Evaluating the Numerical values for the absolute error using Monte Carlo methods

حساب القيمة العددية للخطأ المطلق باستخدام طرائق مونت كارلو

Dr. Abdul – Kareem I. Sheet
Univ. of Mosul – Remote Sensing Center
Nadia Adeel Saeed / Univ. of Mosul - College of Science- Dep. Of Biophysics
E. mail: kareem59y@yahoo.com

Abstract

The aim of this research is to present a rapid survey of numerical methods for solving linear Fredholm integral equations of the second kind- that is, for equations of the form [6]:

$$\Phi(x) = f(x) + \lambda \int_a^b K(x,t)\Phi(t)dt$$

where the inhomogeneous term f, and the kernel K are given. In addition to use the Monte Carlo multidimensional integration method to estimate the norm [6], [7]:

$$\left\| \Phi - \tilde{\Phi} \right\| \le \| \Lambda \| .(1+\| R_K \|).(1+\| R_L \|).\| f \|$$

The efficiency of Monte Carlo simulation technique was obvious from the results obtained.

الملخص:

أن الهدف من هذا البحث هو التقديم لطريقة عددية سريعة ومختصرة لحل معادلات فردهولم التكاملية الخطية من النمط الثاني التي يعبر عنها بالصيغة الرياضية التالية:

$$\Phi(x) = f(x) + \lambda \int_a^b K(x,t)\Phi(t)dt$$

للمقدار التالي [6]: إضافة إلى استخدام طريقة تكامل مونت كارلو المتعدد الابعاد لحساب القيمة العددية

$$\left\| \Phi - \tilde{\Phi} \right\| \le \| \Lambda \| .(1+\| R_K \|).(1+\| R_L \|).\| f \|$$

لقد اتضحت كفاءة أسلوب محاكاة المونت كارلو من خلال النتائج التي تم الحصول عليها

1. Introduction

An integral equation is an equation in which the unknown function appears within an integral. It sometimes occurs that the same problem can be expressed both as an integral equation, and as a differential equation, but this is not always the case [3].

Integral equations form an important class of problems arising frequently in engineering, and mathematical and scientific analysis [4]. Section 2 a rapid survey of numerical methods for linear Fredholm integral equations, section 3 present a basic Monte Carlo simulation to approximate the solution of the linear Fredholm integral equations studied [7].

2. Approximate Methods of Solving Linear Integral Equations

There are number methods of finding approximate solutions for linear integral equations. In all cases a set of functions is defined in some way, and the actual solution is a member of this set [1]. The technique for solving Fredholm equations of the second kind can be classified into five broad categories:

1. Analytical and semi analytical methods [5].
2. Projection methods [5].
3. Quadrature methods [4].
4. Volterra and initial value methods [2].
5. Kernel approximation methods [6].
   5.1 Replacing the kernel by a degenerate kernel.
   5.2 The method of successive approximations.

In this section we present a rapid survey of kernel approximation method known as replacing the kernel by a degenerate kernel.

2.1 Replacing the kernel by a degenerate kernel

Given the following linear Fredholm integral equation of the second kind [6]:

$$\Phi(x) = f(x) + \lambda \int_a^b K(x,t)\Phi(t)dt \qquad (1)$$

where $\Phi(x)$ is the unknown function, f(x) and K(x, t) are known functions, x and t are real variables varying in the interval (a, b), and λ is a numerical factor. The function K(x, t) is called the kernel of the integral equation (1); it is assumed that the kernel K(x, t) is defined in the square $\Omega\{a \le x \le b,\ a \le t \le b\}$ in the (x, t) plane and is continuous in Ω. The simplicity of finding a solution to an equation with a degenerate kernel leads one to think of replacing K(x, t) approximately by a degenerate kernel L(x, t) and taking the solution $\tilde{\Phi}(x)$ of the new equation [6]:

$$\tilde{\Phi}(x) = f(x) + \int_a^b L(x,t)\tilde{\Phi}(t)dt \qquad (2)$$

as an approximation to the solution of (1), we can estimate the norm $\left\|\Phi - \tilde{\Phi}\right\|$ by using the formula:

$$\left\| \Phi - \tilde{\Phi} \right\| \le \|\Lambda\|.(1+\|R_K\|).(1+\|R_L\|).\|f\| \qquad (3)$$

where $\|\Lambda\|$, $\|R_K\|$ and $\|R_L\|$ are the norms of the operators with corresponding kernel.

$$\|R_K\| \le \frac{\|K\|}{1-|\lambda|.\|K\|} \qquad (4.a)$$

$$\|R_L\| \le \frac{\|L\|}{1-|\lambda|.\|L\|} \qquad (4.b)$$

where λ is a numerical factor, $\Lambda(x,t)$ has a small norm in some metric. The norm in formula (3) can be taken in any function space. And in the space of $C(0,1)$ of continuous functions on [0, 1] [6];

$$\|K\| = \max_{0\le x\le 1} \int_0^1 |K(x,t)|\,dt \qquad (5.a)$$

$$\|f\| = \max_{0\le x\le 1} |f(x)| \qquad (5.b)$$

While in the space of quadratically summable function over the basic square $\{a \le x,t \le b\}$

$$\|K\| \le \left(\int_a^b \int_a^b K^2(x,t)\,dx\,dt \right)^{1/2} \qquad (6.a)$$

$$\|f\| \le \left(\int_a^b f^2(x)\,dx \right)^{1/2} \qquad (6.b)$$

$$\|L\| \le \left(\int_a^b \int_a^b L^2(x,t)\,dx\,dt \right)^{1/2} \qquad (6.c)$$

3. Basic Monte Carlo Simulation

A Monte Carlo simulation is a generic numerical method using a computer utilizing pseudo-random numbers to simulate stochastic inputs to a process and generate large numbers of possible outcomes. It solves a problem by generating suitable random numbers and observing that fraction of the numbers obeying some property or properties (Sabelfeld, 2004). It is commonly used for simulations in many fields that require solutions for problems that are impractical or impossible to solve by traditional analytical or numerical methods (Yang, 2002). The Monte Carlo simulation is mainly used to predict the final consequence of a series of occurrences, each having its own probability. A major requirement in Monte Carlo simulation is that the mathematical system be described by probability density functions. Once the density functions are known, the Monte Carlo simulation can proceed by random sampling from these density functions for use within the simulation. Given that a particular simulation outcome is based randomness performing the same simulation again would, most likely result is a different outcome. Many simulations are performed and the desired result is taken as an average over the number of outcomes (Smaldone, 2001),(Sheet, 2008)

The main steps of Monte Carlo Simulation are (Stokes, 2004), (Sheet, 2008):

1) Model the inputs and process.
2) Draw a vector of random varieties.
3) Evaluate the function of interest.

Repeat the last two steps many times, aggregating the results

4. Monte Carlo Simulation for Solving Integral Equations

The Monte Carlo Simulation can be used to estimate $\left\|\Phi - \tilde{\Phi}\right\|$, consider the following linear Fredholm integral equation of the second kind [6]:

$$\Phi(x) = \sin(x) + \int_0^1 (1 - x\cos(x))\,\Phi(t)\,dt \qquad (7)$$

From equation (7) we find that

$f(x) = \sin(x)$,

$$K(x,t) = 1 - x\cos(xt) \quad = 1 - x + \frac{x^3 t^2}{2} - \frac{x^5 t^4}{24} + \cdots$$

Let us take the first three terms of the expansion of K(x, t) for the degenerate kernel L(x, t),

$$L(x,t) = 1 - x + \frac{x^3 t^2}{2}$$

The approximated solution of (7) by using replacing the kernel by a degenerate kernel is given by [6]:

$$\tilde{\Phi}(x) = 1.003(1-x) + 0.1674x^3 + \sin(x) \qquad (8)$$

The exact solution of the equation (7) $\Phi(x) \equiv 1$. In the metric space of L_2 - space we find that:

$$\|\Lambda\| \le \frac{1}{24} \left\{ \int_0^1 \int_0^1 x^{10} t^8 \,dx\,dt \right\}^{1/2} = 0.00418$$

$$\|K\| = 0.5968$$

$$\|L\| = 0.5976$$

$\| f \| = 0.5222$

$\| R_K \| = 1.480159$ and $\| R_L \| = 1.485089$

using formula (3) we find that $\left\| \Phi - \tilde{\Phi} \right\|$ = 0.013486.

Table 1 summarize the results of the Monte Carlo simulation of solving the integral equation (7) using four sequences of $n = 10, 10^2, 10^3, 10^4$ random numbers using the Monte Carlo Hit or Miss integration method to approximate the value of $\left\| \Phi - \tilde{\Phi} \right\|$ given by formula (3) and its related formulas [7].

n	$\|K\|$	$\|L\|$	$\|f\|$	$\|\Lambda\|$	$\|R_K\|$	$\|R_L\|$	$\left\|\Phi - \tilde{\Phi}\right\|$
10	0.54120	0.51035	0.55986	0.00248	1.179599	1.042275	0.00618
10^2	0.58488	0.58147	0.51630	0.00418	1.408942	1.389315	0.012422
10^3	0.58485	0.59531	0.52820	0.00412	1.408768	1.471027	0.012953
10^4	0.59479	0.59725	0.52196	0.00419	1.467856	1.482930	0.013423

Table 1: Approximated value of $\left\| \Phi - \tilde{\Phi} \right\|$ using Monte Carlo simulation.

Figure 1 plots the approximated value of the norm $\left\| \Phi - \tilde{\Phi} \right\|$ using replacing the kernel by a degenerate kernel approximated method (= 0.013486, upper line) and Monte Carlo Hit or Miss integration method [7], using four sequences of $n = 10, 10^2, 10^3, 10^4$ random numbers (the second lower curve) .

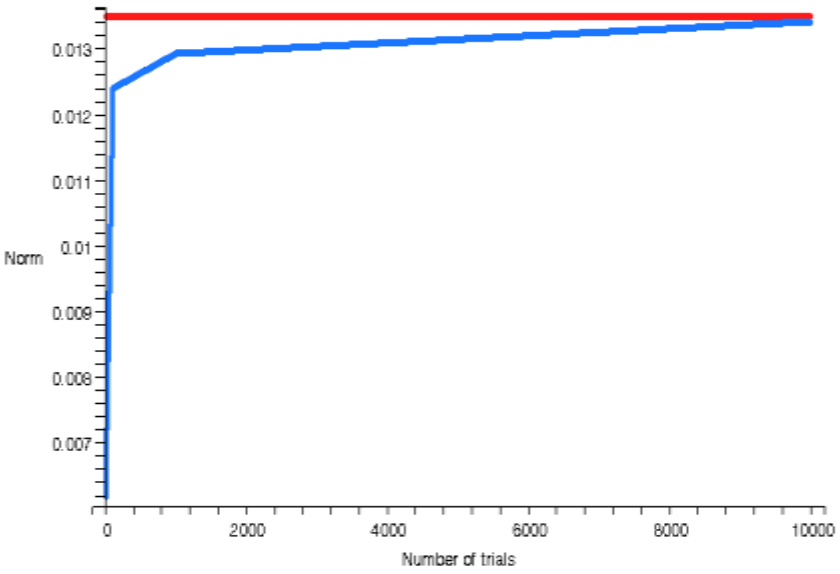

Figure 1: A comparison between the approximated values of $\left\| \Phi - \tilde{\Phi} \right\|$.

From figure 1 we see that the approximated value of the norm $\left\|\Phi - \tilde{\Phi}\right\|$ using the Monte Carlo Hit or Miss integration method is converges fast to that value using replacing the kernel by a degenerate kernel approximated method as the number of trials ***n*** increase.

References

1. Anderson Robert S., Honk Frank R. and Lukas Mark A., The Application and Numerical Solution of Integral Equations, Sijthoff and Noordhoff International Puplishers, 1980.
2. Brunner H. and Vander Houwen P. J., The Numerical Solution of Volterra Equations, North-Holland,1986.
3. Chambers L1. G., Integral Equations: A short Course, International Textbook Company Limited, 1976.
4. Delves L. M. and Mohamed J. L., Computational Methods for Integral Equations, Cambridge University Press,1985.
5. Golberg Michael A., Solution Methods for Integral Equations, Plenum Press, 1979.
6. Kransnov M., Kiseler A. and Makarenko G., Problems and Exercises in Integral Equations, MIR Publishers, 1971.
7. Sheet Abd alkareem I.: Numerical Integration using Monte Carlo Methods and Variance Reduction Technique, Un published PhD. Thesis, Univ. of Mosul, 2006.
8. Stokes, V. P., Monte Carlo simulation: basics , Survey Report, Uppsala Univ. Press. ,2004

9- Sabelfeld, K. K. Monte Carlo methods and applications Monte Carlo Methods, Vol. 10., 2004

10- Smaldone, Tony, Modeling and Simulation , Mc- Graw Hill, New York, 2001

11- Yang, Guan, A Monte Carlo Method of Integration , The Chemical Rubber Company Press, 2002.

12 – Sheet abd alkareem I. , Evaluating the Area of a Circle and the Volume of a Sphere by Using Monte Carlo Simulation ,Iraqi journal of Statistical science, no. 14,vol. 8, 2008.