Existence of a Periodic Solution for Nonlinear Systems of Differential Equations with Retarded Argument

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Received 6/9/2003 Accepted 20/12/2003

الخلاصة

يتضمن البحث دراسة وجود وتقارب الحلول الدورية لنظم من المعادلات التفاضلية غير الخطية ذات تأخر المتغير المستقل وذلك باستخدام طريقة الحلول الدورية للمعادلات التفاضلية الاعتيادية لـ A. M. Samoilenko.

ABSTRACT

In this paper we investigate the existence and approximation of the periodic solutions for nonlinear systems of differential equations with retarded argument, by using the method of periodic solutions of ordinary differential equation which are given be A. M. Somilenko .

INTRODUCTION

Consider the following system of differential equations with retarded argument:

$$\frac{dx(t)}{dt} = f(t, x(t), x(t-\tau), \frac{dx(t)}{dt}, \frac{dx(t-\tau)}{dt}), \qquad (1)$$

where the function $f(t, x, x_\tau, y, y_\tau)$ is defined on the domain:

$$-\infty < t < \infty, \quad a \le x(t) \le b, \quad a \le x(t-\tau) = x_\tau \le b,$$

$$c \le y = \frac{dx(t)}{dt} \le d, \quad c \le \frac{dx(t-\tau)}{dt} = y_\tau \le d, \quad 0 \le \tau \le T, \qquad (2)$$

which is continuous in t, x, x_τ, y, y_τ and periodic in **t** of period **T**.

let the function $f(t, x, x_\tau, y, y_\tau)$ satisfies the following inequalities:

$$|f(t,x,x_\tau,y,y_\tau)| \le M, \tag{3}$$

$$|f(t,x',x'_\tau,y',y'_\tau) - f(t,x'',x''_\tau,y'',y''_\tau)| \le K_1|x'-x''| + K_2|x'_\tau - x''_\tau| + K_3|y'-y''| + K_4|y'_\tau - y''_\tau|, \tag{4}$$

where M is a positive constant and K_1, K_2, K_3, K_4 are Lipeshitz constants.

The constants a, b, c, d, M and K_1, K_2, K_3, K_4, T are connected by the conditions:

1- $b - a \ge \frac{MT}{2}, \quad c \le -2M \le 2M \le d,$ (5)

2- $T(K_1 + K_2) + 4(K_3 + K_4) < 1$ (6)

Lemma:

Let f(t) be a continuous function defined on the interval $0 \le t \le T$. Then:

$$\left|\int_0^t \left(f(s) - \frac{1}{T}\int_0^T f(s)ds\right)ds\right| \le \alpha(t)\|f(t)\|$$

where $\alpha(t) = 2t(1 - \frac{t}{T})$ and $\|f(t)\| = \max_{0 \le t \le T}|f(t)|$. For the proof see [3].

We use the linear operator L acting of the function f(t) continuous for $t \in [0,T]$ by the formula

$$(Lf)(t) = Lf(t) = \int_0^t \left(f(s) - \frac{1}{T}\int_0^T f(s)ds\right)ds.$$

It is obvious that if f(t) is continuous function on the interval [0,T], then Lf(t) is also continuous on the same interval.
By the lemma, we get

$$|Lf(t)| \le \alpha(t)\ \|f(t)\|$$

for all $t \in [0,T]$ and $\alpha(t) \le \frac{T}{2}$

APPROXIMATE SOLUTION

The investigation of approximate solution of the system (1) will be introduced by the following theorem.

Theorem 1: If the system of differential equations with retarded argument (1) satisfy the inequalities (3), (4) and the conditions (5), (6) has a periodic solution $x = x(t, x_0)$, passing through the point $(0, x_0)$, then the sequence of functions:

Finally we have to show that $x(t,x_0)$ is unique solution of (1). Assume that $r(t,x_0)$ is another solution of (1), i.e.

$$r(t,x_0)=x_0+Lf(t,r(t,x_0),r(t-\tau,x_0),\dot{r}(t,x_0),\dot{r}(t-\tau,x_0)).$$

Now we can prove that $x(t,x_0) = r(t,x_0)$ in a similar way to that of theorem 1 [1].

EXISTENCE OF SOLUTION

The problem of existence of periodic of the system (1) is uniquely connected with the existence of zeros of the function $\Delta(x_0)$ it's with form

$$\Delta(x_0)=\frac{1}{T}\int_0^T f(t,x_\infty(t,x_0),x_\infty(t-\tau,x_0),\dot{x}_\infty(t,x_0),\dot{x}_\infty(t-\tau,x_0))dt, \quad (17)$$

Since this function is approximately determined from the sequence of functions:

$$\Delta_m(x_0)=\frac{1}{T}\int_0^T f(t,x_m(t,x_0),x_m(t-\tau,x_0),\dot{x}_m(t,x_0),\dot{x}_m(t-\tau,x_0)dt, \quad (18)$$

We proof the following theorem taking in to account that the following inequality

$$\| \Delta(x_0) - \Delta_m(x_0) \| \le \xi_m \quad (19)$$

where $\xi_m = Q^{m+1}(1-Q)^{-1}\dfrac{M}{4}$ will be satisfied for all $m \ge 1$.

<u>Theorem 2</u>: Let the function $f(t, x, x_\tau, y, y_\tau)$ of the system (1) is defined on the interval $[a,b]$ in R^1. Assume that for any integer $m \ge 1$ the function (18) satisfies the inequalities:

$$\inf_{a+\frac{MT}{2}\le x\le b-\frac{MT}{2}} \Delta_m(x) \le -\xi_m, \quad (20)$$

$$\sup_{a+\frac{MT}{2}\le x\le b-\frac{MT}{2}} \Delta_m(x) \ge \xi_m$$

Then (1) has a periodic solution $x = x(t)$, where

$$x(0)\in\left[a+\frac{MT}{2},b-\frac{MT}{2}\right],$$

<u>Proof</u>:

Let x_1 and x_2 be any points of the interval $[a,b]$ such that

$$\Delta_m(x_1) = \inf_{a+\frac{MT}{2}\le x\le b-\frac{MT}{2}} \Delta_m(x), \Delta_m(x_2) = \sup \Delta_m(x)$$

By using the inequalities (19) and (20),we have

$$\Delta(x_1) = \Delta_m(x_1) + (\Delta(x_1) - \Delta_m(x_1)) \le 0,$$
$$\Delta(x_2) = \Delta_m(x_2) + (\Delta(x_2) - \Delta_m(x_2)) \ge 0 \quad (21)$$

the continuity of $\Delta(x)$ and from (21) there exists a point x_∞, $x_\infty \in [x_1,x_2]$, such that $\Delta(x_\infty) = 0$, and this proves the theorem.

Similar result can be obtained for other class of different equation with retarded argument. In particular, the system of equations which has the form:

$$\frac{dx(t)}{dt} = f(t, x(t), x(t-h(t)), \dot{x}(t), \dot{x}(t-h(t))) \tag{22}$$

In the system (22), let $x = (x_1, x_2, \ldots, x_n) \in D \subset R^n$ where D is a closed bounded domain. The vector function $f(t, x, x_h, y, y_h)$ is defined on the domain:

$$f(t, x, x_h, y, y_h) \in R^1 \times D \times D \times D_1 \times D_1$$

which is continuous function in t, x, x_h, y, y_h and periodic in **t** of period **T**, where D_1 is bounded domain subset of Euclidean spaces R^m.
The scalar function $h(t)$ is a periodic in **t** with period **T**, defined and continuous in R^1.

Suppose that the function $f(t, x, x_h, y, y_h)$ satisfies the following inequalities

$$|f(t, x, x_h, y, y_h)| \le M,$$
$$|f(t, x', x'_h, y', y'_h) - f(t, x'', x''_h, y'', y''_h)| \le K_1|x - x'| +$$
$$+ K_2|x'_h - x''_h| + K_3|y - y'| + K_3|y'_h - y''_h|$$

Where $M = (M_1, M_2, \ldots, M_n)$ is a positive constant vector and K_1, K_2, K_3, K_4 are $(n \times n)$ constant matrices.

We define the non-empty sets as follows:

$$D_f = D - \frac{MT}{2}$$

and

$$D_{1f} = D_1 - 2M.$$

Furthermore, we suppose that the largest eigen value λ_{max} of the following matrix:

$$\Lambda = \begin{pmatrix} TK & 4\overline{K} \\ TK & 4\overline{K} \end{pmatrix}$$

is less then unity, where $K = K_1 + K_2$ and $\overline{K} = K_3 + K_4$

Theorem 3: If the system of equations (22) satisfies the above hypotheses and conditions has a periodic solution $x = \phi(t)$ passing at $t = 0$ through the point $x_0 \in D_f$ then the unique solution is the limit function of a uniformly convergent sequence which has the form:

$$x_{m+1}(t,x_0) = x_0 + Lf(t,x_m(t,x_0),x_m(t-h(t),x_0),\dot{x}_m(t,x_0),\dot{x}_m(t-h(t),x_0))$$

with

$x_0(t,x_0) = x_0$, $m = 0,1,2,\ldots$.

The proof is the same as in theorem 1 [1]. When we discuss the following mapping:

$$\Delta_m(x_0) = \frac{1}{T}\int_0^T f(t,x_m(t,x_0),x_m(t-h(t),x_0),\dot{x}_m(t,x_0),\dot{x}_m(t-h(t),x_0))dt.$$

Remark

We can state a theorem similar to the theorem 2, if

$\lambda_{max} = TK + 4\overline{K}$

For this remark see [2].

REFERENCES

1-Butris, R. N. Educ. and Sc. J., Vol8 (1994).

2- Mitropolsky, Yn. A. and Martynyuk, D. I.Ukrain Mat. J.,No5 (1985).

3- Samoilenko, A. M .and Ronto,N.I. Anumericl-analytic methods for investigtions periodic solutions , Ukrain ,Kiev,(1976).