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Existence and Uniqueness Theorems For Non- Linear Fractional Differential Equations

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الملخص

في هذا البحث درسنا وجود و وحدانية الحل لمعادلة تفاضلية لاخطية ذات رتبة كسرية معينة ومن الصيغة أدناه:

$$D^{\alpha}[y - T_{m-1}(y)](x) = f(x, y(x)) \qquad m-1 \le \alpha < m \qquad m \ge 2$$

مع

$$y^{(k)}(0) = y_0^{(k)} \qquad k = 0,1,2,\ldots\ldots\ldots,m-1$$

وذلك باستخدام طريقة شاودر للنقطة الثابتة المعطاة في[4] .

ABSTRACT

In this paper, we study the existence and uniqueness solution for non-linear fractional differential equation which has the form

$$D^{\alpha}[y - T_{m-1}(y)](x) = f(x, y(x)) \qquad m-1 \le \alpha < m \qquad m \ge 2$$

With

$$y^{(k)}(0) = y_0^{(k)} \qquad k = 0,1,2,\ldots\ldots\ldots,m-1$$

By using Schauder fixed point theorem [4].

INTRODUCTION

The history of the concept of fractional derivative and integral can be traced back to Leibnitz and Euler. Many authors since then have published important papers on this concept, with different approaches. The earliest paper was published by Liouville in which he defined the derivative of any order for a given function as a series of exponentials Riesz showed some properties of the integrals of fractional order which is a generalization of the Rimann integral to more than one dimension. Bassam[3] has shown the equivalence between the two definite integrals given by Holmgren and Riesz[8] , and thus he established one combined definition and which we have used in our work, Our work is to extend some results of [4] to prove the existence and uniqueness solution for certain non-linear fractional differential equation which has the form

$$D^{\alpha}[y - T_{m-1}(y)](x) = f(x, y(x)) \qquad m-1 \le \alpha < m \qquad \text{..........(1).}$$

With

$$y^{(k)}(0) = y_0^{(k)} \qquad k = 0,1,2,\ldots\ldots\ldots, m-1 \qquad \text{..........(2).}$$

Where $T_{m-1}(y)$ is the Taylor polynomial of order $(m-1)$ for y, centered at 0 , and let the function f is continuous and bounded on the domain

$$D = [0, \eta^*] \times \left[\sum_{k=0}^{m-1} \frac{y_0^{(k)}}{k!} x^k - a, \sum_{k=0}^{m-1} \frac{y_0^{(k)}}{k!} x^k + a \right] \qquad \text{..........(3).}$$

with some $\eta^* > 0$ and some a>0.Define the norm $\|f\|$ by:

$$\|f\| = \sup_{x \in I} \left\{ \exp\left(-\mu \left| \int_c^x (x-t)^{\alpha-1} z(t) dt \right| \right) |f(x)| \right\} \qquad \text{..........(4)}$$

We define the set U as follows:

$$U = \{ y : y \in C(I) : \|y - y_0\| \le a \} \qquad \text{..........(5)}$$

PRELIMINARIES

In this section we set some definitions and lemmas to be used in this work..

Definition 1:

Let f be a function which is defined a.e (almost every where) on [a, b].For $\alpha > 0$, we define:

$${}_a^b I^{\alpha} f = \frac{1}{\Gamma(\alpha)} \int_a^b f(t)(b-t)^{\alpha-1} dt$$

Provided that this integral (Lebesgue) exists , where Γ is the Gamma function.

Definition 2:

Let p, q >0, then the Beta function $\beta(p,q)$ is defined as

$$\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx$$

Remark1:

For p,q >0, then following identity holds

$$\beta(p,q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$$

Lemma 1:

Let $\alpha, \beta \in R$, $\beta > -1$. If $x > a$, then

$$ {}_{a}^{b}I^{\alpha} \frac{(t-a)^{\beta}}{\Gamma(\beta+1)} = \begin{cases} \dfrac{(x-a)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} & \alpha+\beta \neq \text{ negative integer} \\ 0 & \alpha+\beta = \text{ negative integer} \end{cases} $$

Lemma 2:

If $\alpha > 0$ and $f \in L(a,b)$, then ${}_{a}^{x}I^{-\alpha}\, {}_{a}^{t}I^{\alpha} f = f(x)$ a.e on $a \le x \le b$.

Lemma 3:

If the function f is continuous, then the initial value problem (1),(2) is equivalent to the nonlinear Volterra integral equation of the second kind

$$ y(x) = \sum_{k=0}^{m-1} \frac{x^k}{k!} y^k(0) + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t, y(t))dt \qquad \text{.........}(6) $$

With $m-1 < \alpha \le m$. In other words, every solution of the Volterra equation (3) is also a solution of our original initial value problem (1),(2).

Lemma 4:

Let

$$ E_{\alpha}(A:\eta) = \sum_{n=1}^{\infty} \frac{A^{n-1}\eta^{n\alpha-1}}{\Gamma(n\alpha)} \qquad \text{.........}(7) $$

where $A \in R$ then

i. The series converges for $x \neq 0$ and $\alpha > 0$.

ii. The series converges everywhere when $\alpha \ge 1$.

iii. If $\alpha = 1$ in (4) then $E_1(A;\eta) = \exp[A\eta]$

Lemma 5: (Arzela-Ascoli)

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions defined on the bounded interval I such that it is uniformly bounded and equi-continuous, then there exists subsequence $\{f_{nk}\}_{n=1}^{\infty}$ is uniformly convergent on I .

Theorem 1 (Schauder's fixed-point theorem):

If K is a closed, bounded and convex subset of a Banach space E , and the mapping $T: K \to K$ is completely continuous , then T has a fixed point in K .

Theorem 2:

Let U be a nonempty closed subset of a Banach space E, and let $\alpha_n \geq 0$

For every and such that $\sum_{n=0}^{\infty} \alpha_n$ converges. Moreover, let the mapping $A: U \to U$ satisfy the inequality

$$\|A^n u - A^n v\| \leq \alpha_n \|u - v\| \qquad \text{..........(8)}$$

For every $n \in N$ and every $u, v \in U$. Then, A has a uniquely defined fixed point u^*. Furthermore, for any $u_0 \in U$, the sequence $(A^n u_0)_{n=1}^{\infty}$ converges to this fixed point u^*.

Theorem 3:

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real – valued functions on the set E, then $\{f_n\}_{n=1}^{\infty}$ is uniformly convergent on E if and only if given $\varepsilon > 0$ there exist $N \in Z^+$ such that

$$|f_m(x) - f_n(x)| \in \qquad (m, n \geq N, \quad x \in E)$$

Theorem 4:

If $\{f_n\}_{n=1}^{\infty}$ is a sequence of continuous real – valued functions on a metric space X that converge uniformly to f on X. then f is also continuous on X.

Theorem 5:

A sequence $\{f_n\}_{n=1}^{\infty}$ in C[a,b] converges to $f \in C[a,b]$ if and only if it is converge uniformly to f on [a,b].

Remark2:

For the proof of definitions, lemmas and theorems see [1], [2], [3], [4, [5], [6], and [7].

THE MAIN THEOREMS

In this section we shall prove the existence and uniqueness theorem.

Theorem 6:

Let the function $f: D \to R$ be defined in the domain (3), define

$\eta = \min\left\{\eta^*, \left(a\Gamma(\alpha+1)/\|f\|\right)^{\frac{1}{\alpha}}\right\}$, then there exists a function $y: [0, \eta] \to R$ solving the initial value problem (1) and (2).

Proof:

Let $y_0(x)=\sum_{k=0}^{m-1}\frac{y_0^{(k)}}{k!}x^k$, $I=[0,\eta]$ and we shall prove that the norm (4) satisfies the following conditions .

i. $\|y\|=0 \quad \Leftrightarrow \quad \sup_{x\in I}\left\{\exp\left(-\mu\left|\int_c^x (x-t)^{\alpha-1}z(t)dt\right|\right)|y(x)|\right\}=0$

$$\Leftrightarrow \left\{\exp\left(-\mu\left|\int_c^x (x-t)^{\alpha-1}z(t)dt\right|\right)|y(x)|\right\}=0$$

$$\Leftrightarrow \frac{|y(x)|}{\exp\left(-\mu\left|\int_c^x (x-t)^{\alpha-1}z(t)dt\right|\right)}=0$$

Since $\exp(u)\neq 0$ for any u

$$\Leftrightarrow |y(x)|=0 \qquad \Leftrightarrow y=0$$

and so

$$\| y \|=0 \quad \Leftrightarrow \quad y=0$$

ii. For all $\alpha \in R$ and $y \in C(J)$

$$\| \alpha y \|=\sup_{x\in I}\left\{\exp\left(-\mu\left|\int_c^x (x-t)^{\alpha-1}z(t)dt\right|\right)| \alpha y(x)|\right\}$$

$$=\sup_{x\in I}\left\{\exp\left(-\mu\left|\int_c^x (x-t)^{\alpha-1}z(t)dt\right|\right)|\alpha||y(x)|\right\}$$

$$=|\alpha| \sup_{x\in I}\left\{\exp\left(-\mu\left|\int_c^x (x-t)^{\alpha-1}z(t)dt\right|\right)|y(x)|\right\}$$

$$=|\alpha|\|y\|$$

$$\| \alpha y\|=|\alpha|\|y\|$$

iii. For all $f,g \in C(I)$ then

$$\|f+g\|=\sup_{x\in I}\left\{\exp\left(-\mu\left|\int_c^x (x-t)^{\alpha-1}z(t)dt\right|\right)|(f+g)(x)|\right\}$$

$$\le \sup_{x\in I}\left\{\exp\left(-\mu\left|\int_c^x (x-t)^{\alpha-1} z(t)dt\right|\right)|f(x)|+|g(x)|\right\}$$

$$\le \sup_{x\in I}\left\{\exp\left(-\mu\left|\int_c^x (x-t)^{\alpha-1} z(t)dt\right|\right)|f(x)|\right\}+\sup_{x\in J}\left\{\exp\left(-\mu\left|\int_c^x (x-t)^{\alpha-1} z(t)dt\right|\right)|g(x$$

$$=\| f \|+\| g \|$$

$$\| f+g \| \le \| f \|+\| g \|$$

This implies that $\| y \|$ is a norm.

Next, we shall prove that the space C (I) with the norm (4) is a Banach space.

Let $\{y_n(x)\}_{n=1}^{\infty}$ be a Cauchy sequence in C (I), $\forall \varepsilon>0$ there is an $m_0 \in I$ such that

$$\| y_n - y_m \| \le \varepsilon \qquad , \qquad (m,n>m_0)$$

$$\sup_{x\in I}\left\{\exp\left(-\mu\left|\int_c^x (x-t)^{\alpha-1} z(t)dt\right|\right)|y_n(x)-y_m(x)|\right\}\le\varepsilon$$

Thus

$$\left\{\exp\left(-\mu\left|\int_c^x (x-t)^{\alpha-1} z(t)dt\right|\right)|y_n(x)-y_m(x)|\right\}\le\varepsilon$$

or

$$|y_n(x)-y_m(x)|\le\frac{\varepsilon}{\exp\left(-\mu\left|\int_c^x (x-t)^{\alpha-1} z(t)dt\right|\right)}$$

$$|y_n(x)-y_m(x)|\le\varepsilon_1 \qquad \text{Where } \varepsilon_1=\frac{\varepsilon}{\exp\left(-\mu\left|\int_c^x (x-t)^{\alpha-1} z(t)dt\right|\right)}$$

It follows from the theorem (3) that $\{y_n(x)\}_{n=1}^{\infty}$ is convergent uniformly to the function y .By the theorem (4) $y\in C(I)$ therefore from the theorem (5) $\{y_n(x)\}_{n=1}^{\infty}$ convergent to the y.

This implies that C(I) is a Banach space .

We shall prove the set (5) is a closed subset in C (I). If $\{y_n\}_{n=1}^{\infty}$ be a sequence in U such that $\lim_{n\to\infty} y_n = y$ then $\forall\varepsilon>0$, $\exists N\in I^{+}$ such that

$$\|y_n - y\|<\varepsilon \qquad , \quad n\ge N$$

$$\|y-y_0\| = \sup_{x\in I}\left\{\exp\left(-\mu\left|\int_c^x (x-t)^{\alpha-1} z(t)dt\right|\right)|y(x)-y_0(x)|\right\}$$

$$= \sup_{x\in I}\left\{\exp\left(-\mu\left|\int_c^x (x-t)^{\alpha-1} z(t)dt\right|\right)|y(x)-y_n+y_n-y_0(x)|\right\}$$

$$\le \sup_{x\in I}\left\{\exp\left(-\mu\left|\int_c^x (x-t)^{\alpha-1} z(t)dt\right|\right)|y(x)-y_n(x)|\right\}+$$

$$+\sup_{x\in I}\left\{\exp\left(-\mu\left|\int_c^x (x-t)^{\alpha-1} z(t)dt\right|\right)|y_n(x)-y_0(x)|\right\}$$

$$= \|y_n-y\| + \|y_n-y_0\|$$

$$\le \ \varepsilon + \ a$$

Since $\varepsilon > 0$ arbitrary then $\|y-y_0\| \le a$

So $y \in U$, and therefore U is a closed set in $C(I)$, and hence U is a closed subset of a Banach space of all continuous function on I.

Now, we shall prove that (5) is convex set. Let $y_1\, y_2 \in U$, $0<\lambda\le 1$, then

$$\|(\lambda y_1+(1-\lambda)y_2)-y_0\| = \sup_{x\in I}\left\{\exp\left(-\mu\left|\int_c^x (x-t)^{\alpha-1} z(t)dt\right|\right)|(\lambda y_1(x)+(1-\lambda)y_2(x))-y_0(x)|\right\}$$

$$= \sup_{x\in I}\left\{\exp\left(-\mu\left|\int_c^x (x-t)^{\alpha-1} z(t)dt\right|\right)|\lambda y_1(x)-\lambda y_0(x)+y_2(x)-\lambda y_2(x)-y_0(x)+\lambda y_0(x)|\right\}$$

$$= \sup_{x\in I}\left\{\exp\left(-\mu\left|\int_c^x (x-t)^{\alpha-1} z(t)dt\right|\right)|\lambda(y_1(x)-y_0(x))+(1-\lambda)y_2(x)-(1+\lambda)y_0(x)|\right\}$$

$$= \sup_{x\in I}\left\{\exp\left(-\mu\left|\int_c^x (x-t)^{\alpha-1} z(t)dt\right|\right)|\lambda(y_1(x)-y_0(x))+(1-\lambda)(y_2(x)-y_0(x))|\right\}$$

$$\le \sup_{x\in I_n}\left\{\exp\left(-\mu\left|\int_c^x (x-t)^{\alpha-1} z(t)dt\right|\right)|\lambda(y_1(x)-y_0(x))|\right\}+$$

$$+\sup_{x\in I}\left\{\exp\left(-\mu\left|\int_c^x (x-t)^{\alpha-1} z(t)dt\right|\right)|(1-\lambda)(y_2(x)-y_0(x))|\right\}$$

$$\le \lambda\sup_{x\in I}\left\{\exp\left(-\mu\left|\int_c^x (x-t)^{\alpha-1} z(t)dt\right|\right)|(y_1(x)-y_0(x))|\right\}+$$

$$+(1-\lambda)\sup_{x\in I}\left\{\exp\left(-\mu\left|\int_c^x (x-t)^{\alpha-1} z(t)dt\right|\right)|(y_2(x)-y_0(x))|\right\}$$

$$=\lambda\ \|y_1-y_0\| + (1-\lambda)\ \|y_2-y_0\|$$

Since $y_1, y_2 \in U$ then

$$\|y_1 - y_0\| \le a \quad , \quad \|y_2 - y_0\| \le a$$

$$\|(\lambda y_1 + (1-\lambda)y_2) - y_0\| \le \lambda a + (1-\lambda)a$$

$$= a$$

So $\lambda y_1 + (1-\lambda)y_2 \in U$

Therefore U is convex.

On U we define the operator T by

$$(Ty)(x) = y_0(x) + \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t, y(t))dt \qquad \text{..........(9)}$$

We shall prove that T is continuous.

Since f is continuous on the compact set D, it is uniformly continuous there. Thus, give an arbitrary $\varepsilon > 0$, we can find $\delta > 0$ such that

$$|f(x,y) - f(x,z)| < \kappa_1 \qquad \text{whenever} \qquad |y-z| < \delta \qquad \text{..........(10)}$$

$$\text{Where } k_1 = \frac{\varepsilon}{\eta^\alpha}\Gamma(\alpha+1)$$

Now, let $y, \tilde{y} \in U$ such that $\|y - \tilde{y}\| < \delta$. Then from the equation (10)

$$|f(x, y(x)) - f(x, \tilde{y}(x))| < k_1 \qquad \text{..........(11)}$$

For all $x \in I$. Hence

$$|(T\tilde{y})(x) - (Ty)(x)| = \frac{1}{\Gamma(\alpha)}\left|\int_0^x (x-t)^{\alpha-1} f(t, \tilde{y}(t))dt - \int_0^x (x-t)^{\alpha-1} f(t, y(t))dt\right|$$

$$\le \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} |f(t, \tilde{y}(t)) - f(t, y(t))|dt$$

$$\le \frac{k}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} dt$$

$$= \frac{\varepsilon\Gamma(\alpha+1)}{\Gamma(\alpha+1)\eta^\alpha} x^\alpha \qquad = \frac{\varepsilon x^\alpha}{\eta^\alpha}$$

Since $0 \le x \le \eta$, $m-1 \le \alpha < m$ then $x^\alpha \le \eta^\alpha$

$$|(T\tilde{y})(x) - (Ty)(x)| < \frac{\varepsilon\eta^\alpha}{\eta^\alpha} = \varepsilon$$

This implies that T is continuous on U.

Let $y \in U$, $x \in I$

$$\|(Ty)(x) - y_0\| = \frac{1}{\Gamma(\alpha)} \left\| \int_0^x (x-t)^{\alpha-1} f(t, y(t))dt \right\|$$

$$= \frac{1}{\Gamma(\alpha)} \sup_{0 \le w \le x} \left\{ \exp\left(-\mu \left| \int_c^w (w-t)^{\alpha-1} z(t)dt \right| \right) \left| \int_0^w (w-s)^{\alpha-1} f(s, y(s))ds \right| \right\}$$

$0 < w < x \quad \Rightarrow \quad w - t < x - t$

Since $\alpha > 1 \quad \Rightarrow \quad (w-t)^{\alpha-1} \le (x-t)^{\alpha-1}$

$$\le \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \sup_{0 \le w \le t} \left\{ \exp\left(-\mu \left| \int_c^w (w-t)^{\alpha-1} z(t)dt \right| \right) |f(w, y(w))| \right\} dt$$

$$\le \frac{1}{\Gamma(\alpha)} \sup_{0 \le w \le x} \left\{ \exp\left(-\mu \left| \int_c^w (w-t)^{\alpha-1} z(t)dt \right| \right) |f(w, y(w))| \right\} \int_0^x (x-t)^{\alpha-1} dt$$

$$\le \frac{\|f\|}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} dt \qquad \le \frac{\|f\|}{\alpha \Gamma(\alpha)} x^\alpha$$

$$\le \frac{\|f\|}{\Gamma(\alpha+1)} \eta^\alpha \quad = \quad a$$

Therefore $Ty \in U$ when $y \in U$, then *T maps* from U into itself.

Then we look at the set of functions T*(U)* which is define by

$$T(U) = \{Ty : y \in U\}$$

For $z \in T(U)$ we find that , for all $x \in I$,

$$\|z(x)\| = \|(Ty)(x)\|$$

$$= \left\| y_0(x) + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t, y(t))dt \right\|$$

$$\le \left\| \sum_{k=0}^{m-1} \frac{y_0^{(k)}}{k!} x^k \right\| + \frac{1}{\Gamma(\alpha)} \left\| \int_0^x (x-t)^{\alpha-1} f(t, y(t))dt \right\|$$

$$\le \sum_{k=0}^{m-1} \frac{\|y_0^{(k)}\|}{k!} x^k + \frac{1}{\Gamma(\alpha)} \sup_{0 \le w \le x} \left\{ \exp\left(-\mu \left| \int_c^w (w-t)^{\alpha-1} z(t)dt \right| \right) \left| \int_0^w (w-t)^{\alpha-1} f(t, y(t))dt \right| \right\}$$

$$\le \sum_{k=0}^{m-1} \frac{\|y_0^{(k)}\|}{k!} x^k + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \sup_{0 \le w \le t} \left\{ \exp\left(-\mu \left| \int_c^w (w-t)^{\alpha-1} z(t)dt \right| \right) |f(w, y(w))| \right\} dt$$

$$=\sum_{k=0}^{m-1}\frac{\left\|y_0^{(k)}\right\|}{k!}x^k+\frac{1}{\Gamma(\alpha)}\sup_{0\le w\le x}\left\{\exp\left(-\mu\left|\int_c^w (w-t)^{\alpha-1}z(t)dt\right|\right)\left|f(w,y(w))\right|\right\}\int_0^x (x-t)^{\alpha-1}dt$$

$$=\sum_{k=0}^{m-1}\frac{\left\|y_0^{(k)}\right\|}{k!}x^k+\frac{\| f \|}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}dt$$

$$=\sum_{k=0}^{m-1}\frac{\left\|y_0^{(k)}\right\|}{k!}x^k+\frac{\| f \|}{\Gamma(\alpha+1)}x^k$$

$$\le\sum_{k=0}^{m-1}\frac{\left\|y_0^{(k)}\right\|}{k!}x^k+\frac{\| f \|}{\Gamma(\alpha+1)}\eta^\alpha$$

$$\le\sum_{k=0}^{m-1}\frac{\left\|y_0^{(k)}\right\|}{k!}x^k+a$$

Which means that T(U) is bounded. Moreover, for $0\le x_1\le x_2\le\eta$ then

$$\|(Ty)(x_1)-(Ty)(x_2)\|=\frac{1}{\Gamma(\alpha)}\left\|\int_0^{x_1}(x_1-t)^{\alpha-1}f(t,y(t))dt-\int_0^{x_2}(x_2-t)^{\alpha-1}f(t,y(t))dt\right\|$$

$$=\frac{1}{\Gamma(\alpha)}\left\|\int_0^{x_1}(x_1-t)^{\alpha-1}f(t,y(t))dt-\left[\int_0^{x_1}(x_2-t)^{\alpha-1}f(t,y(t))dt+\int_{x_1}^{x2}(x_2-t)^{\alpha-1}f(t,y(t))dt\right]\right\|$$

$$\le\frac{1}{\Gamma(\alpha)}\left[\left\|\int_0^{x_1}\left[(x_1-t)^{\alpha-1}-(x_2-t)^{\alpha-1}\right]f(t,y(t))dt\right\|+\left\|\int_{x_1}^{x_2}(x_2-t)^{\alpha-1}f(t,y(t))dt\right\|\right]$$

$$=\frac{1}{\Gamma(\alpha)}\left[\sup_{0\le w\le x}\left\{\exp\left(-\mu\left|\int_c^w(w-t)^{\alpha-1}z(t)dt\right|\right)\left|\int_0^{w_1}\left[(w_1-t)^{\alpha-1}-(w_2-t)^{\alpha-1}\right]f(t,y(t))dt\right|\right\}+\right.$$

$$\left.+\sup_{0\le w\le x}\left\{\exp\left(-\mu\left|\int_c^w(w-t)^{\alpha-1}z(t)dt\right|\right)\left|\int_{w_1}^{w_2}(w_2-t)^{\alpha-1}f(t,y(t))dt\right|\right\}\right]$$

$$\le\frac{1}{\Gamma(\alpha)}\left[\int_0^{x_1}\left[(x_1-t)^{\alpha-1}-(x_2-t)^{\alpha-1}\right]\sup_{0\le w\le t}\left\{\exp\left(-\mu\left|\int_c^w(w-t)^{\alpha-1}z(t)dt\right|\right)f(w,y(w))\right\}dt+\right.$$

$$\left.+\int_{x_1}^{x_2}(x_2-t)^{\alpha-1}\sup_{0\le w\le t}\left\{\exp\left(-\mu\left|\int_c^w(w-t)^{\alpha-1}z(t)dt\right|\right)f(w,y(w))\right\}dt\right]$$

$$=\frac{1}{\Gamma(\alpha)}\left[\sup_{0\le w\le x}\left\{\exp\left(-\mu\left|\int_c^w (w-t)^{\alpha-1}z(t)dt\right|f(w,y(w))\right)\right\}\int_0^{x_1}\left[(x_1-t)^{\alpha-1}-(x_2-t)^{\alpha-1}\right]dt+\right.$$

$$\left.+\sup_{0\le w\le x}\left\{\exp\left(-\mu\left|\int_c^w (w-t)^{\alpha-1}z(t)dt\right|\right)f(w,y(w))\right\}\int_{x_1}^{x_2}(x_2-t)^{\alpha-1}dt\right]$$

$$=\frac{\|f\|}{\Gamma(\alpha)}\left[\int_0^{x_1}\left[(x_1-t)^{\alpha-1}-(x_2-t)^{\alpha-1}\right]dt+\int_{x_1}^{x_2}(x_2-t)^{\alpha-1}dt\right]$$

$$=\frac{\|f\|}{\Gamma(\alpha+1)}\left\{\left[(x_1-t)^{\alpha}-(x_2-t)^{\alpha}\right]_0^{x_1}+\left[(x_2-t)^{\alpha}\right]_{x_1}^{x_2}\right\}$$

$$\le\frac{\|f\|}{\Gamma(\alpha+1)}\left[2(x_2-x_1)^{\alpha}+x_1^{\alpha}+x_2^{\alpha}\right] \qquad \text{..........(12)}$$

$$\le\frac{2\|f\|}{\Gamma(\alpha+1)}(x_2-x_1)^{\alpha}$$

Thus, if $|x_2-x_1|<\delta$, then

$$\|(Ty)(x_1)-(Ty)(x_2)\|\le\frac{2\|f\|}{\Gamma(\alpha+1)}\delta^{\alpha} \qquad \text{..........(13)}$$

We see that the right-hand side of the inequality (13) is independent of y.This means the set T(U) is equicontinuous, then from lemma (5) every sequence of function in T(U) has a uniformly convergent subsequence .

Therefore T (U) is relatively compact .then from Schauder's fixed-point theorem T has a fixed point

$$(Ty)(x)=y(x)$$

$$y(x)=y_0(x)+\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t,y(t))dt$$

From the lemma (3), y(x) satisfies the initial value problem (1)and (2).

Theorem 7 :

Let all assumptions of theorem (6) be given, and let f satisfies a Lipschitz condition i.e.

$$|f(x,y)-f(x,z)|\le A|y-z| \qquad \text{..........(14)}$$

With some constant A>0 independent of x, y and z .Then there exists at most one function $y:I\to R$ solving the initial value problem (1),and (2).

Proof:

From the theorem (6) U is a closed subset of the Banach space of all continuous function on I, and it is not empty, and from the equation (12) then (Ty) is continuous function when $0 \le x_1 \le x_2 \le \eta$.

The next step is to prove that, for every $n \in N_0$ and every $x \in I$, we have

$$\| T^n y - T^n \tilde{y} \| \le \frac{(A\eta^n)^n}{\Gamma(1+\alpha n)} \| y - \tilde{y} \| \qquad \text{...........(15)}$$

This can be seen by induction. In the case n=0 the statement is trivially true.

Suppose that the statement is true in the case (n-1).

For the induction step $n-1 \to n$, we write

$$\| T^n y - T^n \tilde{y} \| = \| T(T^{n-1} y) - T(T^{n-1} \tilde{y}) \|$$

$$= \frac{1}{\Gamma(\alpha)} \left\| \int_0^x (x-t)^{\alpha-1} f(t, T^{n-1} y(t)) dt - \int_0^x (x-t)^{\alpha-1} f(t, T^{n-1} \tilde{y}(t)) dt \right\|$$

$$\le \frac{1}{\Gamma(\alpha)} \sup_{0 \le w \le x} \left\{ \exp\left(-\mu \left| \int_c^w (w-t)^{\alpha-1} z(t) dt \right| \right) \left| \int_0^w (w-t)^{\alpha-1} \left[f(t, T^{n-1} y(x)) - f(t, T^{n-1} \tilde{y}(t)) \right] dt \right| \right\}$$

$$\le \frac{1}{\Gamma(\alpha)} \sup_{0 \le w \le x} \left\{ \exp\left(-\mu \left| \int_c^w (w-t)^{\alpha-1} z(t) dt \right| \right) \int_0^w (w-t)^{\alpha-1} \left| f(t, T^{n-1} y(t)) - f(t, T^{n-1} \tilde{y}) \right| dt \right\}$$

We use (12) and the induction hypothesis and find

$$\| T^n y - T^n \tilde{y} \| \le \frac{A}{\Gamma(\alpha)} \sup_{0 \le w \le x} \left\{ \exp\left(-\mu \left| \int_c^w (w-t)^{\alpha-1} z(t) dt \right| \right) \int_0^w (w-t)^{\alpha-1} \left| T^{n-1} y(t) - T^{n-1} \tilde{y}(t) \right| dt \right\}$$

$$\le \frac{A}{\Gamma(\alpha)} \sup_{0 \le w \le x} \left\{ \exp\left(-\mu \left| \int_c^w (w-t)^{\alpha-1} z(t) dt \right| \right) \left| T^{n-1} y(w) - T^{n-1} \tilde{y}(w) \right| \right\} \int_0^x (x-t)^{\alpha-1} dt$$

$$\le \frac{A}{\Gamma(\alpha)} \| T^{n-1} y - T^{n-1} \tilde{y} \| . \int_0^x (x-t)^{\alpha-1} dt$$

$$\le \frac{A}{\Gamma(\alpha)\Gamma(1+\alpha(n-1))} \| y - \tilde{y} \| \int_0^x (x-t)^{\alpha-1} t^{\alpha(n-1)} dt$$

$$\| T^n y - T^n \tilde{y} \| \le \frac{A^n}{\Gamma(\alpha)\Gamma(1+\alpha(n-1))} \| y - \tilde{y} \| \left[-\int_0^1 u^{\alpha-1} x^{\alpha-1} x^{\alpha(n-1)} (1-u)^{\alpha(n-1)} (-x\, du) \right]$$

$$\le \frac{A^n}{\Gamma(\alpha)\Gamma(1+\alpha(n-1))} \| y - \tilde{y} \| \int_0^1 u^{\alpha-1} (1-u)^{\alpha(n-1)} x^{\alpha n} du$$

$$\le \frac{A^n}{\Gamma(\alpha)\Gamma(1+\alpha(n-1))} \| y - \tilde{y} \| \beta(\alpha, \alpha(n-1)+1) x^{\alpha n}$$

$$\leq \frac{A^n}{\Gamma(\alpha)\Gamma(1+\alpha(n-1))} \| y-\tilde{y} \| \frac{\Gamma(\alpha)\Gamma(\alpha(n-1)+1)}{\Gamma(\alpha n+1)} x^{\alpha n}$$

$$\leq \frac{A^n}{\Gamma(\alpha n+1)} \| y-\tilde{y} \| \eta^{\alpha n}$$

$$\leq \frac{(A\eta^\alpha)^n}{\Gamma(\alpha n+1)} \| y-\tilde{y} \|$$

This completes the proof of the statement (15).

We have now shown that the operator *T* satisfies the assumptions of theorem (2) with

$$a_n = (A\eta^\alpha)^n / \Gamma(1+\alpha n)$$

To apply that theorem, we only need to verify that the series $\sum_{n=0}^{\infty} a_n$ converges.

From the lemma (4)

$$E_\alpha(A:\eta) = \sum_{n=1}^{\infty} \frac{A^{n-1}\eta^{n\alpha-1}}{\Gamma(n\alpha)}$$

$$= \sum_{n=0}^{\infty} \frac{A^n \eta^{n\alpha}}{\Gamma(n\alpha+1)}$$

Since $\alpha > 1$ then the series $\sum_{n=0}^{\infty} \frac{A^n \eta^{n\alpha}}{\Gamma(n\alpha+1)}$ converges.

Then T *has* a unique fixed point

$$(Ty)(x) = y(x)$$

$$y(x) = y_0(x) + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t, y(t))dt$$

Remark3: We note here that our result is an extension of Diethelm, K. and Neville J. F when $\alpha = 1$.

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