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(k,n,f) - Arcs in Galois Plane of Order Five

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الخلاصة

قمنا في هذا البحث بدراسة الأقواس -(k,n,f) من نوع (n-5,n) في المستوي الاسقاطي PG(2,q) ووجدنا مثال على القوس -(11,7,f) من نوع (2,7) عندما يكون عدد النقاط التي لها وزن (صفر) تشكل القوس -(20,5) وكذلك وجدنا مثال آخر على القوس -(6,6,f) عندما يكون عدد النقاط التي لها وزن (صفر) تشكل القوس – (25,5).

ABSTRACT

In this paper we construct (k,n,f) - arcs of type (n-5,n) in PG(2,5), and we prove that a(11,7,f) - arc of type (2,7) exist where the points of weight zero form (20,5)– arc and a (6,6,f) - arc of type (1,6) exist where the points of weight zero form (25,5)-arc.

1. Introduction:

A (k,n) – arc in the finite projective plane PG(2,q) is defined to be the set K which is composed of k points such that there is a line passes through n points but no line can pass through more than n points.
Following [1] , a (k,n) – arc is called maximal if k=(n-1)q +n.
If f is a function from the set θ of points of the projective plane in to the set of natural number N, the value f(p) is called the weight point p and if F is a function from the set θ of lines in to N, the value $F(\ell)$ is called the weight line ℓ i.e $F(\ell) = \sum_{p\in\ell} f(p)$. see [3].
A(k,n,f)–arc K of θ is a set of k points such that K does not contain any points of weight zero. The (k,n,f)– arcs in a projective plane were studied in the papers of D'Agostini [2] and Wilson [6].

The line ℓ of θ is called i-secant if the total weight of ℓ is i, Pj denotes the number of points having weight j for $j = 0,1,2,\ldots, w$ (where $w = \max_{p \in k} f(p)$)

and we used V_i^j for the number of lines of weight i through a point of weight j, we also denote the number of lines of weight i by s_i, the integers s_i are called the characters of K.

$$f(p) = \begin{Bmatrix} 1 & if \;\; p \in K \\ 0 & if \;\; p \notin K \end{Bmatrix}$$

i.e if the points in the plane are only of weight zero and one, then K is a (k,n) – arc.

Let W denote the total weight of K, so by [2] we have:

$$m(q+1) \leq W \leq (n-w)(q+1)+w \qquad \ldots\ldots \text{(i)}$$

Arcs for which equality holds on the right are called maximal and arcs for which equality holds on the left are called minimal.

Also [2] has proved to be a necessary condition for the existence of a (k,n,f)–arc K of type (m.n), 0 <m<n is that

$$q \equiv 0 \bmod (n-m) \qquad \ldots\ldots \text{(i i)}$$

$$\text{and } \; w \leq n-m \qquad \ldots\ldots \text{(i i i)}$$

The case of m = n – 2 was discussed at length in [2]. By (ii) we must have $q = 2^h$ and then (iii) requires that w≤2. In order to have an arc which is not simply a (k,n)–arc we thus must have w=2 so that (i) gives (n-2)(q+1)≤W≤ (n-2) (q+1)+2.

It may easily be shown that $W \neq (n-2)(q+1)+1$ and the other two possible values of W are discussed in [2]. Such arcs have points having possible weights 0,1 and 2.

2. The projective plane PG (2,5):

In PG(2,5) which can be constructed by the irreducible polynomial $F(x) = x^3 + 2x^2 + x - 1$.So the cyclic projectivity

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$ gives the 31 points of this plane as follows:

If the first point is $p_0 = (0,0,1)$, then pi = $p_0 T^i$, i=0,1,..., 30, and if the first line L_1 so we have 31 lines defined as $L_1 T^i = L_i$, i=1,2,......,31.

That is if the first line L_1 has the points p_0, p_3, p_5, p_{12}, p_{20} and p_{30}, so by multiplying L_1 by T we have L_2 which has the points p_1, p_4, p_6, p_{13}, p_{21}, and p_0 and so on, we construct all 31 lines in θ.

From now for convenience we use the numbers 0,1,2,......30 instead of the points $p_0, p_1, p_2, \ldots\ldots p_{30}$.

3. The cases of (k,n,f)–arcs of type (n-5,5):

For a (k,n,f) – arcs of type (n-5,5) we need to consider further discussion of maximality and minimality of the arcs.

There are four cases:

(1) $L_0>0, L_1>0, L_2>0, Lj = 0$, for $j = 3,4,5$.
(2) $L_0>0, L_1>0, L_2>0, L3 > 0, Lj= 0$, for $j = 4,5$.
(3) $L_0>0, L_1>0, L_2>0, L3>0, L4>0, L5=0$.
(4) $L_0>0, L_1>0, L_2>0, L3>0, L4>0 =L5>0$.

In particular we discuss (1). In this case

$(n-5)(q+1)\leq W\leq(n-5)(q+1)+5$

Using the arguments in [2] the values of V_n^j and V_{n-5}^j, j=0,1,2, are fixed in dependently of the point under consideration.

For W maximal, i.e $W = (n-5)(q+1)+5$ we have in particular

$$V_{n-5}^0 = q \qquad V_n^0 = 1$$
$$V_{n-5}^1 = \frac{4}{5}q \qquad V_n^1 = \frac{1}{5}q+1$$
$$V_{n-5}^2 = \frac{3}{5}q \qquad V_n^2 = \frac{2}{5}q+1$$
$$V_{n-5}^3 = \frac{2}{5}q \qquad V_n^3 = \frac{3}{5}q+1$$
$$V_{n-5}^4 = \frac{1}{5}q \qquad V_n^4 = \frac{4}{5}q+1$$
$$V_{n-5}^5 = 0 \qquad V_n^5 = q+1$$

For these results we can prove the following lemma:

Lemma: There is no point of weight 5 that lies on (n–5)- secant of a (k,n,f)– arc of type (n – 5, n).

Now for W minimal, i.e $W = (n-5)(q+1)$ we have in particular

$$V_{n-5}^0 = q+1 \qquad V_n^0 = 0$$
$$V_{n-5}^1 = \frac{4}{5}q+1 \qquad V_n^1 = \frac{1}{5}q$$
$$V_{n-5}^2 = \frac{3}{5}q+1 \qquad V_n^2 = \frac{2}{5}q$$
$$V_{n-5}^3 = \frac{2}{5}q+1 \qquad V_n^3 = \frac{3}{5}q \qquad (1)$$
$$V_{n-5}^4 = \frac{1}{5}q+1 \qquad V_n^4 = \frac{4}{5}q$$
$$V_{n-5}^5 = 1 \qquad V_n^5 = q$$

from the above we get

Lemma: There is no point of weight zero on any n – secant of a (k,n,f)– arc .

4. (k,n,f)– arcs of type (n–5,n) with $L_0 > 0$, $L_1 > 0$, $L_2 > 0$, $L_j = 0$, for $j = 3,4,5$.

Let s_{n-5} be the number of the lines of weight n–5 and s_n be the number of lines of weight n, then

$$s_{n-5} + s_n = q^2 + q + 1 \qquad (2)$$

$$(n-5)\, s_{n-5} + n\, s_n = (n-5)(q+1)^2 \qquad (3)$$

Solving (2) and (3) give

$$s_n = \frac{1}{5} q(n-5) \qquad (4)$$

$$s_{n-5} = \frac{1}{5} (5q^2 + 10q - nq + 5) \qquad (5)$$

Now let M be an n – secant which has no point of weight zero and suppose that on M there are α_1 points of weight 1 and α_2 points of weight 2, then counting points of M gives:

$\alpha_1 + \alpha_2 = q+1.$

and the weight of points on M gives :

$\alpha_1 + 2\alpha_2 = n.$

So

$$\left.\begin{aligned} \alpha_1 &= 2(q+1) - n \\ \alpha_2 &= n - (q+1) \end{aligned}\right\} \qquad (6)$$

Counting the incidences between points of weight 1 and n – secant gives :

$L_1 V_n^1 = s_n\, \alpha_1$

By using (1) and the equations (4) and (6) we have:

$$L_1 = (n-5)(2q + 2 - n) \qquad (7)$$

Simillarly, counting incidences between points of weight 2 and n – secant gives:

$L_2 V_n^2 = s_n\, \alpha_2$

Hence, using (1) and the equations (4) and (6) we have

$$L_2 = [(n-5)(n-q-1)] / 2 \qquad (8)$$

Since $L_0 + L_1 + L_2 = q^2 + q + 1$

then by equation (7) and (8) we get

$$2q^2 + (17 - 3n)\, q + n^2 - 8n + 17 - 2L_0 = 0 \qquad (9)$$

Thus a necessary condition such that (9) with the solution, $(n-19)^2 - (208 - 16L_0)$, be a sequare.

An example was found in PG (2, 5).

Lines	Points				
L_1	0	3	5	12	20	30
L_2	1*	4*	6*	13*	21*	0
L_3	2*	5	7	14*	22*	1*
L_4	3	6*	8*	15	23*	2*
L_5	4*	7	9*	16*	24*	3
L_6	5	8*	10*	17*	25	4*
L_7	6*	9*	11*	8*	26	5
L_8	7	10*	12	19*	27*	6*
L_9	8*	11*	13*	20	28*	7
L_{10}	9*	12	14*	21*	29	8*
L_{11}	10*	13*	15	22*	30	9*
L_{12}	11*	14*	16*	23*	0	10*
L_{13}	12	15	17*	24*	1*	11*
L_{14}	13*	16*	18*	25	2*	12
L_{15}	14*	17*	19*	26	3	13*
L_{16}	15	18*	20	27*	4*	14*
L_{17}	16*	19*	21*	28*	5	15
L_{18}	17*	20	22*	29	6*	16*
L_{19}	18*	21*	23*	30	7	17*
L_{20}	19*	22*	24*	0	8*	18*
L_{21}	20	23*	25	1*	9*	19*
L_{22}	21*	24*	26	2*	10*	20
L_{23}	22*	25	27*	3	11*	21*
L_{24}	23*	26	28*	4*	12	22*
L_{25}	24*	27*	29	5	13*	23*
L_{26}	25	28*	30	6*	14*	24*
L_{27}	26	29	0	7	15	25
L_{28}	27*	30	1*	8*	16*	26
L_{29}	28*	0	2*	9*	17*	27*
L_{30}	29	1*	3	10*	18*	28*
L_{31}	30	2*	4*	11*	19*	29

where * assign the weight zero to the points while _ assign the weight 2 to the points

From this example we get 20 point of weight zero which the table bellow explain these points:

i	p(i)	i	p(i)	i	p(i)	i	p(i)
1	(1, −1, −2)	9	(1, 2, 1)	16	(1, −2, −2)	22	(1,1, 1)
2	(1 , 2, 0)	10	(1, −2, 1)	17	(1, 2, 2)	23	(1, −2, 2)
4	(1, −1, 0)	11	(1, −2, 0)	18	(1, 1, 2)	24	(1, 1, −1)
6	(1, −1, −1)	13	(1, −1, 1)	19	(1, 1, 0)	27	(1, 2, −2)
8	(1, 1, −2)	14	(1, −2, −1)	21	(1, −1, 2)	28	(1, 2, −1)

where the points of weight zero form an (20,5) – arc.
From equation (8) we get 1 point of weight 2 and this point is

0	(0,0,1)

Then the remaining points of PG(2,5) are assigned weight 1 and these points are

i	p(i)	i	p(i)
3	(0 , 1, 2)	20	(0,1, 1)
5	(0,1, −1)	25	(1,0, −1)
7	(1,0, 2)	26	(1,0, −2)
12	(0 , 1, −2)	29	(1,0, 0)
15	(1,0, 1)	30	(0,1, 0)

So the sum of all these points are $q^2 + q + 1$ and the points of PG (2 , 5) of non – zero weight will give an (11 ,7, f) – arc of type (2 , 7).

An another example was found in PG(2,5) which is:

Lines	Points				
L_1	0	3	5	12	20	30
L_2	1*	4*	6*	13*	21*	0
L_3	2*	5	7*	14*	22*	1*
L_4	3	6*	8*	15*	23*	2*
L_5	4*	7*	9*	16*	24*	3
L_6	5	8*	10*	17*	25*	4*
L_7	6*	9*	11*	18*	26*	5
L_8	7*	10*	12	19*	27*	6*
L_9	8*	11*	13*	20	28*	7*
L_{10}	9*	12	14*	21*	29*	8*
L_{11}	10*	13*	15*	22*	30	9*
L_{12}	11*	14*	16*	23*	0	10*
L_{13}	12	15*	17*	24*	1*	11*
L_{14}	13*	16*	18*	25*	2*	12
L_{15}	14*	17*	19*	26*	3	13*
L_{16}	15*	18*	20	27*	4*	14*
L_{17}	16*	19*	21*	28*	5	15*
L_{18}	17*	20	22*	29*	6*	16*
L_{19}	18*	21*	23*	30	7*	17*
L_{20}	19*	22*	24*	0	8*	18*
L_{21}	20	23*	25*	1*	9*	19*
L_{22}	21*	24*	26*	2*	10*	20
L_{23}	22*	25*	27*	3	11*	21*
L_{24}	23*	26*	28*	4*	12	22*
L_{25}	24*	27*	29*	5	13*	23*
L_{26}	25*	28*	30	6*	14*	24*
L_{27}	26*	29*	0	7*	15*	25*
L_{28}	27*	30	1*	8*	16*	26*
L_{29}	28*	0	2*	9*	17*	27*
L_{30}	29*	1*	3	10*	18*	28*
L_{31}	30	2*	4*	11*	19*	29*

From this example we get 25 point of weight zero which explain in table bellow:

i	p(i)	i	p(I)	i	p(i)	i	p(i)
1	(1, –1, –2)	9	(1, 2, 1)	16	(1, –2, –2)	23	(1, –2, 2)
2	(1 , 2, 0)	10	(1, –2, 1)	17	(1, 2, 2)	24	(1, 1, –1)
4	(1, –1, 0)	11	(1, –2,0)	18	(1, 1, 2)	25	(1, 0, –1)
6	(1, –1, –1)	13	(1, –1, 1)	19	(1, 1, 0)	26	(1, 0, –2)
7	(1 ,0, 2)	14	(1, –2,–1)	21	(1, –1, 2)	27	(1, 2, –2)
8	(1, 1, –2)	15	(1, 0, 1)	22	(1, 1, 1)	28	(1, 2, –1)
						29	(1, 0, 0)

where the points of weight zero form an (25,5) – arc.
From equation (8) we get 6 point of weight 1 and these points are:

i	p(i)	i	p(i)
0	(0 , 0, 1)	12	(0,1,–2)
3	(0,1, 2)	20	(0,1, 1)
5	(0 , 1, –1)	30	(0,1, 0)

and there is no point of weight 2,
So the sum of all these points are $q^2 + q + 1$ and the points of PG(2 , 5) of non – zero weight in this example will give an (6,6,f) –arc of type (1,6).

5. Some cases of (k, n, f) –arcs of type(n –5 ,n)having at least one point of weight 5.

Lets consider the following four cases:

(1) $L_0 > 0, L_1 = 0, L_2 = 0, L_3 = 0, L_4 > 0, L_5 > 0$;

(2) $L_0 > 0, L_1 = 0, L_2 = 0, L_3 > 0, L_4 = 0, L_5 > 0$;

(3) $L_0 > 0, L_1 = 0, L_2 > 0, L_3 = 0, L_4 = 0, L_5 > 0$;

(4) $L_0 > 0, L_1 > 0, L_2 = 0, L_3 = 0, L_4 = 0, L_5 > 0$;

we now discussion the case (1) :
In this case there are no points of weight 1,2 and 3 with respect to the (k,n,f)–arc. Let us consider the two cases for L_0.

- Case (1), L_0=1,

Let ℓ be a line of weight n–5 and let R be a unique point of weight zero, such that $R \in \ell$.Let F be a point of weight 4 (possibly on ℓ).Since $V^4_{n-5} \geq 2$ and $V^0_n = 0$, there is at least another (n–5)–secant r through F

besides FR. Let O be a point of weight 5 but this point does not belong to r, in that case r and OR would be (n–5)–secants and this is impossible because $V^{5}_{n-5}=1$.
Hence, every point of the line r has weight 4, since r is an (n–5) – secant we get (n–5)=4 (q+1) whence n=4q+9.Every line through R has weight n–5=4(q+1).Since there are only points of weight 4 or 5 possible a part from R, we have a contradiction.

Theorem: When $L_0=1$, $L_1=0$, $L_2=0$, $L_3=0$, $L_4>0$, $L_5>0$,there is no (k,n,f)–arc of type (n–5,n) with W = (n –5)(q +1).
By similar proof for this when $L_5=1$, we have the following results:

Theorem: When $L_0>0$, L1= 0 , L2= 0 , L3= 0, L4> 0, L5> 0, there is no (k,n,f)–arc of type (n –5,n) with W=(n – 5)(q+1)+5.

- Case (2), with $L_0>1$
Let R and P be distinct points of weight o, both R and P lie on a line ℓ.If X is a point , it will not belong to ℓ(where ℓ is a line of weight n–5). Since $V^{0}_{n}=0$, then XR and XP are both (n–5)– secants; and because $V^{5}_{n-5}=1$, X is not of weight 5, hence X is a point of weight 4. Thus all point not on ℓ have weight 4. Hence, considering a(n–5)–secants through R we have n–5=4q, so n = 4q + 5.
Let A be a point on ℓ with f (A) = 4, any line through A different from ℓ will have 4(q + 1),which is impossible .Thus, the points on ℓ have only weight 5 or weight 0. Therefore K consists of (4/5)q collinear points, each of weight 5 and the q^2 points not collinear with them , each of weight 4.
Hence, we deduce the following theorem:

Theorem: Let K be a (k,n,f) – arc having $L_0>0$, $L_1=0$, $L_2=0$, $L_3=0$,$L_4>0$,$L_5>0$ of type (n–5,n) with W = (n–5) (q+1). Then K consists of (4/5)q collinear points, each of weight 5 and q^2 points not collinear with them, each of weight 4.
By similar proof for this case when $L_5>1$, we have the following results:

Theorem:
Let K be a (k,n,f)– arc having $L_0>0$, $L_1>0$, $L_2=0$,$L_3=0$,$L_4>0$, $L_5>0$ of type (n–5,n) with W= (n–5) (q+1) +5. Then K consists of (4/5)q collinear points each of weight 0 and the q^2 points not collinear with them each of weight 4.
Further n = 4q+5.

Theorem: When L_0=1,for all cases (2)–(4) there is no (k,n,f)–arc of type (n–5,n) with W=(n –5) (q+1).

Theorem: When L_5= 1, for all cases (2)–(4) there is no (k,n,f)–arc of type (n–5,n) with W = (n–5) (q+1) + 5.

Theorem: when L_0>1, for all cases (2) – (6) a (k,n,f) – arc of type (n–5,n) with W =(n–5) (q +1) exists.

Theorem: When L_5>1, for all cases (2) – (6) a (k,n,f)–arc of type (n–5,n) with W=(n–5) (q +1) + 5 exists.

REFERENCES

1- Ball, S; Blokhuis, A. and Mazzocca, F. (1997) "Maximal arcs in Dsarguesian planes of odd order do not exist" Combinatorica 17 (1), p.p. 31 – 41.

2- D' Agostini, E. "Sulla caratterizzazione delle (k,n,f)–calottedi tipo (n–2,n)" Attisem. Mat. Fis. Univ. Modena, XX1X, (1980), 263 – 275.

3- Hameed. F. k. " Weighted (k,n) – arcs in the projective plane of order nine" PH.D. Thesis. University of London (1989).

4- Hirschfeld J. W. P. "Projective Geometries over finite fields" Oxford, (1979).

5- Mohammed, M. J. "Classification of (k,3)– arcs and (k,4)–arcs in Projective Plane over Galois field" M. Sc. Thesis, University of Mosul, Iraq, (1988).

6- Wilson, B. J. "(k,n,f)–arc and caps in finite projective spaces "Annals of Discrete Mathematics 30 (1986), 355–362.