

## Method of Lines With Haar Wavelet For Solving Parabolic Differential Equation

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### الخلاصة

تم في هذا البحث عمل خوارزمية لحل بعض المعادلات التفاضلية الجزئية بإضافة (Haar wavelet) إلى طريقة الخطوط (Method of lines) وقد تبين من هذه الخوارزمية أن الوقت المستغرق في التنفيذ قد اختزل الوقت وان (flops) قد قل. وقد طبقت هذه الخوارزمية على المعادلة التفاضلية الجزئية المكافئة ثم قورنت هذه النتيجة بطريقة الخطوط (Mol) وتبينه أسرع منها.

### ABSTRACT

In this paper we present a theoretical framework and numerical comparisons for a wavelet-based algorithm associated with both method of lines and wavelets for solving some partial differential equations. In particular, we consider a wavelet-based algorithm using Method of Lines (MOL) analysis. The advantage is in the simplicity of the boundary modification, and relatively simple and small representing the differential operators, in contrast to other wavelet-based algorithms. The time of calculations and number of flops were reduced using *Haar* wavelets, and as a demonstration, an example for solving the diffusion equation.

**Key words:** Method of Lines, partial differential equations, Haar wavelet

## 1. Introduction:

During the past years, wavelet-based algorithms have been proven to be optimal efficient numerical schemes for solving system of the linear equation [1],[6],[7] and [8]. This means that the resulting linear systems of equations can be solved with an overall amount of work, which is of the order of the number of unknowns. In particular, this implies that introducing Haar wavelets can reduce the number of iterations, which possesses small dimension extracted from the original dimension of a matrix. Discretizations allowing us to use new methods, such as Method of Lines (MOL), as powerful tools to convert the complex theoretical systems of equations into simple and direct numerical schemes. This availability of direct numerical schemes together with additional analytic properties provide a powerful tool to prove the theoretical asymptotically optimality of the resulting numerical schemes for a whole range of operator equations, including parabolic partial differential equations.

Therefore, we aimed at designing a wavelet-based algorithm that realizes all promising features of wavelet-based algorithms for this class of problems. In this paper we describe new ideas that we have implemented during the last year and which might be helpful as a new attempt in the area of this kind of problems. We have a solving system of the linear equation in papers [1],[6] and [7]. Then we continue the work with solving parabolic differential equations using wavelet also.

## 2. Method of Lines:

Let us consider a partial differential equation defined on  $\Omega: 0 \leq x, t \leq 1$ , Subject to the boundary conditions:

$$u_{xx} = g(x, t, u, u_x, u_t)$$

$$u(x, 0) = \phi_{lo}(x),$$

$$u(x, 1) = \phi_{hi}(x),$$

$$u(0, t) = \phi_{left}(t),$$

$$u(1, t) = \phi_{right}(t),$$

Limiting our scope to the method of lines (MOL), we introduce  $n$  equally spaced lines in  $\Omega$  parallel (in our case) to the  $x$ -axis, separated by distance

$h = 1/(n+1)$ . Along particular line  $k$ , we replace  $u_{xx}$  is replaced by  $u_k''$ ,  $u_x$

by  $u_k'$ , and we also replace the derivatives in the normal direction is also replaced by a finite difference approximation, e.g.

$$\begin{aligned}
 u_t &= \frac{u_{k+1} - u_{k-1}}{2h} + o(h^2); \\
 u_{tt} &= \frac{u_{k+1} - 2u_k + u_{k-1}}{h^2} + o(h^4); \\
 u_{xt} = u_{tx} &= \frac{u'_{k+1} - u'_{k-1}}{2h} + o(h^2)
 \end{aligned}$$

Under this arrangement, the partial differential equation is now approximated by a system of n second order ordinary differential equation in  $U_k$ ,  $k = 1 \dots n$ , with x as the independent variable,

$$U''_k = g_k(x, u_{k-1}, u'_{k-1}, u_k, u'_k, u_{k+1}, u'_{k+1}).$$

Subject to the boundary conditions

$$\begin{aligned}
 u_k(0) &= \varphi_{left}(t_k); \\
 u_k(1) &= \varphi_{right}(t_k);
 \end{aligned}$$

For simplicity we call it the MOL-model of partial differential equation. Note that now  $u_0 = \phi_{l0}(x)$  and  $u_{n+1} = \phi_{ri}(x)$  are now function. For our computational purposes we convert this system into a system of the first order equations by introducing new variables  $V_k = U'_k$ , rearrange the variables so that it is transformed into the set of 2n first order equations:

$$\frac{d}{dx} \begin{bmatrix} U_1 \\ U_2 \\ \cdot \\ U_n \\ V_1 \\ V_2 \\ \cdot \\ \cdot \\ V_{n-1} \\ V_n \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ \cdot \\ V_n \\ f_1 \\ f_2 \\ \cdot \\ \cdot \\ f_{n-1} \\ f_n \end{bmatrix}$$

$$f_k = f_k(h, U_{k-1}, V_{k-1}, U_k, V_k, U_{k+1}, V_{k+1}), \quad k=1,2,\dots,n.$$

The boundary conditions are adjusted accordingly. It is well known that at any value of  $n$ , the solution of the MOL-model would be  $O(h^2)$  approximation to the exact solution of the original partial differential equation, and we have used this fact to decide on the correctness of the computed solution. In terms of solving the partial differential equation, large value of  $n$  is desirable. However, as  $n$  increases, the computed solution of the MOL-model becomes more sensitive to rounding error, as will be shown below. Within the limited range of practical values of  $n$ , this scheme allows us to create a large set of problems.

### 3. Parabolic PDEs:

Some very simple types of PDEs are common that were assigned special names. Given the following PDE in two variables  $x$  and  $t$ :

$$A(x,t)\frac{\partial^2 u}{\partial x^2} + B(x,t)\frac{\partial^2 u}{\partial x \partial t} + C(x,t)\frac{\partial^2 u}{\partial t^2} + f(x,t, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}) = 0 \quad (1)$$

Which characteristic of field problems. In physics,  $x$  and  $t$  can be either spatial or temporal variables and  $A, B, C$  and  $f$  can be arbitrary function of  $x, t, \frac{\partial u}{\partial x}$  and  $\frac{\partial u}{\partial t}$ . Such a PDE is called quasi-linear, since it is linear in the highest derivatives.

Depending on the numerical relationship between  $A, B$ , and  $C$  Eq(1) is classified as either being parabolic hyperbolic, or elliptic. The classification is as follows

$$B^2 - 4AC > 0 \rightarrow \text{PDE is hyperbolic}$$

$$B^2 - 4AC = 0 \rightarrow \text{PDE is parabolic}$$

$$B^2 - 4AC < 0 \rightarrow \text{PDE is elliptic}$$

The classification makes sense, the numerical methods most suitable for these three types of PDEs are vastly different. In this paper, we shall deal with PDEs of the parabolic type exclusively. Parabolic PDEs are very common. For example, all thermal problems are of that nature. The simplest example of a parabolic PDE is the one-dimensional heat diffusion problem, where  $K$  is the thermal diffusivity.

$$K\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + G(x,t) \quad (2)$$

$$u(0,t) = d_0(t) \quad u(1,t) = d_1(t) \quad (3)$$

$$u(x,0) = f(x), \quad 0 < x < 1, t \geq 0. \quad (4)$$

The notion  $u_t$  and  $u_{xx}$  refers to partial derivatives with respect to  $t$  and  $x$ , respectively. Then unknown function  $u(x,t)$  depends on the time  $t$  and a spatial variable  $x$ . The condition (3) are called boundary condition, and (4) is called an initial condition. The solution  $u$  can be interpreted as temperature of the an insulated rod of length 1, with  $u(x,t)$  the temperature at position  $x$  and time  $t$ ; thus (2) is often called heat equation. The functions  $G, d_0, d_1,$  and  $f$  are assumed to be given and smooth. For a development of the theory of (2)-(4), See [9] or any standard introduction to partial differential. We gave the method of lines for solving for  $u$ , a numerical method that has become much more popular in the past ten to fifteen years. It will also lead to the solution of a stiff system of ordinary differential equations. Let  $n > 0$  be integer, and define  $h = 1/n$ .

$$x_k = kh, \quad k = 0, 1, 2, \dots, n.$$

We discretize Eq.(2) by approximating the partial derivative. Using the following this,

$$u_{xx} = \frac{u(x_{k+1}, t) - 2u(x_k, t) + u(x_{k-1}, t))}{h^2} - \frac{h^2}{12} \frac{\partial^4 u(x_k, t)}{\partial x^4}, \quad k = 1, \dots, n-1$$

Substituting into (2)

$$u_t(x_k, t) = K \frac{u(x_{k+1}, t) - 2u(x_k, t) + u(x_{k-1}, t))}{h^2} + G(x_k, t) - \frac{h^2}{12} \frac{\partial^4 u(x_k, t)}{\partial x^4}, \quad 1 \leq k \leq n-1 \quad (5)$$

Equation is to be approximated at each interior node point  $x_k$ . The unknown  $\zeta_k \in [x_{k-1}, x_{k+1}]$ . Drop the final term in (5), the truncation error in the numerical differential. Forcing equality in the resulting approximate, we obtain

$$u_k'(t) = K \frac{u_{k+1}(t) - 2u_k(t) + u_{k-1}(t))}{h^2} + G(x_k, t) \quad (6)$$

$k = 1, 2, \dots, n-1$ . The function  $u_k(t)$  are intended to be approximations of  $u(x_k, t)$ ,  $1 \leq k \leq n-1$ . This is the method of lines approximation to (6), and it is a system of  $n-1$  ordinary differential equation. Note that  $u_0(t)$  and  $u_n(t)$ , which are needed. For  $k=1$  and  $k=n-1$ , are given using (3):

$$u_0(t) = d_0(t) \quad (7a)$$

$$u_m(t) = d_1(t) \quad (7b)$$

The initial condition for ( 6 ) is given by ( 4):

$$u_j(0) = f(x_j), \quad (8)$$

where  $1 \leq j \leq m - 1$

The name method of lines comes from solving  $u(x,t)$  along the lines  $(x_j,t)$   $t \geq 0$ ,  $1 \leq j \leq m - 1$ , in the  $(x,t)$ -plane.

Under suitable smoothness assumptions on the functions  $d_0, d_1, G$  and  $f$ , it can be shown that :

$$\text{Max}_{\substack{0 \leq j \leq m \\ 0 \leq t \leq T}} |U(x_j, t) - u_j(t)| \leq C_T h^2 \quad (9)$$

Thus to complete the solution process, we need only to solve the system (6). It will be convenient to write (6) in matrix form. Introduce

$$u(t) = [u_1(t), \dots, u_{m-1}(t)]^T \quad u_0 = [f(x_1), \dots, f(x_{m-1})]^T$$

$$g(t) = \left[ \frac{K}{h^2} d_0(t) + G(x_1, t)G(x_2, t), \dots, G(x_{m-2}, t), \frac{K}{h^2} d_1(t) + G(x_{m-1}, t) \right]^T$$

$$A = \frac{K}{h^2} \begin{bmatrix} -2 & 1 & 0 & 0 & \dots & \dots & 0 \\ 1 & -2 & 1 & 0 & & & \cdot \\ 0 & 1 & -2 & 1 & & & \cdot \\ \cdot & & & & \cdot & & \cdot \\ \cdot & & & & & & 0 \\ \cdot & & & & & & \cdot \\ & & & & 1 & -2 & 1 \\ 0 & 0 & \dots & \dots & 0 & 1 & -2 \end{bmatrix}$$

The matrix  $A$  is of order  $m - 1$ . In the definitions of  $u$  and  $g$ , the superscript  $T$  indicates matrix transpose, so that  $u$  and  $g$  are column vectors of length  $m - 1$ . Using these matrices, Eqs. ( 6)-( 8) can be rewritten as

$$u'(t) = Au(t) + g(t)$$

$$x = \begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ u_{3,j+1} \\ \cdot \\ \cdot \\ \cdot \\ u_{n-2,j+1} \\ u_{n-1,j+1} \end{bmatrix} \quad b = \begin{bmatrix} u_{1,j} + \alpha u_0 \\ u_{2,j} \\ u_{3,j} \\ \cdot \\ \cdot \\ \cdot \\ u_{n-2,j} \\ u_{n-1,j} + \alpha u_n \end{bmatrix}$$

vii. Then

$$\begin{aligned} Ax &= b \\ WAx &= Wb \\ WAW^{-1}Wx &= Wb \\ WAW'Wx &= Wb \\ A_w x_w &= b_w \end{aligned}$$

where

$$\begin{aligned} A_w &= WAW' \\ b_w &= Wb \\ x_w &= Wx \end{aligned}$$

Not that  $u_0$  and  $u_n$  are the known boundary condition, assumed to be independent of time .

## 4.2 Haar wavelet

$$\psi_H = \begin{cases} 1 & \text{for } 0 \leq x \leq \frac{1}{2} \\ -1 & \text{for } \frac{1}{2} \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Haar wavelet is defined as

Following Fourier, any wavelet could be used basis block to build any wave  
With

$$\psi_{j,k}(x) = 2^{j/2} \psi(2^j x - k), \text{ for all } j, k \in \mathbb{Z}.$$

The coefficients  $c_{j,k}$  are computable from :

$$c_{j,k} = \langle f, \psi_{j,k} \rangle$$

Also following the idea of Fourier transform  $W_\psi$  of any wave  $f(x)$  can now be defined as following

$$(W_\psi f)(b, a) = |a|^{-1/2} \int_{-\infty}^{\infty} f(x) \psi\left(\frac{x-b}{a}\right) dx$$

The coefficients  $c_{j,k}$  are now computable following relation [3]

$$c_{j,k} = (W_\psi f)\left(\frac{k}{2^j}, \frac{1}{2^j}\right)$$

### 5. Numerical Result

Here the algorithm we built is applied to an example, namely heat equation, and to show the time consumed by the calculations, a comparison with the MOL is given.

#### Example

A brick wall with 0.3 m thick, is initially at a uniform temperature of 100°C and its thermal conductivity,  $K=5 \cdot 10^{-7} \text{ m}^2/\text{s}$ . If the temperature of the both surfaces is suddenly lowered to 20°C and kept at this temperature. Plot the temperature distribution of the wall after 7.33 min (440s).

To study this problem we will use a mesh with 15 subintervals of  $x$  and 50 subdivisions of  $t$ . This corresponds to  $\alpha=0.55$ . See( figure.1)

	MOL	Algorithm by using wavelet
Flops	1632819	32940
Time	5.4900	0.2000



solving Parabolic by Using Wavelet

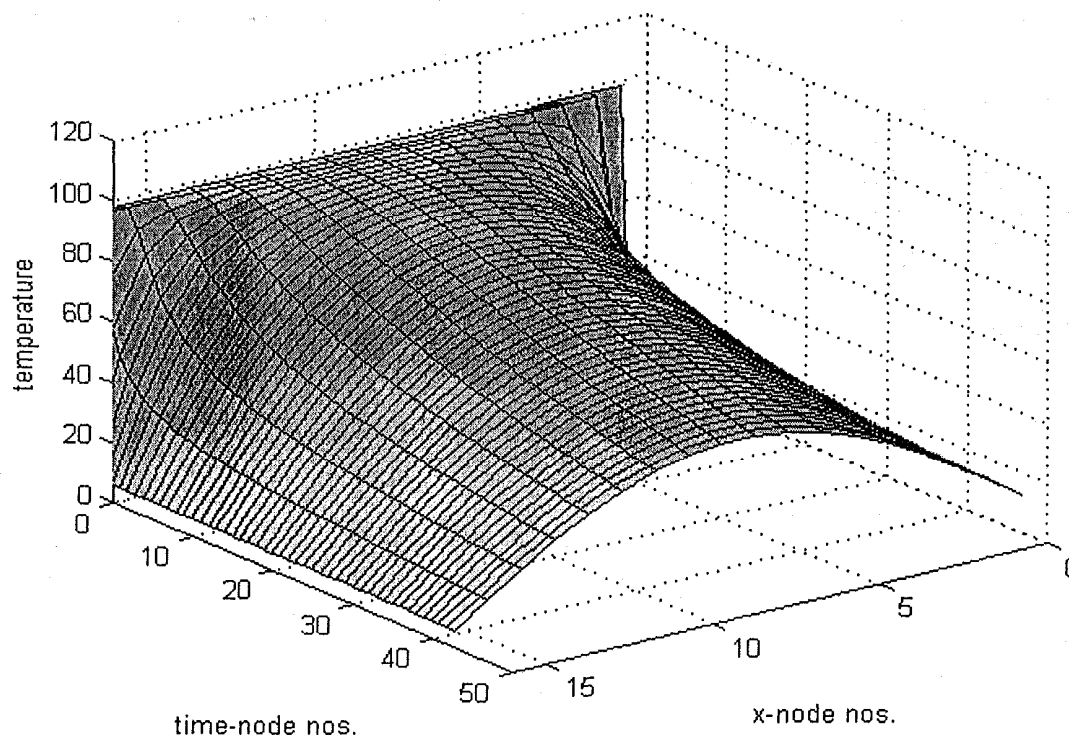


Figure .1 solving parabolic by using wavelet

### 5. Conclusion

The principle of wavelet is used to a speed the MOL for solving parabolic differential equations, the new Technique examined by solving well know problems and results are compared with result obtained by MOL. From the table in our discussions we can conclude that. The accuracy of both methods are the same Wavelet method is able to solve parabolic differential equations in shorter time and fewer flops.

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من (1.41) نجد أن

$$(1.43) \|x_{\infty}(t, x_0^1) - x_{\infty}(t, x_0^2)\| \leq (1 - \Lambda)^{-1} \|x_0^1 - x_0^2\| (1 + B_2 T)$$

وبتعويض (1.43) في (1.41) نحصل على (1.40).

## ملاحظة 1.2

المبرهنة 1.4 تؤكد استقرارية الحل للنظام (1.3) و ذلك عندما يحدث تغيير طفيف في النقطة  $x_0$  يقابله تغيير طفيف في الدالة  $\Delta = (0, x_0)$ .

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