1} f(x)\,dx - \sum_1^{k+1} f(n) \cong \eta$ for negative number $\eta > \varepsilon$, but $f(x) \cong 0$ for all x, therefore

$$\int_0^{k+1} f(x)\,dx \cong \sum_1^{k+1} f(n) \text{ for standard } k \qquad \ldots(\text{I})$$

By Cauchy principle $\exists \omega \in \mathbf{N}$ s.t

$$\int_0^{w} f(x)\,dx \cong \sum_0^{\omega} f(n) \qquad \ldots(\text{II})$$

Now, the remainder terms of $\langle f(n) \rangle_{n \in \mathbf{N}}$ is given by: $R_{n-1}(x) = \sum_{k=0}^{\infty} f(k) - \sum_{k=o}^{n-1} f(k)$.

From (I) and (II) we get

$$R_{n-1}(x) \cong \int_0^{\infty} f(x)\,dx - \int_0^{n-1} f(x)\,dx = \int_n^{\infty} f(x)\,dx$$

Theorem2.2

If f is a positive decreasing function, then $\exists$ an unlimited $\omega \in \mathbf{N}$ such that $d_\omega \le f(0)$, where $0 < d_\omega(x) = \sum_{k=0}^{\omega} f(k) - \int_0^{\omega} f(x)\,dx$

Proof:

The integral $\int_0^{n} f(x)\,dx$ represents the area under the curve $f(x)$ over the interval

[0, k+1], where n, k$\in \mathbf{Z}^+$, and $n \le$k+1. With out lose of generality we have, $\sum_{n=0}^{k+1} f(n) \le \int_0^{k+1} f(x)\,dx \le \sum_{n=0}^{k} f(n)$.

Since f is positive, therefore $\sum_{n=0}^{k+1} f(n) - f(0) \leq \int_0^{k+1} f(x)\,dx < \sum_{n=0}^{k+1} f(n)$

$$\Rightarrow 0 < \sum_{n=0}^{k+1} f(n) - \int_0^{k+1} f(x)\,dx \leq f(0)$$

$$\Rightarrow -f(0) \leq \int_0^{k+1} f(x)\,dx - \sum_{n=0}^{k+1} f(0) < 0 \quad \text{for standard value of } k$$

By Cauchy Principle there exist an unlimited $\omega \in \mathbf{N}$ such that $0 < \sum_{n=0}^{\omega} f(n) - \int_0^{\omega} f(x)\,dx \leq f(0)$, which implies that $0 < d_\omega \leq f(0)$. Moreover by using lemma (**2.1**), we get $0 \cong d_\omega \leq f(0)$

The sensitivity power of the theorem appears when $f(0) \cong 0$. The following corollary will treat such a case.

Corollary 2.3

If $f(x)$ is a function satisfying the same properties of theorem (**2.2**) and $f(0) \cong 0$ then: -

1- Either $d_\omega(x) \in f(0) - gal(0)$ *or* $d_\omega(x) \in f(0) - mon(0)$

2- $d_\omega(x)$ can not exceeding $\sqrt{f(0)}$

3- $d_\omega(x)$ can not be in $mon(f(0))$

4- $d_\omega(x) \in f(0) - micromond(0)$

Proof:

1) We prove this theorem by contradiction.

Now by definition of $\alpha - mon(x)$ and $\alpha - gal(x)$ we have $\frac{d_\omega(x)}{f(0)}$ is neither infinitesimal nor limited, so $\frac{d_\omega(x)}{f(0)}$ is unlimited which is contradiction to that $\frac{d_\omega(x)}{f(0)} < 1$

2) Suppose that $d_\omega(x)$ exceeding $\sqrt{f(0)}$, then at least one can claim that $d_\omega(x)$ is slightly exceed $\sqrt{f(0)}$ such that $d_\omega(x)$ in a positive of $monad(\sqrt{f(0)}$ then $d_\omega(x) = \sqrt{f(0)}$ or $d_\omega(x) = \sqrt{f(0)} + \eta$ for $0 < \eta \cong 0$, in the first case $\frac{d_\omega(x)}{f(0)} = \frac{1}{\sqrt{f(0)}}$ is unlimited which is contradiction.

The second case immediately impossible since $d_\omega \leq f(0)$.

3) The prove is obvious from 2.

4) Since $d_\omega(x) \leq (f(0))^n$ for all standard n, then $d_\omega(x) \in f(0) - micromond(0)$

Corollary 2.4

If f is a function satisfying the same property of theorem **2.2** except that f is defined for all $x \geq m$ where $m \in \mathbf{N}$, then $d_\omega(x) \leq f(m)$.

Proof:
By taking Riemann integration on the interval [m, n+1] with the partition $\pi = \{x_0 = m < m+1 < ... < n-1 < n = x_n\}$, then $\sum_{i=m}^{n+1} f(x) \leq \int_m^{n+1} f(x)\,dx \leq \sum_{i=m}^{n} f(n)$

By continuing this process we get that $d_\omega = \sum_{i=m}^{\omega} f(x) - \int_m^{\omega} f(x)\,dx \leq f(m)$

Example 2.5

For any continuous function f defined on the closed interval [0,1], we can see that

$$\int_0^1 f(x)\,dx \cong \eta \sum_{n=1}^{\omega-1} f(n\eta) \ \ \eta \cong 0$$

Proof: Since f is continuous on the closed interval [0,1], then $\int_0^1 f(x)$ exists. Let

$\pi = \{x_0 = 0 < \frac{1}{n} < \frac{2}{n} < ... < \frac{n}{n} = 1\}$ be a partition of the interval [0,1].

By using nonstandard definition of integration we get $\int_0^1 f(x) \cong \frac{1}{n} \sum_{k=1}^{n-1} f(\frac{k}{n})$.

Thus for unlimited n we conclude that $\int_0^1 f(x) \cong \eta \sum_{k=1}^{\omega-1} f(k\eta)$

Lemma 2.6

Let f be a function such that $f(x) \cong 0$ for all $x \in D_f$ then f is s-uniform continuous function.

Proof: The prove is obvious.

Next we give some condition of uniform continuity for linear functions

Theorem 2.7

Let f be any linear function such that $f(x) \cong 0,\ \forall x \cong 0$ then f is s-uniformly continuous

Proof:

Let $x, y \in D_f\ s.t\ x \cong y$, then $f(x) - f(y) = f(x-y)$, since $x \cong y$ then $x - y \cong 0$ and thus there exist $\alpha \cong 0$ such that $x = y + \alpha$ $\Rightarrow f(x) - f(y) = f(\alpha) \cong 0$ $\Rightarrow f(x) \cong f(y)$

Remark: -

The above theorem shows that continuity of linear real valued functions not remain true if it's defined on a nonstandard domains, for example a linear function $f(x)=\omega x$ where ω is an unlimited real number such that $\omega=\frac{3}{\varepsilon}$ for $\varepsilon \cong 0$ is not continuous in the sense of nonstandard analysis for this take $x=1+\varepsilon$, and $y=1$, its clear that $x \cong y$, but $f(x) \cong f(y)$ since $f(x)-f(y)=3 \cong 0$.

Theorem 2.8

If f is a standard continuous function at $x \in [\alpha,\beta]$ then f is a limited function for all $x \in [\alpha,\beta]$

Proof:

First Method

Suppose that f is unlimited for some $x \in (\alpha,\beta)$, that is there exist $x^* \in (\alpha,\beta)$ such that $f(x^*)>k$ for all k standard.

Since f is continuous then $\forall x$ such that $x \cong x^*$ we get $f(x) \cong f(x^*)$, take $k=f(x)$ $\Rightarrow f(x^*) \cong k$ is limited since any standard number is infinite close to a limited number (or because only limited numbers can be infinite close to a standard numbers)

Second Method

Suppose that f is unlimited for some $x \in (\alpha,\beta)$, since for all k there exist x such that $f(x)>k$, then there is a sequence of elements of x_k corresponds to the values of k.

Since $x_k \in (\alpha,\beta)\ \forall k$, so x_k is a bounded sequence and there exist a convergent subsequence of $x_k\ in(\alpha,\beta)$ say x_{k_p} which has a limit λ in (α,β), i.e $x_{k_p} \cong \lambda$ for $p \in \overline{\mathbf{N}}$.

But f is a standard continuous function at $\lambda \Rightarrow f(x_{k_p}) \cong f(\lambda)\ as\ p \in \overline{\mathbf{N}}$..

Note that $p \in \overline{\mathbf{N}}$ means that $n_p \in \overline{\mathbf{N}}$, by our assumptation f is unlimited, so $f(x_{kp})$ is unlimited $\Rightarrow f(\lambda)$ is also unlimited, which is contradiction.

Remark: -

The above theorem tell us that if f is not standard then f may be continuous and is not bounded function. For example $f(x)=\frac{1}{x+\varepsilon}$ on $[0,1]$ s.t $\varepsilon \cong 0$, so f is not standard but continuous for all $x \in [0,1]$ and not limited at 0.

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