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Semi-Standard Tableaux of Young Between Algebra and Computer Programming

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الخلاصة

منذ ان بدء موضوع نظرية التمثيل بوضع اولى خطواته، أخذ علم الرياضيات وتحديداً في الجبر بفتح آفاق بحثية جديدة وهكذا هو الحال في بحثنا هذا حيث ننطلق من موضوع جبر شور فـي تقديمـه الجديد من قبل كرين في [6] مروراً بعلاقته مع الالواح الشبه قياسية حسب صيغة كرين. تبقى معرفة عدد هذه الالواح غير معروف لحد الآن، فحاولنا جاهدين معرفة الصيغة الرياضية لكن المشكلة تبقى كلما كبر حجم هذه الالواح، أصبح حساب العدد عسيراً جداً لهذا قمنا بالاستعانة بالحاسوب عن طريق وضع برنامج وخوارزمية تقوم بالمهمة وبالتالي وضع الصيغة الرياضية المناسبة.

ABSTRACT

Since the representation theory start, the mathematics science especially in algebra began to open many new researches such our this one, we start from Schur algebra in his new representation described by Green [6], passing through his relation with semi-standard tableaux according to Green.

The number of these tableaux still unknown, so we tried hardly to find the mathematical rule for this problem, but as these tableaux become great, the counting of them become difficult. Therefore, we tried to write an algorithm and a program to do so. As a result, we find the appropriate mathematical rule.

INTRODUCTION

Let n and r be any positive integers. k an infinite field and $\Gamma=GL_n(k)$ of all non-singular n×n matrices over k. Let $\mu,\nu \in \underline{n}=\{1,2,\dots,n\}$ and $c_{\mu\nu}\in k[\Gamma]$ be the function which associates to each $g\in\Gamma$ its (μ,ν)-coefficients $g_{\mu\nu}$. The k-subalgebra of k^Γ generated by the function $c_{\mu\nu}$ is denoted by A or $A_k(n)$. The elements of A are the polynomial function on Γ. Since k is infinite, the $c_{\mu\nu}$ are algebraically independent over k, so that A can be regarded as the algebra of all polynomials over k in n^2 indeterminates. For each $r\geq 0$ we denote the subspace of A consisting of the elements expressible as polynomials which are homogeneous of degree r in the $c_{\mu\nu}$ by $A_k(n,r)$. In particular $A_k(n,0)=k.1_A$, where 1_A denotes the constant function $1_A:g\to 1_k$ $(g\in\Gamma)$.

For the integers $n,r\geq 1$, we write

$$I(n,r)=\{i=(i_1,\dots,i_r)\mid i_\rho\in n \text{ and } 1\leq\rho\leq r\}.$$

The symmetric group G_r act on the right on I(n,r) by

$$i\cdot\pi=(i_{1,\pi},\dots,i_{r,\pi}) \text{ where } \pi\in G_r.$$

Also G_r act on the set I(n,r)×I(n,r) by $(i,j)\cdot\pi=(i\cdot\pi,j\cdot\pi)$.

We write $i\sim j$ to indices that the elements $i,j\in I(n,r)$ are the same G_r-orbit. Similarly for $(i,j)\sim(e,l)$, with this notation

$$A_k(n,r)=\left\langle c_{i,j}=c_{i_1j_1}c_{i_2j_2}\dots\dots c_{i_rj_r}\mid i,j\in I(n,r)\right\rangle_k$$

where $c_{i,j}=c_{e,l}$ if and only if $(i,j)\sim(e,l)$. $A_k(n,r)$ has as k-basis the set of distinct monomial $c_{i,j}$ and these are in bijective correspondence with the G_r-orbits of I(n,r)×I(n,r). Then

$$\dim_k(A_k(n,r))=\binom{n^2+r-1}{r}.$$

The Schur algebra defined by $S_k(n,r)=A_k(n,r)^*=\mathrm{Hom}_k(A_k(n,r),k)$.

For $i,j\in I(n,r)$, $\xi_{i,j}$ is the element of $S_k(n,r)$ given by

$$\xi_{i,j}(c_{p,q})=\begin{cases}1 & \text{if } (i,j)\sim(p,q)\\ 0 & \text{if not}\end{cases} \quad \dots\dots\dots (1.1)$$

We have $\xi_{i,j}=\xi_{e,l}$ if and only if $(i,j)\sim(e,l)$. Also we have

$$\dim_k(S_k(n,r))=\binom{n^2+r-1}{r}.$$

Mahmood in [8] designed a program in GAP (see [9]), for this dimension of $S_k(n,r)$.

Green in [6], give the following result:

$$\xi_{i,j}.\xi_{e,l} = \sum_{(p,q)\in I(n,r)} Z(i,j,e,l,p,q).\xi_{p,q}$$

where $Z(i,j,e,l,p,q) = \left|\{s \in I(n,r) \mid (i,j) \sim (p,s) \text{ and } (e,l) \sim (s,q)\}\right|$.

Mahmood in [8] designed another program in GAP to find Z(i,j,e,l,p,q) for the $\xi_{i,j} \cdot \xi_{e,l}$.

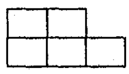
The standard modules of $S_k(n,r)$

A composition μ for r is a sequence $(\mu_1, \mu_2, \ldots)$ of non-negative integers such that $|\mu| = \sum_i \mu_i = r$. The integers μ_i, for $i \geq 1$, are the parts of μ; if $\mu_i=0$ for $i>m$, we identify μ with $(\mu_1, \mu_2, \ldots, \mu_m)$. A composition μ is a partition if $\mu_i \geq \mu_{i+1}$ for all $i \geq 1$.

The diagram of Young of a composition μ is the subset

$$[\mu] = \{(x,y) \mid 1 \leq y \leq \mu_x \text{ and } x \geq 1\} \text{ of } N \times N.$$

The elements of $[\mu]$ are called the nodes of μ; more generally, a node is any element of N×N. It is useful to represent the diagram of μ as an array of boxes in the plane. For example, if $\mu=(2,3)$ then $[\mu]=$



If μ is a composition of n then a μ-tableau is a bijection

$$t: [\mu] \rightarrow \{1,2,\ldots,n\}$$

A μ-tableau **t** is row standard (*resp. row semi-standard*) if the entries in **t** increase from left to right in each row (*resp. if the entries in each row in **t** are non-decreasing*), **t** is standard (*resp. semi-standard*) if μ is a partition and the entries in **t** increase from left to right in each row and from top to bottom in each column (*resp. μ is a partition, **t** is row semi-standard and the entries in each column of **t** are strictly increasing*). We denote Std(μ) be the set of standard μ-tableaux and SStd(μ) be the set of semi-standard μ-tableaux.

Since $[\mu]$ has r elements, we denote $\underline{r}=\{1,2,\ldots,r\}$. Then there exists at least one bijection $\mathbf{T}:[\mu]\rightarrow\underline{r}$, we shall arbitrarily choose one such bijection and we call it the basic μ-tableau $\mathbf{T}=\mathbf{T}^{\mu}$. If $i=(i_1,\ldots,i_r)\in I(n,r)$, we denote the μ-tableau $i_o\mathbf{T}^{\mu}:[\mu]\rightarrow\underline{n}$ by $\mathbf{T}_i$.

For $i,j \in I(n,r)$, an element $(\mathbf{T}_i\!:\!\mathbf{T}_j) \in A_k(n,r)$ is defined by bi-determinant (see [1]). For the partition μ, let $\mathbf{T}_l$ be the tableau such that

$$T_l(x,y) = x \quad \forall (x,y) \in [\mu].$$

As the following example shows:

$\mathbf{T}_l =$

1	1	1	1	1	1	1	......	1
2	2	2	2	......	2			
3	......	3						
...								

We define a subspace $D_{\mu,k} \subseteq A_k(n,r)$ by

$$D_{\mu,k} = \langle (T_l : T_i) \mid i \in I(n,r) \rangle_k .$$

Then $D_{\mu,k}$ is a left $S_k(n,r)$-submodule of $A_k(n,r)$; (see [6, p.54]), where the operation is defined by

$$\xi \cdot (T_l : T_j) = \sum_{i \in I(n,r)} \xi(c_{i,j})(T_l : T_i), \text{ for } \xi \in S_k(n,r) \text{ and } j \in I(n,r).$$

<u>**Theorem 1**</u>**:** [Green, (4.5a)]

$D_{\mu,k}$ has k-basis consisting of all $(\mathbf{T}_l\!:\!\mathbf{T}_i)$ such that $\mathbf{T}_i \in SStd(\mu)$. In particular $\dim_k(D_{\mu,k}) = |SStd(\mu)|$.

<u>**Theorem 2**</u>**:** [Mahmood]

Let $\mu \in \Lambda^+(n,r)$ consisting of all partitions of r into at most n parts, and $\xi_{i,j} \in S_k(n,r)$ where $i,j \in I(n,r)$. We have explicit the formula for the action of $\xi_{i,j}$ on $D_{\mu,k}$. We have

$B_\mu = \{(T_l\!:\!T_h) \mid h \in I(n,r)$ such that T_h is a semi-standard μ-tableau$\}$.

By the theorem 2.1, B_μ is the basic of $D_{\mu,k}$, then we have

$$\begin{aligned} \xi_{i,j} \cdot (T_l : T_q) &= \sum_{p \in I(n,r)} \xi_{i,j}(c_{p,q})(T_l : T_p), \\ &= \sum_{\substack{p \in I(n,r) \\ (i,j) \sim (p,q)}} (T_l : T_p) \text{ by the relation (1.1)} \end{aligned}$$

Mahmood in [8] with the help of Appendix of Geck in [5] proved that the modules $D_{\mu,k}$ are the standard modules for quasi-hereditary structure in $S_k(n,r)$. Green in [6] proved that, "$D_{\mu,k}$ has a unique minimal submodule", while the standard module has a unique maximal submodule; (for more details with the quasi-hereditary structure in $S_k(n,r)$, see [8]).

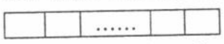
Numeration of semi-standard tableaux

In this section, we shall find the number (rule) of semi-standard tableaux for some n. We start case after case, because it is not simple to find the rule if n is large and the computation will be difficult, therefore we attempt to solve this problem next section.

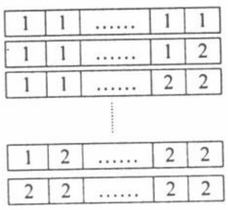
If n=2: we have two cases as the following:

case 1: $\mu=(\mu_1,0)=(\mu_1)$

Young diagram for this case will be as follow:

$[\mu]=$ 

The μ-tableau can easily counted and represented as:

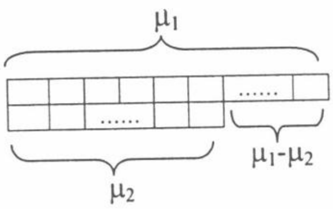


So the general rule for this case is:

$$\binom{\mu_1+1}{\mu_1} \quad \ldots\ldots\ldots (3.1.1)$$

case 2: $\mu=(\mu_1,\mu_2)$

For counting the μ-tableau, the Young diagram will be as follow:

$[\mu]=$ 

By the definition of the tableaux, we should put 1 at the beginning of μ_2 of the boxes in row 1 from the left, and put 2 in row 2. The rest of boxes $(\mu_1-\mu_2)$ in row 1, we shall apply the same way as in case 1 of 3.1. So

1	......	1	1	......	1
2	......	2			

1	......	1	1	......	2
2	......	2			

⋮

1	......	1	2	......	2
2	......	2			

Then the rule will be

$$\binom{\mu_1 - \mu_2 + 1}{\mu_1 - \mu_2} \quad \text{.........} \ (3.1.2)$$

<u>If n=3</u>: we have three cases as the following:

<u>case 1</u>: $\mu=(\mu_1, 0, 0) = (\mu_1)$

This case is not totally different from case1 in 3.1, except that we are going to deal with 1, 2 and 3 to fill the boxes as below:

1	1	......	1	1	1
1	1	......	1	1	2
1	1	......	1	1	3
1	1	......	1	2	2
1	1	......	1	2	3
1	1	......	1	3	3
⋮					
1	3	......	3	3	3

A_1

2	2	......	2	2	2
2	2	......	2	2	3
⋮					
2	3	......	3	3	3
3	3	......	3	3	3

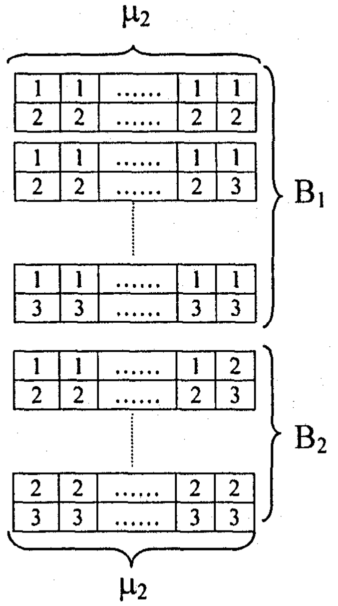
A_2

By conjugating A_1 with A_2, the rule for this case will be:

$$\binom{\mu_1 + 2}{\mu_1} \quad \text{.........} \ (3.2.1)$$

case 2: $\mu=(\mu_1, \mu_2, 0) = (\mu_1, \mu_2)$

The same approach in case 2 of 3.1, we shall divide Young diagram into two partitions, the first part which has μ_2-boxes in row 1, and μ_2-boxes in row 2. From the tableau definition, the boxes will be filled as:



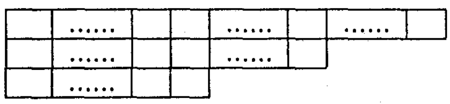
The second part remained of this case that is $(\mu_1-\mu_2)$-boxes, we will follow the same way as in case 1 of 3.2 for each diagram in its first part. Any case of B_1 will take all possible cases of A_1 and A_2, while any case of B_2 will take the possible cases of A_2 only.

By this approach, the main rule for the semi-standard tableaux for μ is:

$$\binom{\mu_2+1}{\mu_2}\binom{\mu_1-\mu_2+2}{\mu_1-\mu_2}+\binom{\mu_2+1}{\mu_2-1}\binom{\mu_1-\mu_2+1}{\mu_1-\mu_2} \quad \text{.........} \quad (3.2.2)$$

case 3: $\mu=(\mu_1, \mu_2, \mu_3)$

We know that Young diagram for this case is:

$[\mu] =$ 

We shall use here the same case 1 and case 2 of 3.2 to find the main rule, as we know and from the definition of semi-standard tableaux we shall put 1 in the first μ_3-box in row 1, then 2 in the first μ_3-box in row 2, then 3 for row 3. It means that we can write the rule for these cases as:

$$\mu=(\mu_1,\mu_2,\mu_3)\rightarrow\mu^*=(\mu_1-\mu_3,\mu_2-\mu_3,0)$$

As we did in case 2 of 3.2, the main rule will be:

$$\binom{\mu_2-\mu_3+1}{\mu_2-\mu_3}\binom{\mu_1-\mu_3-(\mu_2-\mu_3)+2}{\mu_1-\mu_3-(\mu_2-\mu_3)}+\binom{\mu_2-\mu_3+1}{\mu_2-\mu_3-1}\binom{\mu_1-\mu_3-(\mu_2-\mu_3)+1}{\mu_1-\mu_3-(\mu_2-\mu_3)}$$

$$=\binom{\mu_2-\mu_3+1}{\mu_2-\mu_3}\binom{\mu_1-\mu_2+2}{\mu_1-\mu_2}+\binom{\mu_2-\mu_3+1}{\mu_2-\mu_3-1}\binom{\mu_1-\mu_2+1}{\mu_1-\mu_2} \quad \text{......... (3.2.3)}$$

If n=4

There is no differences between this case and the previous cases of 3.1 and 3.2 except that there is sub cases in each four cases for n=4. These cases will be mentioned with no need for explanation because they use the same way that used in 3.1 and 3.2 by considering that the boxes will be filled with 1, 2, 3 and 4. We will explain now some special cases when n=4:

a) If $\mu=(\mu_1, 0, 0, 0)$, then the number of semi-standard μ-tableaux is:

$$\binom{\mu_1+3}{\mu_1} \quad \text{......... (3.3.1)}$$

b) If $\mu=(\mu_1, \mu_2, 0, 0)$

We should consider the following table:

	$L_{(1,\mu_2)}$			$L_{(2,\mu_2)}$			$L_{(3,\mu_2)}$	
$(\mu_1, 0, 0, 0)$	1	$\binom{\mu_1-\mu_2+3}{\mu_1-\mu_2}$	+	0	$\binom{\mu_1-\mu_2+2}{\mu_1-\mu_2}$	+	0	$\binom{\mu_1-\mu_2+1}{\mu_1-\mu_2}$
$(\mu_1, 1, 0, 0)$	3	...	+	2	...	+	1	...
$(\mu_1, 2, 0, 0)$	6	...	+	8	...	+	6	...
$(\mu_1, 3, 0, 0)$	10	...	+	20	...	+	20	...
$(\mu_1, 4, 0, 0)$	15	...	+	40	...	+	50	...
⋮		$\binom{\mu_1-\mu_2+3}{\mu_1-\mu_2}$	+	⋮	$\binom{\mu_1-\mu_2+2}{\mu_1-\mu_2}$	+	⋮	$\binom{\mu_1-\mu_2+1}{\mu_1-\mu_2}$

The main rule here is:

$$\binom{\mu_2+2}{\mu_2}\binom{\mu_1+\mu_2+3}{\mu_1-\mu_2}+2\binom{\mu_2+2}{\mu_2-1}\binom{\mu_1+\mu_3+2}{\mu_1-\mu_2}+X_{\mu_2}\binom{\mu_1-\mu_2+1}{\mu_1-\mu_2} \text{... (3.3.2)}$$

where $X_{\mu_2} = \sum_{d=1}^{3} L_{(d,\mu_2-1)}$

c) If $\mu=(\mu_1, \mu_2, \mu_3)$ and $\mu_2=\mu_3$, then the number of semi-standard μ-tableaux is:

$$\binom{\mu_3+2}{\mu_3}\binom{\mu_1-\mu_2+3}{\mu_1-\mu_2}+\binom{\mu_3+2}{\mu_3-1}\binom{\mu_1-\mu_2+2}{\mu_1-\mu_2} \quad \cdots\cdots\cdots (3.3.3)$$

d) If $\mu=(\mu_1, \mu_2, \mu_3)$, $\mu_2\neq\mu_3$ and $\mu_2-\mu_3=1$, then the rule is:

$$\binom{\mu_3+1}{\mu_3}\binom{\mu_3+3}{\mu_3+2}\binom{\mu_1-\mu_2+3}{\mu_1-\mu_2}+2\binom{\mu_3+3}{\mu_3}\binom{\mu_1-\mu_2+2}{\mu_1-\mu_2}+\ldots$$
$$+\binom{\mu_3+3}{\mu_3}\binom{\mu_1-\mu_2+1}{\mu_1-\mu_2} \quad \ldots(3.3.4)$$

e) If $\mu=(\mu_1, \mu_2, \mu_3)$, $\mu_2\neq\mu_3$ and $\mu_2-\mu_3=2$, then

$$3\binom{\mu_2+1}{\mu_2}\binom{\mu_1-\mu_2+3}{\mu_1-\mu_2}+\binom{\mu_2+1}{\mu_2}\binom{\mu_3+4}{\mu_3+3}\binom{\mu_1-\mu_2+2}{\mu_1-\mu_2}+\ldots$$
$$+\binom{\mu_2+1}{\mu_2}\binom{\mu_3+3}{\mu_3+2}\binom{\mu_1-\mu_2+1}{\mu_1-\mu_2} \quad \ldots(3.3.5)$$

From the above, we see the complexity of finding the rule whenever n is large, and for this reason we will stop, and go to the next section which supports us with the program that we wrote, to count the exact number of semi-standard of μ-tableaux.

Semi-Standard Tableaux Algorithm:

```
Semi-Standard Tableaux Algorithm
Begin {Algorithm}
        Define Condition Con1: Current element in one dimensional Array should
                               be less than or equal to the previous.
        Define Condition Con2 : Every element Should be greater than to the
                               corresponding one in the Previous Array.
        Set N = 0
        Repeat
        L     Enter Element Si to the set S
                 If Si does not satisfy Con1
                        Begin
                           Discard Si
                           Goto L
                        end
              Set N =N+1
        Until End of Input
        Create Array1,Array2 ,.............ArrayN Such that the length of them are
                                           S1,S2,.......,SN respectivly
        Set Array1, Array2,.............., ArrayN with 1's, 2's, .........,N's
                                               As initial values and it is consider as
                                               a first (Ideal) tableau.
        Set Tableau _no =0
        While there is new Case do
             Begin
                If current Case satisfy con2
                    Begin
                        Consider the current case as a new tableau.
                        Add 1 to Tableau _no
                        Print tableau
                    End
                Get Next Case
             End {while}
        Print " number of Tableau is= ", Tableau_no
End {Algorithm}
```

CONCLUSION

We conclude after applying the program that the results were accurate and the relations for the cases (n=2,3 and 4) were sufficient.

By using this program, we could find exactly the number of semi-standard tableaux speedily and accurately whenever the size of n get greater and whenever μ vary.

Program Results:

Ex 1: Enter the Set S such that S1>=S2>=....>=Sn
3 0 0
Sample number 1
1 1 1
Sample number 2
1 1 2
Sample number 3
1 1 3
Sample number 4
1 2 2
Sample number 5
1 2 3
Sample number 6
1 3 3
Sample number 7
2 2 2
Sample number 8
2 2 3
Sample number 9
2 3 3
Sample number 10
3 3 3
Number of tableaux is 10
There are no other tableaux
Press any key to continue

Ex 2: Enter the Set S such that S1>=S2>=....>=Sn
4 2 1
Sample number 1
1 1 1 1
2 2
3
Sample number 2
1 1 1 2
2 2
3
Sample number 3
1 1 1 3
2 2
3
Sample number 4
1 1 2 2
2 2
3
Sample number 5
1 1 2 3
2 2
3
Sample number 6
1 1 3 3
2 2
3
Sample number 7
1 1 1 1
2 3
3
Sample number 8
1 1 1 2
2 3
3
Sample number 9
1 1 1 3
2 3
3
Sample number 10
1 1 2 2
2 3
3

Sample number 11
1 1 2 3
2 3
3
Sample number 12
1 1 3 3
2 3
3
Sample number 13
1 2 2 2
2 3
3
Sample number 14
1 2 2 3
2 3
3
Sample number 15
1 2 3 3
2 3
3
Number of tableaux is 15
There are no other tableaux
Press any key to continue

Ex 3: Enter the Set S such that S1>=S2>=....>=Sn
3 2 2 1
Sample number 1
1 1 1
2 2
3 3
4
Sample number 2
1 1 2
2 2
3 3
4
Sample number 3
1 1 3
2 2
3 3
4

Sample number 4
1 1 4
2 2
3 3
4
Sample number 5
1 1 1
2 2
3 4
4
Sample number 6
1 1 2
2 2
3 4
4
Sample number 7
1 1 3
2 2
3 4
4
Sample number 8
1 1 4
2 2
3 4
4
Sample number 9
1 1 1
2 3
3 4
4
Sample number 10
1 1 2
2 3
3 4
4
Sample number 11
1 1 3
2 3
3 4
4

Sample number 12
1 1 4
2 3
3 4
4
Sample number 13
1 2 2
2 3
3 4
4
Sample number 14
1 2 3
2 3
3 4
4
Sample number 15
1 2 4
2 3
3 4
4
Number of tableaux is 15
There are no other tableaux
Press any key to continue

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