ON γ-REGULAR RINGS

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المُلخص

الهدف الرئيسي في هذا البحث هو تعريف ودراسة نمط جديد من الحلقات المنتظمة سميت بالحلقات المنتظمة من النمط - γ . الحلقة R تسمى حلقة منتظمة من النمط-γ اذا كان لكل a في R يوجد b في R وعدد صحيح موجب 1≠n بحيث ان $a=ab^na$.

كذلك درسنا بعض الصفات الرئيسية لهذه الحلقات والتمثيل لعناصرها . اخيرا درسنا العلاقة بين الحلقات المنتظمة من النمط - γ وبعض الحلقات الاخرى

ABSTRACT

The main goal of this work is to introduce and study a new type of regular rings called γ-regular rings. That is, a ring R is said to be γ-regular if for every $a\in R$ there exists $b\in R$ and a positive integer $n\neq 1$ such that $a=ab^na$.

We will study some basic properties of those rings including the representation of their elements.

Finally, we will study the relation between γ-regular rings and other rings.

1 : Introduction :

We conclude that all rings are assumed to be associative with identity.

A ring R is said to be Von Neumann regular if for every $a\in R$, there exists $b\in R$ such that $a=aba$. The concept of regular rings was introduced by J. Von Neumann in 1936[13]. As a generalization of this concept McCoy[1] defined π-regular rings, that is, a ring R with every $a\in R$, there exists $b\in R$ and a positive integer n such that $a^n = a^n b a^n$. In the recent years regularity and π-regularity have been extensively studied by many authors (cf.[1], [10], [11], [12], [15]).

A ring R is said to be strongly regular if for every $a\in R$, there exists $b\in R$ $a = a^2b$. This concept has been defined some sixty years ago by R.F.Arens and I.Kaplansky [2], and was studied in recent years by many authors (cf. [3],[4] ,[9]) . It should be noted that in a strongly regular ring R , $a = ba^2$ if and only if $a = a^2b$ [5] .Azumaya[3] in 1954 defined strongly π-regular rings, that is, a ring R with every $a\in R$, there exists $b\in R$ and a positive integer n such that $a^n = a^{n+1} b$.

In 1968 Ehrlich [6] defined unit regular rings, that is, a ring R with every $a\in R$, there exists a unit $u\in R$ such that $a=aua$.

: γ-Regular Rings:

In this section we introduce the definition of γ-regular rings and some basic properties of them.

Definition 2.1:

An element a of a ring R is said to be γ-regular if there exists b in R and a positive integer $n\neq 1$ such that $a = a b^n a$.

A ring R is said to be γ-regular if every element of R is γ-regular element.

Examples 2.2:

The following rings are γ-regular rings:

1-Z_3, Z_5, Z_{11}, Z_{15}.

2-$R_{2x2}(Z_2)$, the ring of 2x2 matrices over Z_2.

3-Boolean rings.

Its clear that every γ-regular ring is regular ring, however the converse is not true in general, for example the rings (Q , + , .) of rational numbers, the rational(real) Hamilton Quaternion and a quadratic field are regulars but not γ-regulars because 2 is regular element in each of them but not γ-regular element.

Theorem 2.3:

Let R be a γ-regular ring and I be an ideal of R, then R/I is also γ-regular ring.

Proof: Let $a+I \in R/I$, so $a \in R$. Since R is γ-regular ring then there exists $b \in R$ and a positive integer $n \neq 1$ such that $a = a\, b^n a$.

Hence $a+I= a\, b^n a+I=(a+I)(b^n+I)(a+I) =(a+I)(b+I)^n(a+I)$.

ThereforeR/I is γ-regular ring. ■

Definition 2.4:

An ideal I of a ring R is said to be γ-regular if for every element $a \in I$ there exists $b \in I$ and a positive integer $n \neq 1$ such that $a = a\, b^n a$.

Definition 2.5: [7]

A ring R is said to be reduced, if R contains no non-zero nilpotent elements.

Lemma 2.6: [7]:

Every idempotent element in a reduced ring is central.

Proposition 2.7:

In a reduced γ-regular ring, every ideal is γ-regular.

Proof: Let I be any ideal of a reduced γ-regular ring R, and $a \in I$, then there exists $b \in R$ and a positive integer $n \neq 1$ such that $a = a\,b^n\,a$. Let $e = ab^n$, then e is idempotent element and hence it is central.

Now let $y = ab^{n+1}$, so $y \in I$, then

$ay^n a = a(ab^{n+1})^n a = a(ab^{n+1}\ ab^{n+1}\ ab^{n+1} \ldots ab^{n+1})a = a(ab^n b\ ab^{n+1}\ ab^{n+1} \ldots ab^{n+1})a$

$= a(b\ ab^n\ ab^{n+1}\ ab^{n+1} \ldots ab^{n+1})a$ (Because ab^n is central)

$= a(b\ ab^{n+1}\ ab^{n+1} \ldots ab^{n+1})a = a(b\ ab^n b\ ab^{n+1} \ldots ab^{n+1})a$

$= a(b^2\ ab^n ab^{n+1} \ldots ab^{n+1})a = a(b^2\ ab^{n+1} \ldots ab^{n+1})a = \ldots = \ldots = \ldots$

$= a(b^{n-1}\ ab^{n+1})a = a(b^{n-1}\ ab^n b)a = a(b^n ab^n)a = ab^n ab^n a = ab^n a = a$. That is $a = ay^n a$ nd a positive integer $n \neq 1$ Hence I is γ-regular ideal. ■

Theorem 2.8:

A homomorphic image of γ-regular ring is γ-regular ring.

Proof: Let $f: R \to R'$ be a homomorphism from R to R'. Let $y \in f(R)$. Then there exists $x \in R$ such that $y = f(x)$. Since R is γ-regular ring, then there exists $b \in R$ and a positive integer $n \neq 1$ such that $x = xb^n x$.

Now $y = f(x) = f(xb^n x) = f(x)f(b^n)f(x) = f(x)(f(b))^n f(x) = y(f(b))^n y$. Therefore $f(R)$ γ-regular ring. ■

Lemma 2.9:[10]

If R is a reduced ring, and if a is a non-zero element in R. Then $r(a) = r(a^2)$, and $l(a) = r(a)$, where $l(a)$ and $r(a)$ are the left and right annihilators of a respectively.

Theorem 2.10:

Let R be a reduced ring. If $R/r(a)$ is γ-regular ring for all $a \in R$, then R is γ-regular ring.

Proof: Suppose that $R/r(a)$ is γ-regular ring, then for any $a + r(a) \in R/r(a)$, there exists $b + r(a) \in R/r(a)$ and a positive integer $n \neq 1$ such that $a + r(a) = (a + r(a))\,(b + r(a))^n\,(a + r(a)) = ab^n a + r(a)$. Then $a - ab^n a \in r(a)$. So $a(a - ab^n a) = 0$.

That is a^2 $(1- b^n a)=0$. Then $(1- b^n a) \in r(a^2)=r(a)$ [Lemma 2.9]. o a $(1- b^n a)=0$. Hence $a=ab^n a$. Therefore R is γ-regular ring. ■

Lemma 2.11:

If y is an element of a ring R such that $a-ay^\alpha a$ is γ-regular element, then a is regular element, where $1 \neq \alpha$ is a positive integer.

Proof: Suppose that $a-ay^\alpha a$ is γ-regular element, then there exists an element $b \in R$ and a positive integer $n \neq 1$ such that

$a-ay^\alpha a=(a-ay^\alpha a)b^n(a-ay^\alpha a)$.

Now $a-ay^\alpha a=(a-ay^\alpha a)(b^n a- b^n ay^\alpha a)= ab^n a- ab^n ay^\alpha a- ay^\alpha ab^n a+ ay^\alpha ab^n ay^\alpha a$,

then $a = ay^\alpha a +ab^n a- ab^n ay^\alpha a- ay^\alpha ab^n a+ ay^\alpha ab^n ay^\alpha a$

$= a(y^\alpha +b^n- b^n ay^\alpha- y^\alpha ab^n+ y^\alpha ab^n ay^\alpha)a= a z a$.

Where $z= y^\alpha +b^n- b^n ay^\alpha- y^\alpha ab^n+ y^\alpha ab^n ay^\alpha$. Therefore a is a regular lement.

■

Theorem 2.12:

Let R be a ring and let I be a γ-regular ideal such that R/I is γ-regular. Then R is regular ring.

Proof: Let $a \in R$. Then $a+I \in R/I$. Since R/I is γ-regular ring, then there exists $b+I \in R/I$ and a positive integer $n \neq 1$ such that $a+I=(a+I)(b+I)^n(a+I)$. Then $a+I=ab^n a+I$. So $a-ab^n a \in I$. Hence a is regular element [Lemma 2.11]. herefore R is regular ring. ■

Definition 2.13:

A ring R is said to be unit γ-regular if for every a in R there exists a unit u in R and a positive integer $n \neq 1$ such that $a = a u^n a$.

Definition 2.14:[12]

A ring R is said to be a semi-commutative ring if every idempotent element in R is central.

Hence every reduced ring is semi-commutative ring.[12]

Theorem 2.15:

Let R be a semi-commutative γ-regular ring, then R is unit regular ring.

Proof: Let $x \in R$, then there exists $y \in R$ and a positive integer $n \neq 1$ such that $x = xy^n x$. Then xy^n and $y^n x$ are idempotent elements.

Hence $xy^n = x(y^n x)y^n = (xy^n)(y^n x) = y^n(xy^n)x = y^n x$. Let $v = xy^n + xy^{2n} - 1$ and $u = x + xy^n - 1$. Since $xy^n = y^n x$ and $x = xy^n x$, we have

$uv = (x + xy^n - 1)(xy^n + xy^{2n} - 1)$

$= xxy^n + xxy^{2n} - x + xy^n xy^n + xy^n xy^{2n} - xy^n - xy^n - xy^{2n} + 1$

$= xy^n x + xy^n xy^n - x + xy^n xy^n + xy^n xy^n y^n - xy^n - xy^n - xy^n y^n + 1$

$= x + xy^n - x + xy^n + xy^n y^n - xy^n - xy^n - xy^n y^n + 1 = 1.$

Similarly $vu = 1$ and $xvx = x(xy^n + xy^{2n} - 1)x = xxy^n x + xxy^n y^n x - x^2 = x^2 + x - x^2 = x$.

herefore R is unit regular ring. ■

Proposition 2.16:

If R is a ring such that for each nonzero element $a \in R$ there is a unique $b \in R$ such that $a^n = a^n b a^n$, a positive integer $n \neq 1$, then b is γ-regular element.

Proof: Since $a^n = a^n b a^n$ for each $a \in R$, then R has no divisor of zero. Then cancellation law holds.

ow $1 = ba^n \Rightarrow b = ba^n b$. Therefore b is γ-regular element. ■

Proposition 2.17:

If a ring R is γ-regular, then r(a) is direct summand for every a in R.

Proof: Since R is γ-regular, then for each $a \in R$ there exists $b \in R$ and a positive integer $n \neq 1$ such that $a = ab^n a$. Then $a(1 - b^n a) = 0$.

So $1 - b^n a \in r(a)$. Hence $1 - d \in r(a)$, where $d = b^n a$ and $ad = a$.

Now $1 = d + (1 - d)$, then $R = dR + r(a)$. We shall prove that $dR \cap r(a) = 0$.

Let $x \in dR \cap r(a)$, then $x \in dR$ and $ax = 0$. So $x = dc$ for some $c \in R$.

Now $ax = adc = 0$, then $ac = 0$. So $b^n ac = 0$. That is $dc = 0$. Hence $x = 0$.

Now $y=u^{-1}ay=a(u^{-1}y)\in aR$ (Because $a=e+u$ and R is semi commutative, $en^{-1}a=u^{-1}e+1$ and $au^{-1}=eu^{-1}+1=u^{-1}e+1$. So $au^{-1}=u^{-1}a$). Hence $(1-e)R\subseteq aR$. ■

4 : γ-Regular Rings with condition (*):

One of the most important rings was introduced by Kandasamy [14] is quasi-commutative rings that is a ring R with $ab=b^{m}a$ for every pair $a,b\in R$ and for some positive integer m.

Here we restrict the quasi-commutative ring to the condition that has a main role in our proofs and to discuss the connection between γ-regular rings and some other rings. The condition is

(*): Let R be a ring such that for every $1\neq a\in R$ and $b\in R$, there exists a positive integer $m>1$ such that $ab=b^{m}a$.

The reason for 1 that not satisfies condition (*) is $1\cdot b=b^{m}\cdot 1$, this equation is true if m=1, also the identity element $1\in R$ is γ-regular element, strongly γ-regular element.

In this section we discus the connection between γ-regular ring with the other rings which they are commutative, reduced or satisfies condition (*).

Proposition 4.1:

Every reduced γ-regular ring is strongly regular ring.

Proof: Since reduced γ-regular ring implies reduced regular ring, then t's trongly regular ring, [10;Theorem1.3.7]. ■

But the converse of this theorem is not true in general. For example the ring (Q , + , .) of rational number is reduced strongly regular but not γ-regular.

Corollary 4.2:

Let R be a semi-commutative γ-regular ring. Then R is strongly regular ring.

roof: Follows from [12; Proposition 1.2.5]. ■

Corollary 4.3:

If R is duo γ-regular ring, then R is strongly regular ring.

roof: Follows from [12; Proposition 1.2.5]. ■

Theorem 4.4:

Let R be a ring satisfies condition (*), then the following are equivalent:

1- R is γ-regular ring.

2- R is strongly regular ring.

Proof: 1⇒2: For every $a \in R$ there exists $b \in R$ and a positive integer $n \neq 1$ such that $a=ab^n a$. Since R satisfies condition (*), then $ab^n=(b^n)^r a=b^{nr}a$ for some positive integer $r > 1$.

Now $a=ab^n a=b^{nr}a.a=b^{nr}a^2=ca^2$, where $c=b^{nr} \in R$, and if $b^n a=a^m b^n$ for some positive integer $m > 1$, then $a=aa^m b^n=a^2 a^{m-1} b^n=a^2 d$, where $d= a^{m-1} b^n \in R$. Therefore R is strongly regular ring.

2⇒1: For every $a \in R$ there exists $b \in R$ such that $a=a^2 b$. Now $a=a.ab=ab^n a$ for some positive integer $n > 1$ (R satisfies condition (*)). Then R is -
regular ring. ■

From the proof of above theorem also we can shows that **1⇒2** even when R is quasi-commutative ring, as in the following corollary:

Corollary 4.5:

Let R be a quasi-commutative γ-regular ring, then R is strongly regular ring.

Theorem 4.6:

Let R be a reduced ring satisfies condition (*). Then the following are equivalent:

1- R is γ-regular ring.

2- R is regular ring.

Proof: **1⇒2:** Clearly from the definition of γ-regular ring.

2⇒1: Since R is reduced and regular then R is strongly regular ring, and y [Theorem 4.4] R is γ-regular ring. ■

Theorem 4.7:

Let R be a ring satisfies condition (*). Then R is γ-regular ring if and only if every principal ideal of R is generated by an idempotent.

Proof: If R is γ-regular ring, then its clearly that every principal right ideal of R is generated by an idempotent ,[13].

Conversely: If aR=eR, where e is an idempotent element. Then a=er for some r in R. Now $a=er=e^2r=ea$. Let e=ab for some b in R, since R satisfies condition (*), then $e=b^na$ for some positive integer $n > 1$.

Now $a=ea=b^na.a=b^na^2$. Similarly for Ra=Re. Then R is strongly regular. herefore R is γ-regular ring ,[Theorem 4.4]. ■

Proposition 4.8:

Let R be a ring satisfies condition (*). Then the following are equivalent:

1- R is γ-regular ring.

2- Every principal ideal is a direct summand.

Proof: **1⇒2:** Clearly, from [4; Proposition 1.1.3].

2⇒1: Let R=aR⊕K for some ideal K of R, it's clear that $a\neq 1$ because if a=1 then $R=1\cdot R\oplus\{0\}$, the proof being trivial. Since $1\in R$ then 1=ar+k for some $r\in R$ and $k\in K$. Since R satisfies condition (*) then $ar=r^na$; and a positive integer $n\neq 1$.

Then $1=r^na+k$ imply $a=a\,r^na+ak$ and $ak\in aR\cap K=0$. So $a=a\,r^na$. Therefore R s γ-regular ring. ■

Remark 4.9 :

If we add the condition that R satisfies condition (*) in [Corollary 3.3], then the converse holds.

Theorem 4.10:

If R is a reduced ring satisfies condition (*). Then R is γ-regular ring if and only if for every element $a\in R$, $a=eu$, where e is an idempotent and u is unit.

Proof: If R is reduced γ-regular ring, then $a=eu$, where e is an idempotent and u is unit [Corollary 3.3].

Conversely: Let $a=ue$, where e is idempotent and u is unit, then $e=ra$ where r is the inverse of u.

Now $ae=ara$, but $ae=ue.e=ue^2=a$, then $a=ara$ which is regular. Therefore, y [Theorem 4.6] R is γ-regular ring. ■

Proposition 4.11:

Let R be a ring satisfies condition (*), then R is γ-regular if and only if r(a) is direct summand for every a in R.

Proof: If R is γ-regular, then r(a) is direct summand for every a in R [Proposition 2.17].

Conversely: Let $R=aR+r(a)$. In particular $1=ar+d$, where $r\in R$ and $d\in r(a)$, then $a=a^2r+ad$ imply $a=a^2r$ and hence $a=a^2r=aar=ar^na$ for some ositive integer $n>1$. Therefore R is γ-regular ring. ■

Theorem 4.12:

Let R be a ring satisfies condition (*). Then the following are equivalent:

1- R is γ-regular ring.

2- For every right ideal I and left ideal J in R, $IJ=I\cap J$.

Proof: **1⇒2**: Since every γ-regular is regular, then 2 holds by [10; Theorem 1.1.7].

2⇒1: Let x in R, since x is in $xR\cap Rx=xRx$, there is an element y in R such that $xyx=x$. Since R satisfies condition (*), then $x=y^n x^2$ for some positive integer $n > 1$. So R is strongly regular. By [Theorem 4.4] R is -regular ring. ■

Theorem 4.13:

Let R be a reduced ring satisfies condition (*). If every ideal of R is a maximal right ideal, then R is γ-regular ring.

Proof: Since R is reduced and every prime ideal is maximal right ideal, then R is regular ring [8], and since R satisfies condition (*) nd s reduced, then by [Theorem 4.6] R is γ-regular ring. ■

Corollary 4.14:

Let R be a reduced ring satisfies condition (*). Then R is a γ-regular ring if R/P is γ-regular ring for every prime ideal P in R.

Proof: Let P be a prime ideal in R, then R/P is a division ring, because R/P is a γ-regular and has no nonzero divisor. Therefore P is maximal right(left)ideal in R and Ris a γ-regular ring[Theorem4.13] . ■

Theorem 4.15:

Let R be a ring satisfies condition (*). Then R is γ-regular ring if and only if $I=\sqrt{I}$ for each ideal I in R.

Proof: Suppose that R is γ-regular ring, its clearly that $I\subseteq\sqrt{I}$ for each ideal I in R. Now let $b\in\sqrt{I}$ then $b^n\in I$ for some $n\in Z^+$, then there exists $c\in R$ and $1\neq r\in Z^+$ such that $b^n = b^n c^r b^n$. Since R satisfies condition (*), then $b^n c^r=(c^r)^m b^n$ for some positive integer $m > 1$. That is $b^n= c^{mr} b^{2n}$. So b^{n-1} can be written in the form $b^{n-1}= c^{mr}b^{2(n-1)}= c^{mr} b^{n} b^{n-2} \in I$, and we repeat this n-times we get $b\in I$, then $\sqrt{I} \subseteq I$. Hence $I = \sqrt{I}$.

Conversely: Let $I = \sqrt{I}$ for each ideal I in R. Take $I = a^2R = \sqrt{a^2R}$ then $a^2 \in a^2R \Rightarrow a \in \sqrt{a^2R} = a^2R \Rightarrow a \in a^2R$. Hence R is strongly regular ring. herefore by [Theorem 4.4] R is γ-regular ring. ■

Corollary 4.16:

Let R be a ring satisfies condition (*). Then R is γ-regular ring if and only if each ideal I in R is semi-prime.

Theorem 4.17:

If R is a reduced ring satisfies condition (*) and every maximal ideal of R is a right annihilator, then R is γ-regular ring.

Proof: Let $a\in R$, we shall prove that $aR+r(a)=R$. If not, there exists a maximal right ideal M containing $aR+r(a)$. If $M=r(b)$ for some $0\neq b\in R$, we have $b\in l(aR+r(a))\subseteq l(a)=r(a)$[10; Theorem 1.3.10], which implies $b\in M=r(b)$, then $b^2 =0$ and $b=0$, a contradiction. Therefore $aR+r(a)=R$.

In particular, $ac+d =1$, with $c\in R$ and $d\in r(a)$, then $a^2c + ad=a$ implies $a^2c=a$, then R is strongly regular ring.

Therefore R is γ-regular ring [Theorem4.4]. ■

Theorem 4.18:

Let R is a reduced ring satisfies condition (*) such that every principal right ideal of R is a right annihilator, then R is γ-regular ring.

Proof: Since R is reduced and every principal right ideal of R is a right annihilator, then by [10; Theorem 1..3.10] R is strongly regular ring, and since R satisfies condition (*) then by [Theorem 4.4] R is γ-regular ring ■

5:Strongly γ-Regular Rings:

In this section we introduce another new type of rings that [Proposition 4.1], [Corollary 4.2] and [Corollary 4.3] leads us to define it and we shall call those rings as a strongly γ-regular rings.

Definition 5.1:

Let R be any ring. Then R is called rig strongly γ-regular ring if for every element $a \in R$, there exists $b \in R$ and a positive integer $n \neq 1$ such that $a=a^2b^n$.

Hence, in a strongly γ-regular ring R, $a=a^2b^n$ if and only if $a=b^na^2$,[5].

In a commutative ring,the equation $ab^na=a$ may be written as $a^2b^n=a$. That is, a commutative ring R is γ-regular if and only if it is strongly γ-regular. We see that every strongly γ-regular ring is strongly regular ring, however the converse is not true in general, for examples the rings (Q,+ , .) of rational numbers, the rational(real) Hamilton Quaternion and a quadratic field are strongly regulars but not strongly γ-regulars.

Theorem 5.2:

Let R be a strongly γ-regular ring and I be an ideal of R. Then R/I is also strongly γ-regular ring.

Proof: Let $a+I \in R/I$, so $a \in R$. Since R is strongly γ-regular ring then there exists $b \in R$ and a positive integer $n \neq 1$ such that $a = a^2 b^n$.

Hence $a+I= a^2b^n+I=(a^2+I)(b^n+I)=(a+I)^2(b+I)^n$. Therefore R/I is strongly -regular ring. ■

Theorem 5.3:

A homomorphic image of strongly γ-regular ring is strongly γ-regular ring.

Proof: The proof is similar to the proof of Theorem [2.8]. ■

Here we want to find the condition for strongly regular ring to be strongly γ-regular ring.

Theorem 5.4:

Let R be a ring satisfies condition (*), then the following are equivalent:

1- R is strongly γ-regular ring.

2- R is strongly regular ring.

Proof: **1⇒2:** Clearly from the definition of strongly γ-regular ring.

2⇒1: Since R is strongly regular ring, then for every $a\in R$, there exists $b\in R$ such that $a=a^2b$.

Now since R satisfies condition (*), then for every $a,b\in R$, $ab=b^na$ for some positive integer $n > 1$. Then $a=ab^na$, and since R is strongly regular then R is reduced, implies $a=a^2b^n$. Therefore R is strongly γ-regular ring. ■

Theorem 5.5:

If R is a regular ring satisfies condition (*), then R is strongly γ-regular ring.

Proof: Since R is regular ring, then for every $a\in R$, there exists $b\in R$ such that $a=aba$. Since R satisfies condition (*), then $ab=b^na$ with a positive integer $n\neq 1$for every $a,b\in R$. Then $a=a^2b^n$. Therefore R is strongly γ-regular ring. ■

Here we lead to discus the connection between γ-regular rings and strongly γ-regular rings.

Theorem 5.6: Every strongly γ-regular ring is γ-regular ring.

Proof: Since R is strongly γ-regular ring, then for every $a\in R$ there exists $b\in R$ and a positive integer $n\neq 1$ such that $a=a^2b^n= b^na^2$.

Now if $a=b^na^2$, then $ab^n = (b^na^2)b^n = b^n(a^2b^n) = b^na$. This implies

$ab^na = c^na^2 = a$, then $a = ab^na$. Therefore R is γ-regular ring. ■

The converse of this theorem is not true in general. For example the ring $R_{2x2}(Z_2)$ of 2x2 matrices over the ring Z_2 is γ-regular ring but not strongly γ-regular ring because the element $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is γ-regular element but not strongly γ-regular element.

Theorem 5.7:

Let R be a ring. Then R is strongly γ-regular if and only if R is reduced γ-regular.

Proof: Suppose that R is reduced γ-regular, then for every $a \in R$ there exists $b \in R$ and a positive integer $n \neq 1$ such that $a=ab^n a$.

Now $(a-a^2b^n)^2 = (a-a^2b^n)(a-a^2b^n) = a^2-a^3b^n-a^2b^na+a^2b^na^2b^n =$

$a^2-a^3b^n-a.ab^na+a.ab^na.ab^n = a^2-a^3b^n-a.a+a.a.ab^n = a^2-a^3b^n-a^2+a^3b^n = 0$. Since R is reduced then $a-a^2b^n=0$, and then $a=a^2b^n$. Similarly $(a-b^na^2)^2 = 0$ which implies $a=b^na^2$. Therefore R is strongly γ-regular.

Conversely: Suppose that R is strongly γ-regular ring, then by [Theorem 5.6] R is γ-regular ring. To prove that R is a reduced ring, suppose that there exists a positive integer n, such that $c^n=0$ for some $c \in R$.

Sine $c=c^2d^m$ for sum positive integer $m \neq 1$ gives $0=c^nd^m=c^{n-1}$ and $0=c^{n-1}d^m=c^{n-2}$ and so on $c=0$, then R is a reduced ring. Therefore R is a reduced γ-regular ring. ■

Corollary 5.8:

If R is a strongly γ-regular ring, then R is a unit regular ring.

Proof: Since every strongly γ-regular is reduced γ-regular [Theorem 5.7], and since every reduced ring is semi-commutative ring, then by [Theorem 2.15] R is a unit regular ring. ■

Corollary 5.9:

If a ring R is strongly γ-regular, then a^n is a unit γ-regular element for each $a \in R$ and positive integer $n > 1$.

Proof: Let R be a strongly γ-regular, then R is γ-regular and reduced [Theorem 5.7], and hence R is semi-commutative ring. Therefore by [Theorem 3.1] a^n is a unit γ-regular element. ■

Theorem 5.10:

Let R be a ring. If R is semi-commutative γ-regular ring, then R/N is strongly γ-regular ring.

Proof: since R is γ-regular then for each $a \in R$, there exists $b \in R$ and a positive integer $m \neq 1$ such that $a=ab^m a$. Let $e=ab^m$, then e is idempotent and hence central, then $a=ae=ea$.

Now $a(1-e)=0$ implies $(a(1-e))^n=0$, this means that $a(1-e) \in N$, so $a+N=ae+N$ Thus $a+N=aab^m+N=a^2b^m+N$, yielding $a+N=(a^2+N)(b^m+N)$. Therefore R/N is strongly γ-regular ring. ■

Lemma 5.11:

Let R be a strongly γ-regular ring. Then R is semi-commutative ring.

Proof: From [Theorem 5.7]. ■

Proposition 5.12:

Let R be a semi-commutative γ-regular ring, then R is strongly γ-regular ring.

Proof: Let R be a γ-regular ring, and let a be a non-zero element in R, then there exists b in R and a positive integer $n \neq 1$ such that $a = ab^n a$.

Let $e = ab^n$, then e is an idempotent element, and hence e is central (Since R is semi-commutative ring). So $a = ea = ae = a^2 b^n$, and if $e=b^n a$ then e is also an idempotent element, and hence e is central. So $a=ae=ea=b^n a^2$. Therefore R is a strongly γ-regular ring. ■

Corollary 5.13:

Let R be a duo γ-regular ring, then R is strongly γ-regular ring.

Proof: Since R is duo ring, then every idempotent element is central [18;Lemma1.1.9]. Hence R is semi-commutative ring. Also R is a γ-regular ring, then by [Proposition 5.12], R is strongly γ-regular ring. ■

From [Theorem 2.10] we conclude the following:

Theorem 5.14:

Let R be a reduced ring. If R/r(a) is γ-regular ring for all $a \in R$, then R is strongly γ-regular ring.

Proof: Suppose that R/r(a) is γ-regular ring, then for any $a+r(a) \in R/r(a)$, there exists $b+r(a) \in R/r(a)$ and a positive integer $n \neq 1$ such that $a+r(a) = (a+r(a))\,(b+r(a))^n\,(a+r(a)) = ab^na+r(a)$. Then $a-ab^na \in r(a) \Rightarrow a(a-ab^na)=0 \Rightarrow a^2(1-b^na)=0$.

Then$(1-b^na) \in r(a^2)=r(a)=l(a)$([4:Lemma1.3.6]and[4:Lemma1.3.4])$\Rightarrow (1-b^na)a=0 \Rightarrow a=b^na^2$,and from $(1-b^na)a=0 \Rightarrow (1-b^na) \in l(a)=r(a)$[Lemma 1.2.13], then $a-ab^na=0 \Rightarrow (1-ab^n)a=0 \Rightarrow (1-ab^n) \in l(a)=r(a) \Rightarrow a(1-ab^n)=0$ then $a=a^2b^n$.

Therefore R is strongly γ-regular ring. ■

Theorem 5.15:

Let R be a strongly π-regular ring. Then a^n is strongly γ-regular elements for every $a \in R$ and $n \in Z^+$.

Proof: Let R be a strongly π-regular ring, then for every $a \in R$ there exists $b \in R$ and $n \in Z^+$ such that $a^n = a^{2n}\, b^n$.

Hence a^n is strongly γ-regular element. ■

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