# *On Credibility (sub, super) Martingale*

BY

Noori Farhan AL-Mayahi and Firas Hussean Maghool

AL- Qadisiya University

College of Computer sciences and Mathematic

Department of mathematics.

nafm2005@yahoo.com & h.mfiras@yahoo.com

**Abstract:** In this paper introduced a Credibility (sub, super) Martingale and we define $\theta$-Brownian motion and studied some properties on Credibility martingale.

**Keyword:** Credibility martingale, Credibility measure, $\theta$-Brownian motion , convex function.

## 1-Preliminaries

**Definition(1.1)[1]**:A family $F$ of subsets of a set $\Omega$ is called a $\sigma$-field on a set $\Omega$, if

(1) $\Omega \in F$

(2) If $A \in F$, then $A^c \in F$

(3) If $A_n \in F, n=1,2,...$ then $\bigcup_{n=1}^{\infty} A_n \in F$

A measurable space is a pair $(\Omega, F)$,where $\Omega$ is a set and $F$ is $\sigma$-field on $\Omega$

A subset $A$ of $\Omega$ is called measurable (measurable with respect to the $\sigma$-field $F$) , if $A \in F$ i.e. any member of $F$ is called a measurable set.
It is clear to show that

(1) $\phi \in F$. for $\Omega \in F$ and $\phi = \Omega^c \in F$

(2) If $A_1,...,A_n \in F$, then $\bigcup_{i=1}^{n} A_i \in F$ and $\bigcap_{i=1}^{n} A_n \in F$ .

**Definition(1.2)[1]:**Let $G$ be a collection of subsets of a set $\Omega$.The smallest $\sigma$-field $F$ containing $G$ is called the $\sigma$-field generated by G, and it is denoted by $\sigma(G)$ ,i.e. $F=\sigma(G)$.

Let $(\Omega, \tau)$ be a topological space . the $\sigma$-field generated by $\tau$ is called the Borel $\sigma$-field and it is denoted by $\beta(\Omega)$ ,i.e. $\beta(\Omega)=\sigma(\tau)$, the member of $\beta(\Omega)$ are called Borel sets of $\Omega$ .

**Definition(1.3)** : Let$(\Omega, F)$ be a measurable space, a set function $\theta : F \to [0,1]$ is said to be Credibility measure if it is satisfies the following axioms:

(1) $\theta(\Omega)=1$ (Normality )

(2) $\theta(A)+\theta(A^c)=1$ for all $A \in F$ (self – Duality).

(3) For every sequence $\{A_n\}$ in $F$, we have $\theta(\bigcup_{n=1}^{\infty} A_n) = \sup\{\theta(A_n) : n=1,2,...\}$

if $\theta(A_n) \leq 0.5$ (Maximality).

A Credibility space is a tripe $(\Omega, F, \theta)$ where $\Omega$ is a set, $F$ is a $\sigma$- field, $\theta$ Credibility measure on $F$.

**Definition(1.4)[3]:** Let $(\Omega, F, \theta)$ be a Credibility space, a function $X : \Omega \to R$ is called a Credibility variable if $X$ is measurable function( $X^{-1}(A) \in F \quad \forall A \in \beta(R)$ ).

**Definition(1.5)[3]:** The Credibility variable $X_1, X_2,..., X_m$ are said to be independent if $\theta(\bigcap_{i=1}^{m}(x_i \in \beta_i)) = \min_{1\le i\le m}\theta(x_i \in \beta_i)$ for any Borel sets $\beta_1, \beta_2,..., \beta_m$ of real numbers.

**Definition(1.6):** Let $X$ be a Credibility variable on $(\Omega, F, \theta)$. Then the expected value of $X$ defined by

$$E(X) = \int_{\Omega} X d\theta = \min_{a\in A\subset\Omega}\{X(a), \theta(A)\}$$

and the variance defined by $E(X - E(X))^2$.

**Remark:** we say $X$ is integrable if $E(X) < \infty$.

**Definition(1.7)[1]:** Let $A \subseteq \Omega$ . A function $l_A : \Omega \to R$ defined by

$$l_A(w) = \begin{cases} 1 & ,w \in A \\ 0 & ,w \notin A \end{cases}$$

Is called indicator function or (characteristic function ) of $A$.

**Definition(1.8)[2][4]:** Let $T$ be index set and let $(\Omega, F, \theta)$ be a Credibility space, a Credibility stochastic process is a function from $T \times (\Omega, F, \theta)$ to the set of real number. That is Credibility stochastic process $X_t(w)$ is a function of two variables such that the function $X_{t^*}(w)$ is a Credibility variable for each $t^*$

When $T$ is a countable set the Credibility stochastic process is said to be a discrete-time process, if $T$ is interval of the real line, the Credibility stochastic process is said to be continuous-time process. For instance $\{X_n, n = 0,1,...\}$ is a discrete-time Credibility process indexed by the non negative integers, while $\{X_t, t \ge 0\}$ is a continuous-time process indexed by the non negative real numbers. Thus, a Credibility stochastic process is a family of Credibility variables that describes the evolution through time of some process.

**Definition(1.9 )[2]:** A filtration $\{F_t\}_{t\in T}$ is a sequence of sub $\sigma-$field of $F$ such that for all $S < t, F_s \subseteq F_t$.

**Definition(1.10):**A Credibility stochastic process $X = \{X_t\}_{t \in T}$ is adapted to the filtration $\{F_t\}_{t \in T}$ if for $t \in T$, $X_t$ is $F_t$ –measurable .

**Definition(1.11)[1]:**Let $(\Omega, F, \theta)$ be a Credibility space .A condition is said to be hold almost every where with respect the Credibility measure $\theta$ ,written a.e. $[\theta]$(or simply a.e. if $\theta$ is understood),if there is a set $B \in F$ of $\theta$ –measure 0,(i.e. $\theta(B)$=0)such that the condition holds outside of $B$.

**Definition(1.12 ):**A Credibility stochastic process $X = (X_t\}_{t \in T}$ is a said to be a sub martingale (respectively a super martingale) with respect to the filtration $\{F_t\}_{t \in T}$ if

(1) $X$ is adapted to the filtration $\{F_t\}_{t \in T}$ .

(2) $E(|X_t|) < \infty$ for $t \in T$ i.e. ( $X_t$ is integrable $t \in T$ )

(3) $E(X_t \mid F_s) \geq X_s$ a.e (resp, $E(X_t \mid F_s) \leq X_s$ a.e ) $t \in T$ ,$t > s$.

A Credibility stochastic process $X = (X_t\}_{t \in T}$ is martingale iff it is both a sub-and super martingale with respect to the filtration $\{F_t\}_{t \in T}$ .

**Definition(1.13):**A Credibility stochastic process $X_t$ is said to have independent increment if $X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_k} - X_{t_{k-1}}$ are independent Credibility variable for any time $t_0 < t_1 < \ldots < t_n$.

**Definition (1.14):** A Credibility stochastic process $X$ is called normally distributed if it has function is defined as $\mu(x) = 2\left(1 + \exp\left(\frac{|x-e|}{\sqrt{6}\sigma}\right)\right)^{-1}$, where $x \in R, \sigma > 0$ denoted by $N(e, \sigma^2)$.

**Definition(1.15):** A Credibility stochastic process $B_t$ is said to be $\theta$ –Brownian motion (also called wiener process) if

(1) $B_0 = 0$.

(2) $E(B_t) = 0 \forall t > 0$.

(3) $B_t$ has stationary and independent increment .

(4) every increment $B_{t+s} - B_s$ is a normally distributed Credibility variable with expected value $e_t$ and variance $\sigma^2 t^2$ .

**Definition(1.16):**The standard $\theta-$Brownian motion is Credibility stochastic process $(B_t)_{t\in T}$ such that

(1) $B_0 = 0$.

(2) For any finit sequence of time $t_0 < t_1 < ... < t_n$ the increment

$B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, ....B_{t_n} - B_{t_{n-1}}$ are independent

(3) For any time 0 < s < t , $B_t - B_s$ is normally distributed with $E(B_t - B_s) = 0$ and $\mathrm{var}(B_t - B_s) \le t - s$ .

**Remark**: from define (1.15) and (1.16) have the flowing properties

$E(B_t^2) = t$ , $E(B_t - B_s) = 0$ , $E(B_t - B_s)^2 \le t - s$ , $\forall t > s$ .

**2-Main result**

**Definition(2.1)[1]:**A function $g: R \to R$ I s convex function if

$g(\lambda x + (1-\lambda)y) \le \lambda g(X) + (1-\lambda)g(y)$ , $x, y \in \Re$ and $0 \le \lambda \le 1$

an important property of convex function is that they are always continuous .

We need the following fact a bout convex function for the proof of the next property . Let $x_1 < x_2$ and $y \in R$ then

$$\frac{g(x_2) - g(y)}{x_2 - y} \ge \frac{g(x_1) - g(y)}{x_1 - y} \quad ... \text{ (i)}$$

Now assume that $x_1 < y < x_2$ and let $x_2$ converge to $y$ from a bove . the left side of (i) is bounded below ,and its value decreases as $x_2$ decreases to $y$ .therefore the right derivative $g^+$ exists at $y$ and

$$-\infty < g^+(y) = \lim_{x_2 \to y^+} \frac{g(x_2) - g(y)}{x_2 - y} < +\infty$$

Moreover $g(x) \geq g(y) + g^{+}(y)(x-y) \quad \forall x \in R$

Put $-\infty < g^{+}(y) = K < +\infty$ then

$g(x) - g(y) \geq K(x-y) \quad ...$ (ii)

**Theorem(2.2) :**Let $g: R \to R$ be a convex function and let $X$ and g($X$) be integrable Credibility variable on $(\Omega, F, \theta)$ for any sub $\sigma-$field $G$ of $F$ then $E(g(X)) = g(E(X))$ where $g(X) = g \circ X$ .

**Proof:**

$$g(E(X)) = g(\int_{\Omega} X d\theta) = g(\min_{a \in A \subset \Omega}\{X(a), \theta(A)\}) \geq \min_{a \in A \subset \Omega}\{g(X(a)), \theta(A)\} = E(g(X))$$

Then $g(E(X)) \geq E(g(X))$ . . .(1).

and since $g$ is convex function , for (ii) and Replacing x with $X$ and $y$ with $E(X)$, obtain $g(X) - g(E(X)) \geq K(X - E(X))$

Taking the expected value on both sides

$E(g(X) - g(E(X)) \geq K(E(X) - E(X)) = 0$

$\therefore E(g(X)) \geq g(E(X))$ . . . (2)

Then from (1) and (2) we have $E(g(X)) = g(E(X))$ .

**Theorem(2.3):**Let $X$ be $G$- measurable Credibility variable and $g$ be convex function suppose $X$ and g($X$) be integrable and sub $\sigma-$field $G \subset F$ then $E(g(X \mid G)) = g(E(X \mid G))$.

**Proof:** since $g$ is convex function then g is continuous

$\Rightarrow g(X)$ is measurable then $E(g(X \mid G) = g(X)$

And since $X$ is $G$- measurable then $g(E(X \mid G) = g(X)$ then

$E(g(X \mid G)) = g(E(X \mid G))$ .

**Theorem (2.4)**:Let $X = \{X_t, F_t\}_{t \in T}$ be a martingale (resp -sub martingle ) and $g: R \to R$ a convex such that $g(X_t)$ is integrable for all t , then $\{g(X_t), F_t\}_{t \in T}$ is a martingale.

**Proof**: By theorem (2.3) we have $E(g(X_t)/F_s) = g(E(X_t/F_s))$ , $t > s$

Since $X$ is martingale (resp-sub martingale ) then

$E(g(X_t) \mid F_s) = g(E(X_t \mid F_s)) = g(X_s)$.

**Example (2.5)**: If $\{X_t, F_t,\}_{t \in T}$ is martingale and $|X_t|^r, r \geq 1$ is integrable for all $t$ , then $\{|X_t|^r, F_t,\}_{t \in T}$ is a martingale .

**Solution:**

Since $|X_t|^r, r \geq 1$ is integrable then $E(|X_t|^r) < \infty$

will show that $|X_t|^r$ is convex function

$$g(\lambda x + (1-\lambda)g(y)) = |\lambda x + (1-\lambda)y|^r \qquad x, y \in R\,, \quad 0 \leq \lambda \leq 1$$

$$\leq |\lambda x|^r + |(1-\lambda)y|^r \leq \lambda^r |x|^r + (1-\lambda)^r |y|^r$$

$$\leq \lambda\ |x|^r + (1-\lambda)\ |y|^r \leq \lambda g(x) + (1-\lambda) g(y)$$

Then $|X_t|^r$ is convex function and from theorem (2.4 ) have

$\{|X_t|^r, F_t,\}_{t \in T}$ is a martingale .

**Theorem(2.6)** A $\theta-$Brownian motion is a super martingale .

**Proof :**

(1) $E(|B_t|)_{t \geq 0} < \infty$ .

(2) $\{B_t, F_t\}$ is adapted to the filtration on $\{F_t\}_{t \geq 0}$.

(3) $E(B_t \mid F_s) = E((B_t - B_s) + B_s \mid F_s)$ , $t > s$

$\leq E(B_t - B_s \mid F_s) + E(B_s \mid F_s)$ and $B_t - B_S$ is independent of $F_s$

$\Rightarrow E(B_t - B_s \mid F_s) \leq E(B_t - B_s) = 0 \quad \Rightarrow E(B_t \mid F_s) \leq B_s$

$\therefore B_t$ is a super martingale .

**Theorem(2.7):** If $B_t$ is super martingale with respect to the filtration $F_t$ then $\{\exp(B_t), F_t, t \geq 0\}$ is super martingale .

**Proof:** since $\exp(B_t)$ is convex function and integrable on $(\Omega, F, \theta)$ then By theorem ( 2.4) we have

$E(\exp(B_t) \mid F_s) = \exp(E(B_t) \mid F_s) \le \exp(B_s) \quad , t > s$

$\therefore \exp(B_t)$ is a sub martingale .

**Example(2.8):** If $B_t$ is super martingale with respect to the filtration $F_t$ then $B_t^2$ is not super martingale .

**Solve :**

$E(B_t^2 \mid F_s) = E((B_t - B_s + B_s)^2 \mid F_s)$

$\le E((B_t - B_s)^2 \mid F_s) + E(B_t^2 \mid F_s) + 2E((B_t - B_s)B_s \mid F_s)$

By using independent $E((B_t - B_s)^2 \mid F_s) \le E(B_t - B_s)^2 = t - s$ and $E(B_s^2 \mid F_s) \le B_s^2$,

$E((B_t - B_s)B_s \mid F_s) = B_s E(B_t - B_s \mid F_s) \le B_s E(B_t - B_s) = 0$

thus $E(B_t^2 \mid F_s) \le B_s^2 + t - s$

**Example(2.9):** If $B_t$ is super martingale then $B_t^2 - t$ is super martingale.

**Solve:** $E(B_t^2 - t \mid F_s) = E((B_t - B_s + B_s)^2 - t) \mid F_s), \quad t > s$

$E((B_t - B_s)^2 + 2B_t B_s - B_s^2 - t \mid F_s) \le E((B_t - B_s)^2 \mid F_s) + 2E(B_t B_s \mid F_s) - B_s^2 - t$

since $E((B_t - B_s)^2 \mid F_s) = E(B_t - B_s)^2 = t - s$ and $E(B_t \mid F_s) \le B_s$ then we have

$E(B_t^2 - t \mid F_s) \le t - s + B_s^2 - t = B_s^2 - s$

**Theorem (2.10):** If $X = \{B_t, F_t\}_{t \ge 0}, Y = \{W_t, F_t\}_{t \ge 0}$ are two $\theta -$ Brownian motion and $\lambda_1, \lambda_2$ are positive constants then

$\lambda_1 X + \lambda_2 Y = \{\lambda_1 B_t + \lambda_2 W_t, Ft\}_{t \ge 0}$ is a super martingale.

**Proof :** since $|\lambda_1 B_t + \lambda_2 W_t| \le \lambda_1 |B_t| + \lambda_2 |W_t|$

$E(|\lambda_1 B_t + \lambda_2 W_t|) \le \lambda_1 E(|B_t|) + \lambda_2 E(|W_t|) < \infty$

$E(\lambda_1 B_t + \lambda_2 W_t \mid F_s) \le \lambda_1 E(B_t \mid F_s) + \lambda_2 E(W_t \mid F_s)$

$\le \lambda_1 B_s + \lambda_2 W_s$

$\therefore \lambda_1 X + \lambda_2 Y = \{\lambda_1 B_t + \lambda_2 W_t, F_t\}_{t \geq 0}$ is a super martingale .

# ***Reference***


**[1]** R.B .Ash, "Real Analysis and probability", university of lllinois, Academic press(1972).

**[2]** Steven E. Shreve "Brownian Motion and stochastic Calculus"2 nd,N-389-97655-8 Springer-Verlag New York ,(1997).

**[3]** B. Liu," Foundation of Uncertainty Theory", Beijing 100084, China , UTLAB,(2006).

**[4]** Xiaowei Chen," American Option Pricing Formula for Uncertain Financial Market", National Natural Science Foundation of China Grant No.60874067(2011).