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Wavelet Methods Used to Solve a System of Linear Equations

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الخلاصة

في هذا البحث تم إظهار دراسة المقارنة بين عدة طرق لحل نظام من المعادلات الخطية وهي اعتمدت على مبدا طريقة آلموجة القصيرة ومنها طريقة Daubechies wavelet ، Haarwavelet و Meyer wavelet ، Symlet wavelet ، Maxican hat wavelet و Morlet wavelet . من نتائج هذا البحث ان طريقة Haar wavelet تعتبر افضل طريقة لحل المعادلات الخطية بمقارنتها مع الطرق الأخرى وللمصوفات من نوع Dense و Sparse . بالنسبة للمصفوفات من نوع Three diagonal فان طريقة آل Meyer wavelet أعطت نتائج افضل من بقية الطرق وأيضا استنتج أن طريقة Symlet wavelet تحتاج إلي وقت أطول لحل هذا النظام من المعادلات ولجميع أنواع المصفوفات .

ABSTRACT

In this paper, we study the comparison among many methods to solve a system of linear equations based on the principle of wavelet methods as a Daubechies wavelet, Haar wavelet, Meyer wavelet, Symlet wavelet, Mexican Hat wavelet, Morlet wavelet.

As a result, the Haar wavelet method can be considered the best method to solve the linear equations compared with the other methods especially for the dense and sparse matrix. For the three diagonal matrices the Meyer wavelet gave a good results. Other conclusion was found that the Symlet wavelet needs a long time to solve this system of equations and for all types of the matrix.

Keywords: *Daubechies Wavelet, Haar Wavelet, Meyer Wavelet, Symlet Wavelet, Mexican Hat Wavelet, Morlet wavelet*

INTRODUCTION

Sparse linear equation $Ax = b$, where A being a square matrix of dimension n, are the common model of many contemporary engineering systems, and extensive efforts have been made to design efficient schemes to solve them. One of they well known computing methods, which deserve considerable attention, is the wavelet method.

Wavelet Analysis

Wavelets are new family of basis function that can be used to approxmaite general functions [4]. Wavelet is a waveform of effectively limited duration that has an average value of zero. Compare wavelets with sine waves, which are the basis of Fourier analysis. Sinusoidal do not have a limited duration, which can be extend from minus to plus infinity, where sinusoids are smooth and predictable, wavelets is irregular and not symmetric.

Fourier analysis consists of breaking up a signal into sine waves of various frequencies. Similarly, Wavelet analysis is the breaking up of a signal into a shifted and scaled versions of the original wavelet or mother wavelet [2].

Daubechies Wavelet

Ingrid Daubechies, one of the brightest stars in the world of wavelet research, invented what are called compactly supported orthonormal wavelets, and this making discrete a wavelet analysis practicable.

The names of the Daubechies family wavelets are written db$N$, where N is the order, and db the "surname" of the wavelet. The db1 wavelet, as mentioned above, is the same as Haar. In dbN. Some authors use $2N$ instead of N. This family includes the Haar wavelet, written $db1$, the simplest wavelet imaginable and certainly the earliest.

These wavelets have no explicit expression except for $db1$, which is the Haar wavelet. However, the square modulus of the transfer function of h is explicit and fairly simple.

1. Let, $P(y) = \sum_{k=0}^{N-1} C_k^{N-1+k} y^k$, where C_k^{N-1+k} denotes the binomial coefficients, then:

$$|m_0(\omega)|^2 = \left(\cos^2 \frac{\omega}{2}\right)^N P\left(\cos^2\left(\frac{\omega}{2}\right)\right)$$

where:

$$m_0(\omega) = \frac{1}{\sqrt{2}} \sum_{k=0}^{2N-1} h_k e^{-ik\omega}$$

2. The support length of ψ (ψ is a wavelet function) and ϕ (ϕ is a scaling function) is $2N - 1$. The number of vanishing moments of ψ is N.
3. Most dbN are not symmetrical. For some, the asymmetry is very pronounced.
4. The regularity increases with the order, when N becomes very large, ψ and ϕ belongs to $C^{\mu N}$ since μ is approximately equal to 0.2. For sure, this asymptotic

value is too pessimistic for small order N. Note that the functions are more regular at certain points than at others.

5. The analysis is orthogonal [2].

Haar Wavelet [1]

Haar wavelet (ψ_H) defined as

$$\psi_H(x) = \begin{cases} 1 & \text{for} \quad 0 \le x < \frac{1}{2} \\ -1 & \text{for} \quad \frac{1}{2} \le x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Following Fourier, any wavelet $\psi(x)$ could be used as a basic block to build any wave $f(x)$,

$$f(x) = \sum_{j,k=-\infty}^{\infty} c_{j,k} \psi_{j,k}(x)$$

where

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k), \text{ for all } j,k \in Z.$$

The coefficients $c_{j,k}$ are computable from:

$$c_{j,k} = \langle f, \psi_{j,k} \rangle.$$

Also the following idea of Fourier transform, wavelet transform W_ψ of any wave $f(x)$ can now be defined as follow:

$$(W_\psi f)(b,a) = |a|^{\frac{-1}{2}} \int_{-\infty}^{\infty} f(x)\overline{\psi(\frac{x-b}{a})}dx.$$

The coefficients $c_{j,k}$ are now computable from the relation

$$c_{j,k} = (W_\psi f)\left(\frac{k}{2^j}, \frac{1}{2^j}\right).$$

Meyer Wavelet [2]

Both ψ and ϕ are defined in the frequency domain, starting with an auxiliary function. The Meyer wavelet and scaling function are defined in the frequency domain by:

$$\hat{\psi}(\omega) = 0 \text{ if } |\omega| \notin \left[\frac{2\pi}{3}, \frac{8\pi}{3}\right]$$

$$\hat{\psi}(\omega) = (2\pi)^{-1/2} e^{i\omega/2} \sin\left(\frac{\pi}{2} v\left(\frac{3}{2\pi}|\omega| - 1\right)\right) \text{ if } \frac{2\pi}{3} \le |\omega| \le \frac{4\pi}{3}$$

$$\hat{\psi}(\omega) = (2\pi)^{-1/2} e^{i\omega/2} \cos\left(\frac{\pi}{2} v\left(\frac{3}{4\pi}|\omega| - 1\right)\right) \text{ if } \frac{4\pi}{3} \le |\omega| \le \frac{8\pi}{3}$$

where

$$v(\alpha) = \alpha^4(35 - 84\alpha + 70\alpha^2 - 20\alpha^3),\ \alpha \in [0,1]$$

$$\hat{\phi}(\omega) = (2\pi)^{-1/2} \text{ if } |\omega| \le \frac{2\pi}{3},$$

$$\hat{\phi}(\omega) = (2\pi)^{-1/2} \cos\left(\frac{\pi}{2} v\left(\frac{3}{2\pi}|\omega| - 1\right)\right) \text{if } \frac{2\pi}{3}|\omega| \le \frac{4\pi}{3},$$

$$\hat{\phi}(\omega) = 0 \text{ if } |\omega| > \frac{4\pi}{3}$$

By changing the auxiliary function, one gets a family of different wavelets, for the required properties of the auxiliary function ν, see the list of references. This wavelet ensures an orthogonal analysis and the function ψ does not have a finite support, but ψ decreases to 0 when $x \to \infty$, faster than any inverse polynomial:

$$\forall n \in N, \exists C_n \ni |\psi(x)| \le (1 + |x|^2)^{-n}$$

This property holds also for the derivatives:

$$\forall k \in N, \forall k \in N, \exists C_{k,n}, \text{ such that } |\psi^k x| \le C_{k,n}(1 + |x|^2)^{-n}$$

The wavelet is infinitely differentiable.

Symlet Wavelets [2]

General characteristics: compactly supported wavelets with least asymmetry and highest number of vanishing moments for a given support width. Associated scaling filters are near linear-phase filters.

Symlets with short name (sym) for order *N*, *N* = 2, 3, and orthogonal, biorthogonal and compact support width 2*N*-1. Filters length 2*n*, regularity symmetry near from number of vanishing moments for psi *N*.

The symlets are nearly symmetrical wavelets proposed by Daubechies as modifications to the db family. The properties of the two-wavelet families are similar.

In *symN*, some authors use 2*N* instead of *N*. Symlets is only near symmetric; consequently some authors do not call them symlets.

Daubechies proposes modifications of her wavelets such that their symmetry can be increased while retaining great simplicity. The idea consists of reusing the function m_0 introduced in the *dbN*, considering the $|m_0(\omega)|^2$ as a function *W* of $z = e^{i\omega}$. Then we can factor *W* in several different ways in the form of $W(z) = U(z)\overline{U\left(\frac{1}{z}\right)}$. The roots of *W* with modulus not equal to 1 go in pairs. If z_1=1, then z_1 is a root.

- By selecting *U* such that the modulus of all its roots is strictly less than 1, we build Daubechies wavelets *dbN*. The *U* filter is a "minimum phase filter.
- By making another choice, we obtain more symmetrical filters; these are symlets. The symlets have other properties similar to those of the *dbN*s.

Mexican Hat Wavelet [2]

This wavelet has no scaling function and is derived from a function that is proportional to the second derivative function of the Gaussian probability density function.

$$\psi(x)=\left(\frac{2}{\sqrt{3}}\pi^{-1/4}\right)(1-x^2)e^{-x^2/2}$$

Morlet Wavelet [2]

The classic example of continuous time wavelet analysis uses a windowed complex exponential as the prototype wavelet. This is the Morlet wavelet, as first proposed in Goupillaud, et al, 1984/85) for signal analysis, and is given by

$$\psi(t)=\frac{1}{\sqrt{2\pi}}e^{-j\omega_0 t}e^{-t^2/2}, \quad (1)$$

$$\Psi(\omega)=e^{-(\omega-\omega_0)^2/2}$$

The factor $1/\sqrt{2\pi}$ in (1) ensures that $\|\psi(t)\|=1$. The center frequency ω_0 is usually chosen such that the second maximum of $\mathrm{Re}\{\psi(t),t\}, t>0$, is half the first one at t = 0. This leads to

$$\omega_0=\pi\sqrt{\frac{2}{\ln 2}}=5.336$$

It should be noted that this wavelet is not admissible since $|\Psi(\omega)|_{\omega=0}\neq 0$, but its value at zero frequency is negligible ($\sim 7.10^{-7}$), so it does not present any problem in practice. The Morlet wavelet can be corrected so that $\Psi(\omega)=0$, but the correction term is very small [5].

Wavelet Algorithm

The wavelet algorithm to solve $Ax = b$ is show below:

A matrix could conveniently be considered as row-wise or column-wise arrangement of discrete signals, as such it is amendable to transform analysis. If such an operation is performed on a matrix equation $Ax = b$, a transformed equation $WAx = Wb$ is obtained. From this, one could write $(WAW^{-1})(Wx) = Wb$. Choosing, for instance, orthogonal tran$W$, a relation $(WAW^T)Wx = Wb$, similar to block triangularization operation - which avoids costly inversion operation is now on hand to proceed with the computation of the desired numerical solution. An interesting common property of this method is that a wavelet transform of a dense matrix gives rise to a sparse matrix [1]. Hence an $O(N^3)$ cost of computing could be reduced into much cheaper operation.

There are six common ways in which wavelets can be used to transform values within a matrix. Each of these transformations is a two-dimensional generalization of one-dimensional wavelet transform described previously. We gives a brief presentation on wavelet and wavelet transforms [3].

Numerical Results

Dense Matrices:

We generate five dense matrices A_1, A_2, A_3, A_4 and A_5 of size 8, 16, 32, 64 and 128 respectively as follows:

$$A_1 = C1 \equiv \begin{bmatrix} 10 & 20 & 30 & 90 & 8 & 7 & 6 & 6 \\ 6 & 6 & 6 & 12 & 23 & 56 & 8 & 66 \\ 30 & 64 & 9 & 8 & 7 & 12 & 34 & 45 \\ 8 & 21 & 6 & 98 & 76 & 12 & 9 & 8 \\ 7 & 6 & 8 & 8 & 9 & 41 & 34 & 32 \\ 6 & 6 & 12 & 11 & 21 & 12 & 11 & 90 \\ 6 & 21 & 8 & 9 & 27 & 11 & 21 & 12 \\ 11 & 34 & 21 & 87 & 6 & 7 & 8 & 8 \end{bmatrix}$$

$$A2 = \begin{bmatrix} A1 & A1+2 \\ A1 & 5*A1 \end{bmatrix}$$

$$A3 = \begin{bmatrix} A2 & A2+3 \\ A2 & 3*A2 \end{bmatrix}$$

$$A4 = \begin{bmatrix} A3 & 2*A3 \\ A3 & A3+3 \end{bmatrix}$$

$$A5 = \begin{bmatrix} A4 & 3*A4 \\ A4 & A4+3 \end{bmatrix}.$$

Running similar experiments on five values of n, the following results are obtained:

Table (1) Ther results of experiments on five values of n.

Methods of Wavelet		Orde of Dense Matrices				
		8 x 8	**16 x 16**	**32 x 32**	**64 x 64**	**128 x 128**
Haar	Time	0.3800	0.0600	0.3800	1.5400	6.6400
	Flops	5083	32662	228127	1699220	13085855
Meyer	Time	0.2800	0.3800	0.4500	0.5300	Failed
	Flops	4825	31499	224137	1682681	Failed
Symlet	Time	5.0500	9.0700	106.6200	Long	Long
	Flops	33791	178424	997916	Great number	Great number
Mexican	Time	0.0000	0.0500	0.0600	0.0500	Failed
	Flops	4825	31483	224223	1683273	Failed
Morlet	Time	0.1700	0.2600	0.3000	Failed	Failed
	Flops	4825	31499	224179	Failed	Failed
Daubechies	Time	0.0500	0.0600	0.0800	0.1100	Failed
	Flops	4697	30987	222171	1674769	Failed

Three diagonal Matrices:

A three diagonal matrix *A*

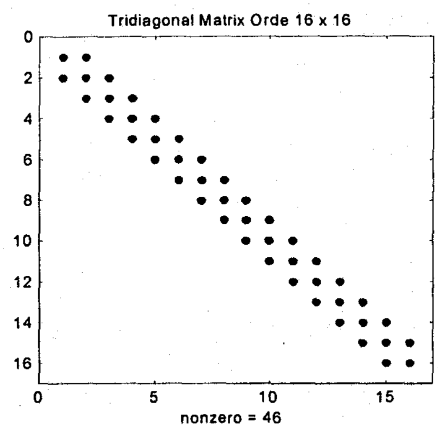

Fig (1) Three diagonal matrix (16 x 16).

Running similar experiments on five values of *n*, the following results are obtained:

Table (2) Ther results of experiments on five values of *n*.

Methods of Wavelet		Orde of Three diagonal Matrices				
		8 x 8	16 x 16	32 x 32	64 x 64	128 x 128
Haar	Time	0.06	0.11	0.39	1.37	5.3200
	Flops	5145	32420	223499	1639936	12455271
Meyer	Time	0.05	0.13	0.20	0.31	0.39
	Flops	4589	27225	176049	1231713	9127617
Symlet	Time	1.43	4.50	22.52	179.16	Long
	Flops	7735	27829	108021	439936	Great number
Mexican	Time	0.1700	0.0500	0.0600	0.0600	0.4300
	Flops	4829	31507	224043	1682199	13019175
Morlet	Time	0.0000	0.0000	0.0000	0.1100	0.3900
	Flops	4813	31449	223921	1681761	13017793
Daubechies	Time	0.0600	0.0500	0.0000	0.0500	0.3300
	Flops	4589	30233	219191	1664327	12950521

We Running similar experiments on a large values of n, the following results are obtained

Methods of Wavelet		**Orde of Three diagonal Matrices**	
		256 x 256	**512 x 512**
Haar	Time	23.4500	128.4700
	Flops	96673684	760125373
Meyer	Time	2.9600	20.7600
	Flops	70077825	548772609
Symlet	Time	Long	Long
	Flops	Great number	Great number
Mexican	Time	4.1700	40.910
	Flops	102027043	812270625
Morlet	Time	4.2900	33.1700
	Flops	102407175	812284893
Daubechies	Time	4.1700	32.1900
	Flops	102137435	811206919

General Sparse Matrices:

As an example of general sparse matrices, we consider a matrix $A1$, $A2$, $A3$, $A4$ and $A5$ of size 8, 16, 32, 64 and 128 respectively as follows:

$$A1=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 10 & 20 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 9 & 0 & 7 & 5 & 4 & 9 & 0 & 0 \\ 12 & 0 & 0 & 0 & 7 & 0 & 0 & 5 \\ 8 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{bmatrix}$$

$$a2=\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 9 & 0 & 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 9 & 5 & 0 & 0 & 0 & 3 \\ 0 & 7 & 0 & 0 & 3 & 0 & 0 & 0 \end{bmatrix} \quad a3=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \\ 0 & 3 & 0 & 0 & 7 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 & 9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad a4=\begin{bmatrix} 7 & 0 & 5 & 6 & 0 & 0 & 0 & 8 \\ 0 & 7 & 0 & 0 & 7 & 0 & 0 & 0 \\ 9 & 3 & 6 & 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 3 & 8 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 & 7 & 4 & 0 \\ 0 & 9 & 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

$$A2=\begin{bmatrix} A1 & a3 \\ a2 & a4 \end{bmatrix}$$

$$A3=\begin{bmatrix} 2*A2 & A2 \\ A2 & A2+I \end{bmatrix}$$

$$A4=\begin{bmatrix} A3 & 2*A3 \\ A3 & A3+I \end{bmatrix}$$

$$A5=\begin{bmatrix} A4 & 2*A4 \\ A4 & A4 \end{bmatrix}$$

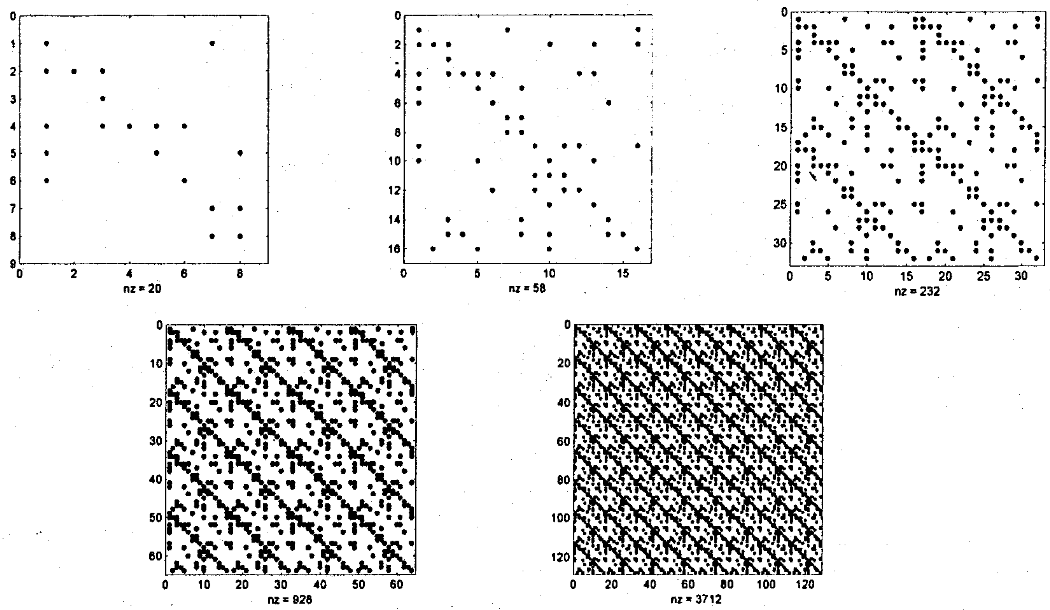

Fig 2: Dots show the number of nonzero elements in a five kind of sparse matrices.

Running similar experiments on five values of *n*, the following results are obtained:

Table (3) Ther results of experiments on five values of *n*.

Methods of Wavelet		Orde of Sparse Matrices				
		8 x 8	16 x 16	32 x 32	64 x 64	128 x 128
Haar	Time	0.0600	0.1100	0.4400	1.6000	5.4400
	Flops	5115	32688	228485	1699674	13089937
Meyer	Time	Failed	Failed	Failed	Failed	Failed
	Flops	Failed	Failed	Failed	Failed	Failed
Symlet	Time	1.5900	5.8700	41.8500	585.9400	Long
	Flops	10230	60182	310579	1533949	Great number
Mexican	Time	0.0000	0.0000	0.0500	0.0600	Failed
	Flops	4861	31527	224121	1683325	Failed
Morlet	Time	0.0600	0.0500	0.0000	Failed	Failed
	Flops	4851	31467	223999	Failed	Failed
Daubechies	Time	0.0000	0.0000	0.0500	Failed	Failed
	Flops	4731	31019	222163	Failed	Failed

CONCLUSIONS

From this paper, we conclude that:

i) Haar wavelet is the best method for dense and sparse matrices.

ii) Meyer wavelet is the best method for three diagonal matrices.

Generally, we can see that Haar wavelet is the best one. This method is convenient for matrices with even element only (2^n x 2^n) where n =1,2,3,..., N, but not for odd element. The Symlet wavelet is convenient for all kind of matrices and all kind of element (even or odd).

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