Properties of The Function ϕ and The Composition Operator C_ϕ Induced by The Function ϕ

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Abstract

Let U denote the unit ball in the complex plane, the Hardy space H^2 is the set of functions $f(z)=\sum_{n=0}^{\infty} f^{\wedge}(n)z^n$ holomorphic on U such that $\sum_{n=0}^{\infty}\left|f^{\wedge}(n)\right|^2<\infty$ with $f^{\wedge}(n)$ denotes then the Taylor coefficient of f.

Let ψ be a holomorphic self-map of U, the composition operator C_ψ induced by ψ is defined on H^2 by the equation

$$C_\psi f = f \circ \psi \quad (f \in H^2)$$

In this paper we have studied the composition operator induced by the automorphism ϕ and discussed the adjoint of the composition of the symbol ϕ .We look also for some known properties on composition operators and tried to see the analogue properties in order to show how the results are changed by changing the function ψ in U.

In order to make the work accessible to the reader, we have included some known results with the details of the proofs for some cases and proofs for the properties .

Introduction

This paper consists of two sections . In section one ,we are going to study the automorphism ϕ and properties of ϕ ,and also discuss the interior and the exterior fixed points of ϕ and also discuss ϕ is a rotation around the origin and ϕ is elliptic and ϕ is a linear fractional transformation .

In section two, we are going to study the composition operator C_ϕ induced by the symbol ϕ and properties of C_ϕ , and also discuss the adjoint of composition operator C_ϕ induced by the symbol ϕ and also discuss C_ϕ is an invertible operator and C_ϕ is a unitary operator and define the eigenvalue of C_ϕ

Section One

The properties of the Function ϕ

we are going to study the automorphism ϕ and properties of ϕ ,and also discuss the interior and the exterior fixed points of ϕ and also discuss ϕ is a rotation around the origin and ϕ is elliptic and ϕ is a linear fractional transformation .

Definition(1.1) : [4]

Let U = $\{z \in C : |z| < 1\}$ which is called unit ball in the complex plane C and $\partial U = \{z \in C : |z| = 1\}$ is called the boundary of U.

Definition (1.2):

For $\beta \in U$, define $\phi(z) = \dfrac{-3z}{3 - 6\bar{\beta}z}$ $(z \in U)$. Since the denominator equal zero only at $z = \dfrac{1}{2\bar{\beta}}$, the function ϕ is holomorphic on the ball $\{|z| < \dfrac{1}{2|\beta|}\}$. Since $\beta \in U$, then this ball contains U. Hence ϕ take U into U and holomorphic on U .

Definition(1.3) : [10]

Let $\psi : U \to U$ be holomorphic map on U. We say that ψ is a conformal automorphism or automorphism of U if and only if ψ is bijective .

Remark (1.4) :

For $\beta \in U$, ϕ is conformal automorphism or automorphism of U .

Definition(1.5) : [10]

A point p $\in$ C is a fixed point for the function ψ , if $\psi(p) = p$.

Proposition (1.6) :

For $\beta \in U$, then 0 , $\dfrac{1}{\bar{\beta}}$ are fixed points for ϕ .

Proof :

Let $\phi(z) = z$ that is $\dfrac{-3z}{3 - 6\bar{\beta}z} = z$, therefore $6\bar{\beta}\, z^2 - 6z = 0$. Hence ϕ has two fixed points

$z_1 = 0$, $z_2 = \dfrac{1}{\bar{\beta}}$

Definition(1.7): [4]

Let $\psi : U \to U$ be holomorphic map on U with the fixed point r, then:

1) r is interior fixed point for ψ if $r \in U$
2) r is exterior fixed point for ψ if $r \notin U$

Proposition (1.8):

If $\phi(z)=\frac{-3z}{3-6\bar{\beta}z}$, then 0 is interior fixed point and $\frac{1}{\bar{\beta}}$ is exterior fixed point for ϕ.

Proof :

Since ϕ has two fixed points $z_1=0$, $z_2=\frac{1}{\bar{\beta}}$, $|z_1|=|0|=0<1$ .Thus z_1 is interior fixed point .

Since $\beta\in U$, then $|\beta|<1$ and $|\bar{\beta}|=|\beta|<1$, therefore $\left|\frac{1}{\bar{\beta}}\right|=\left|\frac{1}{\beta}\right|=\frac{1}{|\beta|}>1$, hence $|z_2|=\left|\frac{1}{\bar{\beta}}\right|>1$. Thus z_2 is exterior fixed point .

Remark(1.9) :

1- For $\beta\in U$, $\phi^{-1}(z)=\frac{-3z}{3-6\bar{\beta}z}=\phi(z)$

2- For $\beta\in U$, then $\phi'(0)=-1$, $\phi'(\beta)=\frac{-9}{\left(3-6|\beta|^2\right)^2}$.

Definition(1.10) : [11]

Let $\psi:U\to U$ be holomorphic map on U. We say that ψ is a rotation around the origin if there exists $\lambda\in\partial U$ such that $\psi(z)=\lambda z$ $(z\in U)$

Proposition (1.11):

If $\beta=0$, $\phi(z)$ is a rotation a round the origin

Proof:

Since $\phi(z)=\frac{-3z}{3-6\bar{\beta}z}$, since $\beta=0$, hence $\phi(z)=-z=\lambda z$,$\lambda=-1\in\partial U$, then $\phi(z)$ is a rotation around the origin .

Theorem (1.12) : [11]

Let $\psi:U\to U$ be holomorphic map on U, then ψ is elliptic if and only if ψ is automorphism that has an interior fixed point.

Proposition (1.13) :

For $\beta\in U$, ϕ is elliptic

Proof :

From (1.4) , ϕ is automorphism , and from (1.8) ϕ has an interior fixed point , hence ϕ is elliptic .

Definition(1.14): [10]

A linear fractional transformation is a mapping of the form $\tau(z)=\frac{az+b}{cz+d}$, where a , b , c , and d are complex numbers, and sometime is denoted it by $\tau_A(z)$ where A is the non-singular 2×2 complex matrix $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Proposition (1.15) :

ϕ is a linear fractional transformation .

Proof :

Since $\phi(z) = \dfrac{-3z}{3-6\bar{\beta}z} = \dfrac{az+b}{cz+d}$ such that a = -3 , b = 0 , c = $-6\bar{\beta}$, d = 3 and a , b , c , and d are complex numbers and $A = \begin{bmatrix} -3 & 0 \\ -6\bar{\beta} & 3 \end{bmatrix}$, hence by (1.14) ϕ is a linear fractional transformation .

Section Two

The Composition Operator C_ϕ Induced by The Function ϕ

we are going to study the composition operator C_ϕ induced by the symbol ϕ and properties of C_ϕ, and also discuss the adjoint of composition operator C_ϕ induced by the symbol ϕ and also discuss C_ϕ is an invertible operator and C_ϕ is a unitary operator and define the eigenvalue of C_ϕ

Definition(2.1): [4]

Let U denote the unit ball in the complex plane, the Hardy space H^2 is the set of functions $f(z) = \sum_{n=0}^{\infty} f\char`^(n)\ z^n$ holomorphic on U, such that $\sum_{n=0}^{\infty} |f\char`^(n)|^2 < \infty$ with $f\char`^(n)$ denotes then the Taylor coefficient of f .

Remark (2.2) : [1]

We can define an inner product of the Hardy space functions as follows:

f(z) = $\sum_{n=0}^{\infty} f\char`^(n)\ z^n$ and $g(z) = \sum_{n=0}^{\infty} g\char`^(n)\ z^n$, then the inner product of f and g is:

$$\langle f, g\rangle = \sum_{n=0}^{\infty} f\char`^(n)\ \overline{g\char`^(z)}$$

Definition (2.3) :[10]

Let $\alpha \in U$ and define $k_\alpha(z) = \dfrac{1}{1-\bar{\alpha}z}$ $(z \in U)$. Since $\alpha \in U$ then $|\alpha| < 1$, hence the geometric series $\sum_{n=0}^{\infty} |\alpha|^{2n}$ is convergent and thus $k_\alpha \in H^2$ and $k_\alpha(z) = \bar{\alpha}\ z^n$.

Definition(2.4) : [4]

Let $\psi : U \to U$ be holomorphic map on U, the composition operator C_ψ induced by ψ is defined on H^2 as follows $C_\psi\ f = f \circ \psi \quad (f \in H^2)$

Definition(2.5) : [2]

Let T be any bounded operator on a Hilbert space H, then the norm of an operator T is defined by $\|T\| = \sup\{\|Tf\| : f \in H, \|f\| = 1\}$.

Littlewood's Subordination Principle (2.6) : [11]

Let $\psi : U \to U$ be holomorphic map on U with $\psi(0) = 0$, then for each $f \in H^2$, $f \circ \psi \in H^2$ and $\|f \circ \psi\| \le \|f\|$.

The goal of this theorem above $C_\psi : H^2 \to H^2$.

Proposition(2.7) :

If $\phi(z) = \dfrac{-3z}{3 - 6\bar{\beta} z}$, then for each $f \in H^2$, $f \circ \phi \in H^2$ and $\|f \circ \phi\| \le \|f\|$

Proof :

Since $\phi : U \to U$ be holomorphic map on U with $\phi(0) = 0$ by (1.6) , then by (2.6) $f \in H^2$, $f \circ \phi \in H^2$ and $\|f \circ \phi\| \le \|f\|$, hence $C_\phi : H^2 \to H^2$

Remark (2.8) : [4]

1) One can easily show that $C_\kappa C_\psi = C_{\psi \circ \kappa}$ and hence $C_\psi^n = C_\psi C_\psi \cdots C_\psi$ $= C_{\psi \circ \psi \circ \cdots \circ \psi} = C_{\psi_n}$

2) C_ψ is the identity operator on H^2 if and only if ψ is identity map from U into U and holomorphic on U.

3) It is simple to prove that $C_\kappa = C_\psi$ if and only if $\kappa = \psi$.

Theorem (2. 9) : [11]

Let $\psi : U \to U$ be holomorphic map on U . C_ψ is an invertible operator on H^2 if and only if ψ automorphism of U and $C_\psi^{-1} = C_{\psi^{-1}}$

Proposition(2.10) :

C_ϕ is an invertible operator on H^2

Proof :

Since ϕ is automorphism of U by (1.4), hence C_ϕ is an invertible operator on H^2.

Theorem (2.11) : [5]

$V_{\alpha \in U} \{K_\alpha\}$ forms a dense subset of H^2.

Definition(2.12): [12]

Let H^∞ be the set of all bounded holomorphic function on U .

Theorem (2.13) : [10]

Let $\psi : U \to U$ be holomorphic map on U, then for all $\alpha \in U$

$C_\psi^* K_\alpha = K_{\psi(\alpha)}$

Definition(2.14): [6]

Let $g \in H^\infty$, the Toeplits operator T_g is an operator on H^2 given by :

$(T_g f)(z) = g(z)\ f(z)\quad (f \in H^2,\ z \in U)$

Remark (2.15) : [7]

For each $f \in H^2$, it is well- know that $T_h^* f = T_{\bar{h}} f$, such that $h \in H^\infty$.

Proposition(2.16) :

Let $\beta \in U$, then $C_\phi^* = T_g\ C_\gamma\ T_h^*$ where $h(z) = 1 - 2\bar{\beta}z$, $g(z) = 1$, $\gamma(z) = \dfrac{6\beta - 3z}{3}$

Proof :

By (2.15), $T_h^* f = T_{\bar{h}} f$ for each $f \in H^2$. Hence for all $\alpha \in U$,

$$\left\langle T_h^*\ f, k_\alpha \right\rangle = \left\langle T_{\bar{h}} f, k_\alpha \right\rangle = \left\langle f, T_{\bar{h}}^*\ k_\alpha \right\rangle \cdots\cdots (2-1)$$

On the other hand ,

$$\left\langle T_h^*\ f,\ k_\alpha \right\rangle = \left\langle f, T_h k_\alpha \right\rangle = \left\langle f, h(\alpha) k_\alpha \right\rangle \cdots\cdots (2-2)$$

From (2-1)and (2-2) one can see that $T_{\bar{h}}^*\ k_\alpha = h(\alpha)\ k_\alpha$. Hence $T_h^*\ k_\alpha = \overline{h(\alpha)}\ k_\alpha$.

Calculation give :

$$C_\phi^*\ k_\alpha(z) = k_{\phi(\alpha)}(z)$$

$$= \frac{1}{1 - \overline{\phi_\beta(\alpha)}\ z} = \frac{1}{1 + \dfrac{3\bar{\alpha}z}{3 - 6\beta\bar{\alpha}}}$$

$$= \frac{1}{\dfrac{3 - 6\beta\bar{\alpha} + 3\bar{\alpha}z}{3 - 6\beta\bar{\alpha}}} = \frac{3 - 6\beta\bar{\alpha}}{3 - \bar{\alpha}(6\beta - 3z)} = \frac{\left(1 - 2\alpha\bar{\beta}\right)}{1 - \bar{\alpha}\left(\dfrac{6\beta - 3z}{3}\right)}$$

$$= \overline{\left(1 - 2\alpha\bar{\beta}\right)}\ \cdot\ (1)\ \cdot\ \frac{1}{1 - \bar{\alpha}\left(\dfrac{6\beta - 3z}{3}\right)}$$

$$= \overline{h(\alpha)}\ \cdot\ T_g\ K_\alpha(\gamma(z)) = T_g\ \overline{h(\alpha)}\ K_\alpha(\gamma(z))$$

$$= T_g\ \overline{h(\alpha)}\ C_\gamma\ K_\alpha(z) = T_g\ C_\gamma\ \overline{h(\alpha)}\ K_\alpha(z)$$

$$= T_g\ C_\gamma\ T_h^*\ K_\alpha(z) \text{ , therefore}$$

$$C_\phi^*\ k_\alpha(z) = T_g\ C_\gamma\ T_h^*\ k_\alpha(z).$$

But $\overline{V_{\alpha \in U}\ \{K_\alpha\}} = H^2$, then $C_\phi^* = T_g\ C_\gamma\ T_h^*$

Definition (2.17) : [3]

Let T be an operator on a Hilbert space H , T is called unitary operator if $T\ T^* = T^*\ T = I$, and T is called normal operator if $T\ T^* = T^*\ T$.

Theorem (2.18) : [9]

Let $\psi : U \to U$ be holomorphic on U, then C_ψ is normal operator if and only if $\psi(z) = \lambda\ z$ for some λ , $|\lambda| \le 1$

Theorem (2.19) :

Let $\psi : U \to U$ be holomorphic map on U, then C_ψ is unitary if and only if $\psi(z) = \lambda\ z$ for some λ , $|\lambda| = 1$

Proof :

Suppose C_ψ is unitary , hence by (2.17) $C_\psi\ C_\psi^* = C_\psi^*\ C_\psi = I$, hence $C_\psi\ C_\psi^* = C_\psi^*\ C_\psi$, hence C_ψ is normal operator, hence by (2.18) $\psi(z) = \lambda\ z$ for some λ , $|\lambda| \le 1$. It is enough to show that $|\lambda| = 1$

$C_\psi^*\ C_\psi\ K_\beta(z) = C_\psi^* K_\beta(\psi(z)) = K_{\psi(\beta)}(\psi(z)) = \dfrac{1}{1 - \overline{\psi(\beta)}\ \psi(z)} = \dfrac{1}{1 - \bar{\lambda}\ \bar{\beta}\ \lambda\ z} = \dfrac{1}{1 - |\lambda|^2\ \bar{\beta}\ z}$. On the other hand $C_\psi^*\ C_\psi\ K_\beta(z) = K_\beta(z)$, hence $\dfrac{1}{1 - |\lambda|^2\ \bar{\beta}\ z} = K_\beta(z) = \dfrac{1}{1 - \bar{\beta}\ z}$. Thus $|\lambda|^2\ \bar{\beta} = \bar{\beta}$,then $|\lambda| = 1$.

Conversely , Suppose is unitary $\psi(z) = \lambda\ z$ for some λ , $|\lambda| = 1$.For $\beta \in U$, for every $z \in U$

$$C_\psi^*\ C_\psi\ K_\beta(z) = C_\psi^* K_\beta(\psi(z)) = K_{\psi\ (\beta)}(\psi(z)) = \frac{1}{1 - \overline{\psi(\beta)}\ \psi(z)} = \frac{1}{1 - \bar{\lambda}\ \bar{\beta}\ \lambda\ z} = \frac{1}{1 - |\lambda|^2\ \bar{\beta}\ z} = \frac{1}{1 - \bar{\beta}\ z} = K_\beta(z).$$

Moreover , for every $z \in U$

$$C_\psi\ C_\psi^*\ K_\beta(z) = C_\psi\ K_{\psi(\beta)}\ (z) = K_{\psi\ (\beta)}(\psi(z)) = \frac{1}{1 - \overline{\psi(\beta)}\ \psi(z)} = \frac{1}{1 - \bar{\lambda}\ \bar{\beta}\ \lambda\ z} = \frac{1}{1 - |\lambda|^2\ \bar{\beta}\ z} = \frac{1}{1 - \bar{\beta}\ z} = K_\beta(z).$$

hence $C_\psi\ C_\psi^* = C_\psi^*\ C_\psi = I$ on the family $\{K_\alpha\}_{\alpha\ \in U}$. But by (2.11) $V_{\alpha\ \in U}\{K_\alpha\}$ forms a dense subset of H^2, hence $C_\psi\ C_\psi^* = C_\psi^*\ C_\psi = I$ on H^2. Therefore C_ψ is unitary composition operator on H^2.

Proposition(2.20) :

If $\beta = 0$, then C_ϕ is unitary composition operator .

Proof :

Since $\phi(z) = \dfrac{-3z}{3 - 6\bar{\beta}z}$, since $\beta = 0$, $\phi(z) = \dfrac{-3z}{3 - 6\bar{\beta}z} = -z = \lambda z$, $\lambda = -1$, $|\lambda| = 1$, hence by (2.19) C_ϕ is unitary composition operator .

Definition (2.21) : [12]

Let $\psi : U \to U$ be holomorphic map on U, the eigenvalue equation for the composition operator κ is define by $C_\psi f = \kappa\ f$ or $f \circ \psi = \kappa\ f$.

Theorem (2.22): [11]

Let $\psi : U \to U$ be holomorphic map on U, and that fixes the point $p \in U$ and suppose that $C_\psi f = \kappa f$ for some non-constant $f \in H^2$ and some $\kappa \in C$. Then $\kappa = (\psi'(p))^n$ for some n = 0, 1, 2, . . .

Proposition(2.23) :

$(-1)^n$ is an eigenvalue of C_ϕ for some n = 0 , 1 , 2 , …..

Proof :

Since $\phi(z)=\dfrac{-3z}{3-6\bar{\beta}z}, \phi'(z)=\dfrac{(3-6\bar{\beta}z)\,(-3)-(-3z)\,(-6\bar{\beta})}{(3-6\bar{\beta}z)^{2}}=\dfrac{-9}{(3-6\bar{\beta}z)^{2}}$, and since ϕ fixed the point $0\in U$, and by (2.22) $\kappa=(\phi'(0))^{n}=(-1)^{n}$ is an eigenvalue of C_{ϕ} for some n = 0 , 1 , 2,...

Definition (2.24) : [7]

The spectrum of an operator T on a Hilbert space H , denoted by $\sigma(T)$ is the set of all complex numbers λ for which $T-\lambda I$ is not invertible . The spectral radius of T, denoted by $r(T)$ is defined as $r(T)=\sup\{|\lambda|:\lambda\in\sigma(T)\}$.

Theorem (2.25) : [5]

If $\psi:U\to U$ be holomorphic map on U, and ψ has interior fixed point p then $r(C_{\psi})=1$.

Proposition (2.26) :

If $\phi(z)=\dfrac{-3z}{3-6\bar{\beta}z}$, then $r(C_{\phi})=1$

Proof :

Since $\phi:U\to U$ be holomorphic map on U , and ϕ has interior fixed point 0 . By (2.25) $r(C_{\phi})=1$.

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خواص الدالة ϕ و المؤثر التركيبي C_{ϕ} المتولد بالدالة ϕ

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المستخلص

ليكن U يرمز إلى كرة الوحدة في المستوى العقدي، إن فضاء هاردي H^2 هو مجموعة كل الدوال $f(z)=\sum_{n=0}^{\infty} f^{\wedge}(n)\ z^n$ التحليلية على U بحيث أن $\sum_{n=0}^{\infty} |f^{\wedge}(n)|^2 < \infty$ ، $f^{\wedge}(n)$ يرمز إلى معاملات تيلر النونية.

لتكن $\psi : U \to U$ دالة تحليلية على U ، المؤثر التركيبي المحتث من ψ يعرف على فضاء هاردي H^2 بواسطة:

$$C_{\psi} f = f \circ \psi \quad (f \in H^2).$$

درسنا في هذا البحث المؤثر التركيبي المحتث من الدالة ϕ حيث ناقشنا المؤثر المرافق للمؤثر التركيبي المحتث من الدالة ϕ . بالإضافة إلى ذلك نظرنا إلى بعض النتائج المعروفة وحاولنا الحصول على نتائج مناظرة لنتمكن من ملاحظة كيفية تغير النتائج عندما تتغير الدالة التحليلية ψ .

ومن أجل جعل مهمة القارئ أكثر سهولة، عرضنا بعض النتائج المعروفة عن المؤثرات التركيبية وعرضنا براهين مفصلة وكذلك برهنا بعض النتائج.