On Free Action of Prime Rings

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Abstract
In this paper we study and investigate a mapping free action on a prime ring and semiprime ring R by using some concepts , when R admits to satisfy some conditions, we give some results about that.

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1. Introduction and preliminaries

This paper has been motivated by the work of M.A.Chaudhry and M.S.Samman [4],F.Ali and M.A.Chaudhry[2].Some researchers have studied the notion of free action on operator algebras,Murray and Von Neumann [9] and Von Neumann [10] introduced the notion of free action on abelian Von Neumann algebras and used it for the construction of certain factors (see Dixmier[6]).Kallman[8] generalized the notion of free action of automorphisms of von Neumann algebras,not necessarily abelian, by using implicitly the dependent elements of an automorphism. Choda, Kashahara and Nakamoto [5] generalized the concept of freely acting automorphisms to C^*-algebras by introducing dependent elements associated to automorphisms.Several other authors have studied dependent elements on operator algebras.Abrief account of dependent elements in W^*-algebras has also appeared in the book of Stratila [11].It is well-known that all C^*-algebras and von Neumann algebras are semiprime rings; in particular, a von Neumann algebra is prime if and only if its center consists of scalar multiples of identity.Thus a natural extension of the notions of dependent elements of mappings and free actions on C^*-algebras and von Neumann algebras is the study of these notions in the context of semiprime rings and prime rings.Laradji and Thaheem [16] initiated a study of dependent elements of endomorphisms of semiprime rings and generalized a number of results of H.Choda, I.Kasahara, R.Nakamoto[5] to semiprime rings.

Vukman and Kosi-Ulbl [12] and Vukman [13] have made further study of dependent elements of various mappings related to automorphisms,derivations,(α, β)-derivations and generalized derivations of semiprime rings.The main focus of the authors of J.Vukman, I.Kosi-Ulbl [13] and [14] has been to identify various freely acting mappings related to these mappings, on semiprime and prime rings.The theory of centralizers (also called multipliers) of C^*-algebras and Banach algebras is well established(see C.A.Akemann,G.K.Pedersen,J.Tomiyama[1]and P.Ara, M.Mathieu [3]).Zalar[15]and Vukman and Kosi-Ulbl [14] have studied centralizers in the general framework of semiprime rings.

Throughout,R will stand for associative ring with center Z(R).As usual,the commutator xy−yx will be denoted by [x,y] and (xoy) stand for anti-commutator xy+yx.We shall use the basic commutator identities [xy,z]=[x,z]y+x[y,z] and [x,yz]=[x,y]z+y[x,z].A ring R is said to be n-torsion free, where n is non-zero an integer, if whenever nx=0,with x∈ R, then x=0.Recall that a ring R is prime if aRb=(0) implies that either a=0 or b=0. We and it is R is semiprime if xRx=(0) implies x= 0. A prime ring is semiprime but the converse is not true in general**..** By Zalar [15],an additive mapping T:R → R is called a left (right) centralizer if T (xy)=T(x)y (T(xy)=xT(y)) for all x,y ∈ R.If a ∈ R, then La(x) = ax and Ra(x)=xa(x∈ R) define a left centralizer and a right centralizer of R, respectively. An additive mapping T:R → R is called a centralizer if T(xy) =T(x)y=xT(y)for all x, y ∈ R. An additive mapping d:R→R is called a derivation if d(xy)=d(x)y+xd(y) holds for all x,y ∈ R and d is called left centralizer if d(xy)=d(x)y for all x,y ∈ R. An additive mapping D:R→R is said to be a generalized derivation if there exists a derivation d:R→R such that D(xy)=D(x)y+xd(y)for all x,y∈ R. However , generalized derivation covers the concept of derivation . Also with d = o , a generalized derivation covers the concept of left multiplier (left centralizer) that is , an additive mapping D satisfying D(xy) = D(x)y for all x,y∈R . [2]). Following A.Laradji, A.B.Thaheem [16],an element a ∈ R is called a dependent element of a mapping T:R → R if T(x)a = ax holds for all x ∈ R.A mapping T:R → R is called a free action or (act freely) on R if zero is the only dependent element of T.For a mapping T:R→ R,D(T) denotes the collection of all dependent elements of T.For other ring theoretic notions used but not defined here we refer the reader to I.N.Herstein [7]. In this paper we study and investigate a mapping free action or (act freely) on a prime ring R ,we give some results about that.

2.The main results

In this section, the main results related to this paper are presented which are given as theorems with their proofs.

Theorem 2.1

Let R be a 3-torsion free prime ring ,d be a left centralizer of R and a mapping $\psi:R\to R$ defined by $\psi(x)=[d(x^2),x]$ for all $x\in R$. Then Ψ is free action on R.

Proof: Let $a\in D(\psi)$,then

$[d(x^2),x]a=ax$ for all $x\in R$. (2.1)

Linearzing (2.1),we obtain

$[d(x^2),x]a+[d(x^2),y]a+[d(xy),x]a+[d(xy),y]a+[d(yx),x]a+[d(yx),y]a+$
$[d(y^2),x]a+[d(y^2),y]a=a(x+y)$ for all $x,y\in R$.

According to (2.1) ,we get

$[d(x^2),y]a+[d(xy),x]a+[d(xy),y]a+[d(yx),x]a+[d(yx),y]a+[d(y^2),x]a=0$
for all $x,y\in R$. (2.2)

Since d is a left centralizer then(2.2) become

$[d(x),y]xa+[d(x),x]ya+[d(x),y]ya+[d(y),x]xa+[d(y),y]xa+[d(y),x]ya=0$
for all $x,y\in R$. (2.3)

Putting x = -x and using (2.3),we obtain

$[d(x),y]xa+[d(x),x]ya+[d(y),x]xa=0$ for all $x,y\in R$.

Replacing y by x,we get

$3[d(x),x]xa=0$ for all $x\in R$. (2.4)

Also, since d is a left centralizer from(2.1),we obtain

$[d(x),x]xa=ax$ for all $x\in R$. According to (2.4),we get

$3ax=0$ for all $x\in R$.

Since R is 3-torsion free prime ring, we obtain

a=0.Thus, Ψ is free action on R.

Theorem 2.2

Let R be a 6-torsion free prime ring ,d be a left centralizer mapping of R and a mapping $\psi:R\to R$ defined by $\psi(x)=[d(x),x^2]$ for all $x\in R$. Then ψ is free action on R.

Proof: Let $a\in D(\psi)$,then

$[d(x),x^2]a=ax$ for all $x\in R$. (2.5)

Linearzing (2.5)(i.e. x =x+y),we obtain

$[d(x),x^2]a+[d(x),xy]a+[d(x),yx]a+[d(x),y^2]a+$

$[d(y),x^2]a+[d(y),xy]a+[d(y),yx]a+[d(y),y^2]a = a(x+y)$ for all $x\in R$. (2.6)
According to(2.5),the equation(2.6) reduced to
$[d(x),xy]a+[d(x),yx]a+[d(x),y^2]a+[d(y),x^2]a+[d(y),xy]a+[d(y),yx]a = 0$
for all $x\in R$.
Replacing y by x and using (2.5),we obtain
$6ax=0$ for all $x\in R$. Since R is 6-torsion free prime ring ,we get
$a=0$. Hence ψ is free action on R.

Theorem 2.3
Let R be a 3-torsion free prime ring ,d be a left centralizer mapping of R and a mapping $\psi:R\rightarrow R$ defined by $\psi(x)=[d(x)^2,x]$ for all $x\in R$.Then ψ is free action on R.
Proof: Let $a\in D(\psi)$,then
$[d(x)^2,x]a= ax$ for all $x\in R$.
Linearzing x by x+y and complete our proof by same method of Theorem 2.1.

Theorem 2.4
Let R be a 2-torsion free prime ring ,d be a left centralizer mapping of R and a mapping $\psi:R\rightarrow R$ defined by $\psi(x)=[d(x)^2,x^2]$ for all $x\in R$.Then ψ is free action on R.
Proof: Let $a\in D(\psi)$,then
$[d(x)^2,x^2]a= ax$ for all $x\in R$. Replacing x by –x and using the result with our relation,we obtain
$2[d(x)^2,x^2]= 0$ for all $x\in R$. According to our relation and R is 2-torsion free, we get
$ax=0$ for all $x\in R$. Again since R is prime ring, then
$a=0$.Hence Ψ is a free action on R.

By the same method we can prove the following theorem.

Theorem 2.5
Let R be a 2-torsion free prime ring ,d be a left centralizer mapping of R and a mapping $\psi:R\rightarrow R$ defined by $\psi(x)=[d(x^2),x^2]$ for all $x\in R$.Then ψ is free action on R.

Corollary 2.6
Let R be a 2-torsion free prime ring , d is additive mapping and let a mapping $\psi:R\rightarrow R$ defined by $\psi(x)=[d(x),x]$ for all $x\in R$.Then ψ is free action on R.

Proof: Let $a\in D(\psi)$.Then

$[d(x),x]a=ax$ for all $x\in R$. (2.7)

Linearizing (2.7)and using the result with (2.7),we obtain

$[d(x),y]a+[d(y),x]a=0$ for all $x,y\in R$.

Replacing y by x with using R is 2-torsion free, we get

$[d(x),x]a=0$ for all $x\in R$ .According to (2.7),we obtain

$ax=0$ for all $x\in R$ .Since R is prime ring ,then

$a=0$.Then ψ is a free action.

Theorem 2.7

Let R be a 2-torsion free prime ring and ,d is a additive mapping and a mapping $\psi:R\to R$ defined by $\psi(x)=[d^2(x)^2,x^2]$ for all $x\in R$. Then ψ is free action on R.

Proof: We have $[d^2(x)^2,x^2]a=ax$ for all $x\in R$. (2.8)

Replacing x by –x and comparing the result with (2.8),we obtain

$2[d^2(x)^2,x^2]a=0$ for all $x\in R$.

Since R is 2-torsion free, we get

$[d^2(x)^2,x^2]a=0$ for all $x\in R$. (2.9)

Substituting (2.9) in (2.8),we obtain

$ax=0$ for all $x\in R$. Since R is prime ring ,then

$a=0$.Then ψ is a free action.

By the same method in Theorem 2.7,we can prove the following

Theorem 2.8

Let R be a 2-torsion free prime ring ,d is additive mapping and a mapping $\psi:R\to R$ defined by $\psi(x)=[d^3(x),x]$ for all $x\in R$. Then ψ is free action on R.

Remark2.9

In the preceding theorems, we cannot exclude the condition "n-torsion free", and the following example demonstrates that

Example 2.10

Let R be a ring of all 2×2 matrices over a field F,R=$\{\begin{pmatrix} na & 0 \\ 0 & 0 \end{pmatrix} / a\in F\}$ and

$q=\begin{pmatrix} nh & 0 \\ 0 & 0 \end{pmatrix}$,where n is a positive integer. Let d be an additive map induced by w

,that is, $d(x)=[w,x]$ for all $x\in R$, $d(x)=\begin{pmatrix}1&0\\0&0\end{pmatrix}x-x\begin{pmatrix}1&0\\0&0\end{pmatrix}$,where $w=\begin{pmatrix}1&0\\0&0\end{pmatrix}$ then d is left centralizer of R .It is readily verified that d satisfies the conditions in preceding theorems where R is n-torsion free i.e. , $q=\begin{pmatrix}0&0\\0&0\end{pmatrix}$ and the mapping ψ is free action on R.

Theorem 2.11

Let R be a semiprime ring,T be a left centralizer and d a non-zero derivation on R ,then a mapping $\psi:R\to R$ defined by $\psi=Tod$ is free action on R.

Proof: Let $a\in D(\psi)$.Then

$\Psi(x)a=ax$ for all $x\in R$. (2.10)

$(Tod)(x)a=ax$ for all $x\in R$.Then

$T(d(x)a)=ax$ for all $x\in R$.

$T(a)d(x)=ax$ (2.11)

Replacing x by xy in (2.11),we obtain

$T(a)d(xy)=axy$ for all $x,y\in R$.Then

$T(a)d(x)y+T(a)xd(y)=axy$ for all $x,y\in R$. (2.12)

According to (2.11) above equation reduces to

$T(a)xd(y)=0$ for all $x,y\in R$. (2.13)

Replacing x by $d(y)rT(a)$ in (2.13),we obtain

$T(a)d(y)rT(a)d(y)=0$ for all $x,y,r\in R$. Then by using the semiprimeness, we obtain

$T(a)d(y)=0$ for all $y\in R$. (2.14)

Substituting (2.14) and (2.13) in (2.12),we get

$axy=0$ for all $x,y\in R$.

Replacing y by a with using the semiprimeness, we obtain

$a=0$. Then ψ is a free action.

The proof of theorem is completes.

Theorem 2.12

Let R be a prime ring, $D:R\to R$ be a generalized derivation and $a\in R$ be dependent element of D then either $a\in Z(R)$ or $D(x)=x$ for all $x\in R$.

Proof: From our hypothesis, we get

$D(x)a=ax$ for all $x\in R$. (2.15)

Replacing x by xy,we obtain

$(D(x)y+xd(y))a=axy$ for all $x,y\in R$. (2.16)

By using the fact that D can be written in the form D=T+d, where T is the left centralizer, then according this we can in (2.16) replacing d(y)a by D(y)a -T(y)a, with using (2.15),we get

$D(x)ya+xD(y)a-xT(y)a=axy$ for all $x,y\in R$. Then

$D(x)ya+[a,x]y-xT(y)a=0$ for all $x,y\in R$. (2.17)

Replacing y by yD(x) in above equation with using (2.15) ,we get

$D(x)yax+ [a,x]yD(x)-xT(y)ax=0$ for all $x,y\in R$. (2.18)

Right-multiplying (2.17) by x, we obtain

$D(x)yax+[a,x]yx-xT(y)ax=0$ for all $x,y\in R$. (2.19)

Subtracting (2.18) and (2.19),we get

$[a,x]y(D(x)-x)=0$ for all $x,y\in R$. Then

$[a,x]R(D(x)-x)=0$.Thus by the primeness of R, we obtain either $[a,x]=0$ for all $x,y\in R$,which gives $a\in Z(R)$ or $D(x)=x$ for all $x\in R$.

The proof of theorem is completes.

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حول الحلقات الأولية طليقة الفعالية

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الملخص:في هذا البحث سوف ندرس ونتحرى بخصوص التطبيقات طليقة الفعالية على الحلقات الأولية وشبه الأولية R باستخدام بعض المفاهيم ،عندما الحلقة الأولية R تسمح بتحقيق بعض الشروط ،سوف نعطي بعض النتائج حول ذلك.