On G.P.P- Rings

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الخلاصة :

في هذا البحث ندرس الحلقات من النمط- P.P المعممة. لقد برهنا أن الحلقات من النمط- P.P المعممة اليمنى تكون حلقات من النمط P.P إذا كان $r(a^n) \subseteq r(a)$ لكل $a \in R$ ولكل عدد صحيح موجب n. كما بينا ان الحلقة المختزلة من النمط P.P المعممة تكون حلقة منتظمة قوية من النمط - π ، اذا كان كل عنصر ليس من قواسم الصفر له معكوس .

Abstract:

In this work we introduce the notion of G.P.P – rings and some of it is basic properties , we prove that if R is a right G.P.P – ring , then R is P.P– ring if $r(a^n) \subseteq r(a)$ for every $a \in R$ and a positive integer n .We also consider that a reduced G.P.P – Ring with every non – zero divisor has inverse is strongly π – Regular .

1- Introduction :

Throughout this paper, R will denote an associative ring with identity, and all modules are unitary R– module. Recall that ;

1) R is called reduced if R has non-zero nilpotent elements[5]. 2) R is right (left) duo if every right(left) ideal of R is an ideal of R. 3) R is strongly

π – regular if for every a∈R, There exists b∈R and a positive integer n such that $a = a^{2n}b$ [4]; 4) A right R–module M is called general right principally injective (briefly right GP-injective) if for any 0≠a∈R there exist a positive integer n, such that $a^n \neq 0$. and any right R-homomorphism of a^nR into M extends to one of R into M. 5) for any element a in R we define a right annihilator of a by r(a) = {x∈R:ax=0} and left annihilator of a, I(a) is similarly defined.

2- Basic Properties:

In this section we introduce the notion of G.P.P– rings, we give some of it is basic properties. Following [2] A ring R is said to be right (left) P.P – ring if for every a∈R, there exist b∈R such that a = ab and r(a)=r(b) (a = ba , l(a)=l(b)).

Definition 2–1:

A ring R is siad to be right G.P.P – ring if for every a∈R there exist e∈R and a positive integer n such that $a^n = a^ne$ and $r(a^n) = r(e)$.

Clearly every P.P – ring is an G P.P – ring , however The converse is not true. we now consider a necessary and sufficent condition for G P.P–ring to be an P.P– ring.

Lemma 2-2:-

Let R be a right GP.P. ring and $r(a^n) \subseteq r(a)$ for every a∈R then R is P.P – ring.

Proof: Assume that R is GP.P. ring , then $a^n = a^ne$ and $r(a^n)=r(e)$. Implies that $a^n(1 - e) = 0$. and hence $1 - e \in r(a^n) \subseteq r(a)$.

There fore (1–e) ∈r(a) where a = ae. and $r(e) = r(a^n) \subseteq r(a)$ Thus a = ae and r(a) = r(e).

Throrem 2-3:-

R is G.P.P-ring if and only if for all a ∈R , $r(a^n)$ is direct summand.

Proof :-

Assum that $r(a^n)$ is direct summand, then there exists a right ideal I such that $r(a^n) + I = R$. In particular , d + b = 1 , for some $d \in r(a^n)$ and b ∈I $a^n.1 = a^n(d+b) = a^nd + a^nb = 0 + a^nb = a^nb$ Implies that $a^nb = a^n$.

Now ,we must prove that $r(a^n) = r(b)$. Let x∈r(b), then bx = 0, and

$a^n bx = 0$. So $a^n x = 0$ and $x \in r(a^n)$.
Hence $r(b) \subseteq r(b^n)$ Now Let $y \in r(a^n)$,then $a^n y = 0$ and $a^n by=0$.
Thus $by \in r(a^n)$. but $by \in I$ implies that $by \in r(a^n) \cap I = 0$, then $y \in r(b)$ and $r(a^n) \subseteq r(b)$. Therefore $r(a^n)=r(b)$.

Conversely :

Assume that R is right G. P.P – ring ,then for every $a \in R$ there exists $b \in R$ and a positive integer n such that $a^n=a^n b$ and $r(a^n)=r(b)$,
Since $a^n(1-b)=0$, then $(1-b) \in r(a^n)$. So $1= b + (1-b)$, hence $R = bR + r(a^n)$.
Let $x \in bR \cap r(a^n)$ implies that $x = by$ for some $y \in R$ and $a^n x=0$,
so $a^n by=0 = a^n y$. Hence $y \in r(a^n) = r(b)$ and $by= 0 = x$.Thus $bR \cap r(a^n)=0$.
Therefore $r(a^n)$ is directed summand.

Lemma 2-4:[3]

If R is a duo ring , then every idempotent element in R is central.

Theorem 2.5 :

Let R be a duo G.P.P- ring and let J_1,J_2 be ideals in R. Then $r(J_1)+r(J_2)$ generated by a central idempotent element.

Proof: Let R be a duo G.P.P – ring and let J_1,J_2 be two ideals in R.
Then $r(J_1)= e_1R$ and $r(J_2)= e_2R$ where e_1,e_2 are idempotent elements.
Since R is duo ring then by Lemma 2 – 4 e_1 ,e_2 are central idempotents.
Also $r(J_1)+r(J_2)=e_1 R+e_2 R = e_1 R + e_2 e_1 R +e_2(1-e_1)R$.
And $e_1 R + e_2 R \subseteq e_1 R + e_2 (1-e_1)R \subset e_1 R + e_2 R= e_1 R + e_2 (1-e_1)R$,
so $r(J_1)+r(J_2)= e_1 R + e_2 (1-e_1)R$.
Let $e_3= e_2(1-e_1)$,we prove that e_3 is idempotent element

$$
\begin{aligned}
e_3^2 &= e_2 (1-e_1) e_2 (1-e_1) \\
&= (e_2-e_2 e_1)(e_2-e_2 e_1) \\
&= e_2^2 - e_2^2 e_1 - e_2 e_1 e_2 + e_2 e_1 e_2 e_1 \\
&= e_2^2 - e_2^2 e_1 - e_2^2 e_1 + e_2^2 e_1^2 \\
&= e_2 - e_2 e_1 \quad \text{(since } e_1, e_2 \text{ are idempotent elements)} \\
&= e_2(1-e_1) \\
&= e_3
\end{aligned}
$$

Hence e_3 is idempotent element. Since R is duo ring,
then by Lemma 2 – 4 , e_3 is central idempotent element.
Now $e_1e_3 =e_1 (e_2 (1-e_1))$.

$=e_1e_2(1-e_1) =e_1e_2-e_1e_2e_1 =e_1e_2-e_1e_2=0$ (since e_1 is idempotent)

Similariy $e_3e_1=0$.
Now let $x \in (e_1+e_3)R$ then $x=(e_1+e_3)r$; $r \in R$

Thus x= $e_1r+e_3r \in e_1R+e_3R$ and $(e_1+e_3)R \subseteq e_1R+e_3R$
Also let $y \in e_1R+e_3R$. Then $y=e_1r_1+e_3r_3$ for some $r_1,r_3 \in R$
$(e_1+e_3)y=(e_1+e_3)(e_1r_1+e_3r_3)$
$= e_1^2r_1+e_1e_3r_3+e_3e_1r_1+e_3e_3r_3$
$= e_1r_1+0+0+e_3r_3$
$= e_1r_1+e_3r_3=y$ implies $y \in (e_1+e_3)R$
Thus $e_1R+e_3R=(e_1+e_3)R$
That is $r(J_1)+r(J_2)=(e_1+e_3)R$, when (e_1+e_3) is central idempotent element.

3-The connection between G. P.P. – Rings and other rings:

Theorem 3-1:-

Let R be a reduced G P.P. - ring with every non-zero divisor has inverse. Then R is strongly π – Regular.

Proof: we must prove $a^n R \cap e R = 0$ for all $a \in R$. Since R is G. P.P-ring, then $r(a^n) = r(e)$ where e is central idempotent element. Let $x \in a^n R \cap e R$ implies that $x= a^n r$, and $x=e r'$ for some $r, r' \in R$.
Now , See that $x=e r' =e. e r' = e x$. Since $e \in e R = r(a^n)$ then $a^n e = e a^n =0$. Also $e x = e a^n r =0$, $x= a^n r$, then $x=e x=0$. Thus $a^n R \cap e R =0$.
Now we must prove that (a^n+e) is non-zero divisor.
Let $(a^n+e) y = 0$ Implies that $a^n y = -e y$. That is $a^n y = -e y \in a^n R \cap e R$.
Since $a^n R \cap e R=0$.
Then $a^n y = e y =0$ and we have $a^n y = 0$ .That is $y \in r(a^n)=e R$.
There exists $r_1 \in R$ such that $y=er_1$,also $0 = e y =e. e r_1 = e^2 r_1 = e r_1 = y$ (e is idempotent), since (a^n+e) is a non-zero divisor.
Let x be the inverse of (a^n+e). Then we have $(a^n+e) x =1$.
implies that $a^n(a^n+e) x = a^n$ implies $(a^{2n} + a^n e) x = a^n$.
Since $a^n e = 0$, then $a^{2n} x = a^n$. Therefore R is strongly π - regular ring.

Theorem 3–2:

Let R be a G.P.P.-ring with $r(a^n) \subseteq r(a)$ for any $a \in R$ and a positive integer n. Then $a^n R$ is idempotent ideal if $R/ a^n R$ is GP-injective ring.

Proof:

Since R be G.P.P.-ring, then for all $a \in R$ there exists $b \in R$ and a positive integer n , such that $a^n = a^n b$ and $r(a^n) = r(b)$.
Now define a right R-homomorphism $f: a^n R \to R/ a^n R$ by $f(a^n x)=bx +a^nR$ for all $x \in R$. Then, f is well-defined, indeed, let $a^nx_1=a^nx_2$ for any two elements x_1,x_2 in R ,then $a^nx_1-a^nx_2=0$. So $a^n(x_1-x_2)=0$.

Thus $(x_1-x_2)\in r(a^n) = r(b)$ then $x_1-x_2\in r(b)$ implies $b(x_1-x_2)=0$.
Hence $bx_1=bx_2$ therefore $f(a^n x_1)= bx_1+a^nR=bx_2+a^nR=f(a^n x_2)$
Now define g: $R/a^nR\to R/(a^nR)^2$ by $g(y+a^nR)=a^ny+(a^nR)^2$ for all $y\in R$ and by the same way we can prove that g is well-defined. Since R/a^nR is GP- injective ring, there exists $c\in R$ Such that

$$f(a^nx) = (c +a^n R)a^nx =ca^nx+a^nR .$$

Now

$$\begin{aligned} g(f(a^nx)) &= g(bx + a^nR) \\ &= g(ca^nx +a^nR) \\ &= a^nca^nx +(a^nR)^2 \\ &= a^nx+(a^nR)^2. \end{aligned}$$

So $a^nx+(a^nR)^2= aca^nx+(a^nR)^2$.
But $a^nx\in a^nR$,then $(a^nR)^2\subseteq a^nR$. Thus $a^nx+(a^nR)^2\in a^nR$ and $a^n c a^n x\in a^nR a^nR =(a^nR)^2$ .This gives $a^n c a^n x + (a^nR)^2\in(a^nR)^2$.
Hence $a^nR\subseteq (a^nR)^2$ .Therefore $a^nR=(a^nR)^2$

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