The Simple Rings of Differential Operators

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الخلاصة:

تم افتراض ان الحلقات من النوع التفاضلي المميز هي حلقة بسيطة simple ring وعلى هذا الافتراض تمكنا من تعميم نتائج جونس والتي حققها عام 1995. ان هذا البحث هو مكمل لبحث ونتائج محمود في عام 2006.

Abstract:

In this work, we propose the rings of differential operators are simple and for this proposition we can generalize the results of Jones in 1995. In fact, this work is complement to the results of Mahmood in 2006.

1. Introduction:

Let k be an algebraically closed field of characteristic zero. For a commutative k-algebra A, we define

$$D(A)=\bigcup_{i=0}^{\infty} D^{i}(A),$$

where $D^0(A)=\mathrm{End}_A(A)$ and $D^i(A)=\{\theta\in \mathrm{End}_k(A)\ \|[\theta,a]\in D^{i-1}(A),\forall a\in A\ \}$.

Then D(A) is a sub-ring of $\mathrm{End}_k(A)$, called the ring of differential operators on A. For an irreducible affine variety X, we define $D_1(x)=D(O(x))$, where O(x) is a ring of regular functions of X and call this ring D(O(x)) of differential operator on x.

We use here all the remarks of Jones in [2] and [3]. For $\wedge$ an arbitrary semi-group such that Z = M, let k be the associated semi-group algebra of $\wedge$ and $D(k\wedge)$ the ring of differential operators. Then

$$\begin{aligned} D(k\wedge) &\subseteq D(kM) \\ &= D(k[x_1^{\pm 1},\ldots\ldots,x_r^{\pm 1}]) \\ &= k[x_1^{\pm 1},\ldots\ldots,x_r^{\pm 1},\partial_1,\ldots\ldots,\partial_r], \quad \text{see}[6] \end{aligned}$$

Clearly $x_i\partial_i$ is in $D(k\wedge)$. Now for $\mu\in Z\wedge$, $x_i\partial_i x^{\mu}=\mu_i x^{\mu}$, where $\mu=(\mu_1,\ldots\ldots,\mu_r)$. Let $W=Q[x_1\partial_1,\ldots\ldots,x_r\partial_r]Qk$. Then $W\in D(k\wedge)$. Thus the element of W define polynomial from $Z\wedge\subseteq M_Q$ to k by the rule $x_i\partial_i(\mu)=\mu_i$ for $\mu\in Z\wedge$. Thus for $f\in W$ and $\mu\in Z\wedge$, $f(x_i\partial_i,\ldots\ldots,x_i\partial_i)*x^{\mu}=f(\mu)x^{\mu}$.

Hence we can regard an element $f\in(M_Q)^*$ as an element of $\otimes[x_1\partial_1,\ldots\ldots,x_r\partial_r]$

…… **(A)**

Also we have

$H_i=\{(a_1,\ldots,a_r)\in M_{\otimes} | a_i\geq 0\ ,\ \forall_i=1,\ldots,t\}$

……**(B)**

with respect to the basis $\{m_1',\ldots\ldots,m_r'\}$ of $M_{\otimes}$, order $\{m_1',\ldots\ldots,m_t'\}$ so that $m_i'\in\bigcap_{j\neq i}\partial H_j$.

2- $D(k\wedge)$ and $D(k\tilde{\wedge})$:

For semi-groups $A, B \in M$ such that $ZA = ZB$, define

$$D(kA, kB) = \{\partial \in D(kZA) | \partial * kA \subseteq kB\}.$$

Then D(kA,kB) is a D(kB)-D(kA) –bi-module, see [5].

Jones in [2] and Musson in [4] proved the following proposition:

Proposition 2.1:

(1) $D(k\tilde{\wedge}, k\wedge) = \underset{\lambda \in M}{\oplus} X^{\lambda} I(\Omega_{\tilde{\wedge},\wedge}(\lambda))$;

(2) $D(k\tilde{\wedge}, k\wedge) = \underset{\lambda \in M}{\oplus} X^{\lambda} I(\Omega_{\wedge,\tilde{\wedge}}(\lambda))$.

where $I(\Omega) = \{f \in w | f(p) = 0, \forall p \in \Omega\}$ and $\Omega_{A,B}(\lambda) = \{\mu \in A | \lambda + \mu \notin B\}$.

In this paper, we propose that $D(k\tilde{\wedge})$ is a simple ring.

Proposition 2.2:

(1) $D(k\tilde{\wedge})$ is a simple ring ;

(2) $D(k\tilde{\wedge}, k\wedge) * k\tilde{\wedge} = k\wedge$;

(3) $D(k\wedge) = End_{D(k\tilde{\wedge})} D(k\tilde{\wedge}, k\wedge)$;

(4) $D(k\wedge, k\tilde{\wedge}) D(k\tilde{\wedge}, k\wedge) = D(k\tilde{\wedge})$. In particular, $D(k\tilde{\wedge}, k\wedge)_{D(k\tilde{\wedge})}$ is a generator.

Proof: See [2, proposition 3.10].

Proposition 2.3:

$$D(k\wedge, k\tilde{\wedge}) * k\wedge = k\tilde{\wedge}.$$

Proof: By the proposition 2.2, we have $D(k\tilde{\wedge}, k\wedge) * k\tilde{\wedge} = k\wedge$. Then

$$\begin{aligned} D(k\wedge, k\tilde{\wedge}) * k\wedge &= D(k\wedge, k\tilde{\wedge}) D(k\tilde{\wedge}, k\wedge) * k\wedge \\ &= D(k\tilde{\wedge}) * k\tilde{\wedge} \\ &= k\tilde{\wedge} \end{aligned}$$

By the relations (A) and (B), and the proposition 2.1, Jones in [2] proved the following theorem.

Theorem 2.4:

$D(k\tilde{\wedge}, k\wedge) D(k\wedge, k\tilde{\wedge}) = D(k\wedge)$. In particular $D(k\tilde{\wedge}, k\wedge)$ is a finitely generated projective right $D(k\tilde{\wedge})$-module.

Definition 2.5: Two rings are Morita equivalence if there is a progenerator W_R, such that $S \cong \text{End}\, W_R$. $R \underset{\sim}{M} S$.

Hence, by the proposition 2.1 and theorem 2.3, we have

Theorem 2.5:
There is Morita equivalence $D(k\wedge) \underset{\sim}{M} D(k\bar{\wedge})$.

3.The new relations:

In fact, by the propositions 2.1-2.4, we can find the following results:

Proposition 3.1:
(1) $D(k\bar{\wedge})D(k\wedge,k\bar{\wedge}) = D(k\wedge,k\bar{\wedge})D(k\wedge)$;
(2) $D(k\bar{\wedge},k\wedge)D(k\bar{\wedge}) = D(k\wedge)D(k\bar{\wedge},k\wedge)$.

Proof:
(1) We have $D(k\wedge,k\bar{\wedge})D(k\bar{\wedge},k\wedge) = D(k\bar{\wedge})$. Then $D(k\wedge,k\bar{\wedge})D(k\bar{\wedge},k\wedge)D(k\wedge,k\bar{\wedge}) = D(k\bar{\wedge})D(k\wedge,k\bar{\wedge})$, hence $D(k\wedge,k\bar{\wedge})D(k\wedge) = D(k\bar{\wedge})D(k\wedge,k\bar{\wedge})$.
(2) The same way of part (1).

Note 3.2: It is easy to prove part (1) of proposition 3.1, it's equal to $D(k\wedge,k\bar{\wedge})$ and part (2) of the same proposition is equal to $D(k\bar{\wedge},k\wedge)$.

Proposition 3.3:
(1) $(D(k\bar{\wedge}))^2 = D(k\bar{\wedge})$:
(2) $(D(k\wedge))^2 = D(k\wedge)$

Proof:
(1)
$$\begin{aligned}(D(k\bar{\wedge}))^2 &= D(k\bar{\wedge})D(k\bar{\wedge}) \\ &= D(k\bar{\wedge})[D(k\wedge,k\bar{\wedge})D(k\bar{\wedge},k\wedge)] \\ &= \underline{[D(k\bar{\wedge})D(k\wedge,k\bar{\wedge})]}\, D(k\bar{\wedge},k\wedge) \\ &= D(k\wedge,k\bar{\wedge})D(k\bar{\wedge},k\wedge), \quad \text{By Note 3.2} \\ &= D(k\bar{\wedge}).\end{aligned}$$
(2) The same way of part (1).

Proposition 3.4:
(1) $D(k\bar{\wedge})D(k\wedge) \subseteq D(k\wedge,k\bar{\wedge})$;
(2) $D(k\bar{\wedge},k\wedge) \subseteq D(k\wedge)D(k\bar{\wedge})$.

Proof: By the note 3.3 (A) and (B) in Mahmood in [3], we have:

$$\Omega_{\wedge,\widetilde{\wedge}}(\lambda) \subseteq \left\{\begin{array}{c}\Omega_{\wedge}(\lambda)\\ \text{or}\\ \Omega_{\widetilde{\wedge}}(\lambda)\end{array}\right\} \subseteq \Omega_{\widetilde{\wedge},\wedge}(\lambda)$$

Then $I(\Omega_{\widetilde{\wedge},\wedge}(\lambda)) \subseteq \left\{\begin{array}{c}I(\Omega_{\wedge}(\lambda))\\ \text{or}\\ I(\Omega_{\widetilde{\wedge}}(\lambda))\end{array}\right\} \subseteq I(\Omega_{\wedge,\widetilde{\wedge}}(\lambda))$,

and therefore $D(k\widetilde{\wedge},k\wedge) \subseteq \left\{\begin{array}{c}D(k\wedge)\\ \text{or}\\ D(k\widetilde{\wedge})\end{array}\right\} \subseteq D(k\wedge,k\widetilde{\wedge})$.

Then $\underline{D(k\widetilde{\wedge})D(k\wedge)} \subseteq D(k\widetilde{\wedge})D(k\wedge,k\widetilde{\wedge}) = \underline{D(k\wedge,k\widetilde{\wedge})}$ and this is the first part.

Also, $D(k\wedge)D(k\widetilde{\wedge},k\wedge) \subseteq \left\{\begin{array}{c}D(k\wedge)\\ \text{or}\\ D(k\wedge)D(k\widetilde{\wedge})\end{array}\right\}$

$\Rightarrow D(k\widetilde{\wedge},k\wedge) \subseteq \left\{\begin{array}{c}D(k\wedge)\\ \text{or}\\ D(k\wedge)D(k\widetilde{\wedge})\end{array}\right\}$, and this is the second part.

Note 3.5:

(1) $D(k\widetilde{\wedge},k\wedge)*k\wedge \subseteq \left\{\begin{array}{c}k\wedge \subseteq k\widetilde{\wedge}\\ D(k\widetilde{\wedge})*k\wedge \subseteq k\widetilde{\wedge}\\ \text{or}\\ k\wedge \subseteq D(k\wedge)*k\wedge\end{array}\right\} \subseteq D(k\wedge,k\widetilde{\wedge})*k\widetilde{\wedge}$;

(2) $(D(k\widetilde{\wedge},k\wedge))^2 \subseteq D(k\widetilde{\wedge},k\wedge) \subseteq \left\{\begin{array}{c}D(k\widetilde{\wedge})\\ \text{or}\\ D(k\wedge)\end{array}\right\} \subseteq D(k\wedge,k\widetilde{\wedge}) \subseteq (D(k\wedge,k\widetilde{\wedge}))^2$

(3) $(D(k\widetilde{\wedge},k\wedge))^2 \subseteq D(k\widetilde{\wedge})D(k\widetilde{\wedge},k\wedge) \subseteq D(k\widetilde{\wedge}) \subseteq D(k\wedge,k\widetilde{\wedge})D(k\widetilde{\wedge}) \subseteq (D(k\wedge,k\widetilde{\wedge}))^2$

(4) $(D(k\widetilde{\wedge},k\wedge))^2 \subseteq D(k\widetilde{\wedge},k\wedge)D(k\wedge) \subseteq D(k\wedge) \subseteq D(k\wedge)D(k\wedge,k\widetilde{\wedge}) \subseteq (D(k\wedge,k\widetilde{\wedge}))^2$

Proof:

(2), (3) and (4) are very clear.

(1) By the proposition 3.4, we have

$$D(k\tilde{\wedge},k\wedge) \subseteq \begin{Bmatrix} D(k\tilde{\wedge}) \\ \text{or} \\ D(k\wedge) \end{Bmatrix} \subseteq D(k\wedge,k\tilde{\wedge}).$$

then $D(k\tilde{\wedge},k\wedge) * k\tilde{\wedge} \subseteq D(k\tilde{\wedge}) * k\tilde{\wedge} \subseteq D(k\wedge,k\tilde{\wedge}) * k\tilde{\wedge}$

$$\Rightarrow k\wedge \subseteq k\tilde{\wedge} \subseteq D(k\wedge,k\tilde{\wedge}) * k\tilde{\wedge}$$

......(C)

Also, $D(k\tilde{\wedge},k\wedge) * k\wedge \subseteq D(k\wedge) * k\wedge \subseteq D(k\wedge,k\tilde{\wedge}) * k\tilde{\wedge}$

$$\Rightarrow D(k\tilde{\wedge},k\wedge) * k\wedge \subseteq k\wedge \subseteq k\tilde{\wedge}$$

...... (D)

Therefore by (C) and (D), we reach

$$D(k\tilde{\wedge},k\wedge) * k\wedge \subseteq k\wedge \subseteq k\tilde{\wedge} \subseteq D(k\wedge,k\tilde{\wedge}) * k\tilde{\wedge}$$

...... (E)

Of another side, $D(k\tilde{\wedge},k\wedge) * k\wedge \subseteq D(k\tilde{\wedge}) * k\wedge \subseteq D(k\wedge,k\tilde{\wedge}) * k\wedge$

$$\Rightarrow D(k\tilde{\wedge},k\wedge) * k\wedge \subseteq D(k\tilde{\wedge}) * k\wedge \subseteq k\tilde{\wedge}$$

...... (C')

Also, $D(k\tilde{\wedge},k\wedge) * k\tilde{\wedge} \subseteq D(k\wedge) * k\tilde{\wedge} \subseteq D(k\wedge,k\tilde{\wedge}) * k\tilde{\wedge}$

$$\Rightarrow k\wedge \subseteq D(k\wedge) * k\tilde{\wedge} \subseteq D(k\wedge,k\tilde{\wedge}) * k\tilde{\wedge}$$

...... (D')

Therefore, by the relations (C), (D) with (E), we have:

$$D(k\tilde{\wedge},k\wedge) * k\wedge \subseteq \begin{Bmatrix} k\wedge \subseteq k\tilde{\wedge} \\ D(k\tilde{\wedge}) * k\wedge \subseteq k\tilde{\wedge} \\ \text{or} \\ k\wedge \subseteq D(k\wedge) * k\tilde{\wedge} \end{Bmatrix} \subseteq D(k\wedge,k\tilde{\wedge}) * k\tilde{\wedge}$$

then the proof is completed.

Now, for $n \in N^*$, we have the following theorem:

Theorem 3.6:

(1) $(D(k\tilde{\wedge},k\wedge)D(k\wedge)^n = (D(k\tilde{\wedge},k\wedge))^n D(k\wedge) \subseteq D(k\wedge)$;

(2) $(D(k\wedge,k\tilde{\wedge})D(k\tilde{\wedge})^n = (D(k\wedge,k\tilde{\wedge}))^n D(k\tilde{\wedge}) \subseteq (D(k\wedge,k\tilde{\wedge}))^{n+1}$;

(3) $(D(k\wedge)D(k\wedge,k\tilde{\wedge}))^n = D(k\wedge)(D(k\wedge,k\tilde{\wedge}))^n \subseteq (D(k\wedge,k\tilde{\wedge}))^{n+1}$;

(4) $(D(k\tilde{\wedge})D(k\tilde{\wedge},k\wedge))^n = (D(k\wedge)(D(k\tilde{\wedge},k\wedge))^n \subseteq D(k\tilde{\wedge})$.

Proof:

(1) For n = 1, its clear by note 3.5. Then we start with n = 2:

$$(D(k\tilde{\wedge},k\wedge)D(k\wedge))^2 = D(k\tilde{\wedge},k\wedge)\underline{D(k\wedge)D(k\tilde{\wedge},k\wedge)}\,D(k\wedge)$$

$= D(k\tilde{\wedge},k\wedge)D(k\tilde{\wedge},k\wedge)D(k\wedge)$, By Proposition 3.1

$= (D(k\tilde{\wedge},k\wedge))^2 D(k\wedge)$

$\subseteq D(k\tilde{\wedge},k\wedge)D(k\wedge)$ By note 3.5 (4)

$\subseteq D(k\wedge)$.

$$(D(k\tilde{\wedge},k\wedge)D(k\wedge))^3 = \left(D(k\tilde{\wedge},k\wedge)D(k\wedge)\right)^2\left(D(k\tilde{\wedge},k\wedge)D(k\wedge)\right)$$

$\subseteq \underline{D(k\wedge)D(k\tilde{\wedge},k\wedge)}\,D(k\wedge)$

$\subseteq D(k\wedge)D(k\wedge)$, By Proposition 3.4

$= D(k\wedge)$, By Proposition 3.3.

We assume that the proposition is satisfy when n=t.

$$(D(k\tilde{\wedge},k\wedge)D(k\wedge))^{t+1} = \left(D(k\tilde{\wedge},k\wedge)D(k\wedge)\right)^t\left(D(k\tilde{\wedge},k\wedge)D(k\wedge)\right)$$

$\subseteq \underline{D(k\wedge)D(k\tilde{\wedge},k\wedge)}\,D(k\wedge)$

$= D(k\wedge)$, By Proposition 3.3.

Similarity for (2),(3)and (4).

For the Proposition 3.1, the Note 3.5 and the Theorem 3.6 we conclude the following theorem:

Theorem 3.7: For $n \in N^*$

(1) $(D(k\tilde{\wedge})D(k\tilde{\wedge},k\wedge)D(k\wedge))^n \subseteq (D(k\wedge,k\tilde{\wedge}))^n$,

(2) $(D(k\wedge)D(k\wedge,k\tilde{\wedge})D(k\tilde{\wedge}))^n \subseteq (D(k\wedge,k\tilde{\wedge}))^{2n+1}$,

(3) $$\left.\begin{matrix}(D(k\wedge,k\tilde{\wedge})D(k\tilde{\wedge})D(k\tilde{\wedge},k\wedge))^n \\ \text{or} \\ (D(k\tilde{\wedge},k\wedge)D(k\wedge)D(k\wedge,k\tilde{\wedge}))^n\end{matrix}\right\} \subseteq (D(k\wedge,k\tilde{\wedge}))^{n+1}.$$

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