# On Essential fuzzy submodule and Uniform Fuzzy Module

Asst. Prof. Hussein H. Abbas
DEPARTMENT OF MATHMATICS, COLLAGE OF EDUCATION OF GIRLS, UNIVERSITY OF KUFA
E-mail: Msc_hussein@yahoo.com
Shukor NeamahMobred
COLLAGE OF PHYSICAL PLANNING, UNIVERSITY OF KUFA
E-mail: sh_aeashi2012@yahoo.com

## Abstract

In this paper, we study the concept of uniform fuzzy module and essential fuzzy submodule. Also we give some of characteristics about it such as the intersection of two essential fuzzy submodule s is essential fuzzy submodule , the inverse and image of essential fuzzy submodule is an essential fuzzy submodule also the intersection of two uniform fuzzy module and the image and inverse image of uniform fuzzy module.

## Introduction

The concept of fuzzy sets was introduced by Zadeh in 1965 [1]. It was first applied to the theory of groups by Rosenfeld in 1971 [2] . Since then, many authors introduced fuzzy subring and fuzzy ideals [3],[4]The concept of fuzzy module was introduced by Negoita and Relescu in 1975 [3]. Since then several authors have studied fuzzy modules. The concept of essential fuzzy submodule was introduced by Hadi 2000[11] . In this paper, we study the concept of uniform fuzzy module essential fuzzy submoduleand give some of characteristics about it.

**Key words**Fuzzy set, fuzzy module, fuzzy submodule, essential fuzzy submodule, uniform fuzzy module.

## 1 SOME BASIC CONCEPTS

In this section, we shall give the concept of fuzzy set with some basic definitions and properties about it which are used in the next sections.

**Definition 1.1 [1]:**

Let S be a non-empty set and I be the closed interval [0, 1] of the real line (real numbers). A fuzzy set A in S (a fuzzy subset of S) is a function from S into I.

Note that [2]

Let $x_t : S \rightarrow [0, 1]$ be a fuzzy subset of S, where $x \in S$ , $t \in [0, 1]$

defined by: $x_t(y) = \begin{cases} t & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$ , for all $y \in S$.

$x_t$ is called a fuzzysingleton or a fuzzy point in S .

If x=0 and t=1 then: $0_1(y) = \begin{cases} 1 & \text{if } y = 0 \\ 0 & \text{if } y \neq 0 \end{cases}$ , for all $y \in S$ is called the fuzzy zero singleton.

**Definition 1.2 [5]:**

Let A and B be two fuzzy sets in S, then:

**1-** A=B if and only if A (x) = B (x), for all $x \in S$.

**2-** $A \subseteq B$ if and only if A (x) ≤ B (x), for all $x \in S$ .If A is a subset of B and there exists $x \in S$ such that A (x) < B (x), then A is called a proper fuzzy subset of B and written $A \subset B$.

**Definition 1.3[5]:**Let A and B be two fuzzy subsets of S, then:

(1) $(A \cap B)(x) = \min\{A(x), B(x)\}$, for all $x \in S$.

(2) $(A \cup B)(x) = \max\{A(x), B(x)\}$, for all $x \in S$.

$A \cap B$ and $A \cup B$ are fuzzy subsets of S,

**Definition 1.4[1]:**

Let f be a mapping from a set M into a set N, let A be a fuzzy subset of M and B be a fuzzy subset of N.

The image of A denoted by f (A) is the fuzzy set in N defined by:

$$f(A)(y) = \begin{cases} \sup\{A(z) \mid z \in f^{-1}(y)\} \text{ if } f^{-1}(y) \neq \phi, \text{ for all } y \in N, \\ 0 \quad \text{otherwise} \end{cases}$$ where $f^{-1}(y) = \{x \in M, f(x) = y\}$.

And the inverse image of B, denoted by $f^{-1}(B)$ is the fuzzy set in M defined by:

$f^{-1}(B)(x) = B(f(x))$, for all $x \in M$.

**Definition 1.5 [7]:**

Let f be a function from a set M into a set N. A fuzzy subset A of M is called f-invariant if A (x) = A (y) whenever f (x) = f (y), where $x, y \in M$.

**Proposition 1.6[7]**

If f is a function defined on a set M, $A_1$ and $A_2$ are two fuzzy subsets of M, $B_1$ and $B_2$ are two fuzzy subsets of f (M). Then:-

1. $A_1 \subseteq f^{-1}(f(A_1))$.
2. $A_1 = f^{-1}(f(A_1))$, whenever $A_1$ is f-invariant.
3. $f(f^{-1}(B_1)) = B_1$.
4. If $A_1 \subseteq A_2$, then $f(A_1) \subseteq f(A_2)$.
5. If $B_1 \subseteq B_2$, then $f^{-1}(B_1) \subseteq f^{-1}(B_2)$.

**Proposition 1.7[4]:**

Let f be a function from a set M into a set N. If $B_1$ and $B_2$ are fuzzy subsets of N,then$f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

**Definition 1.8[8], [9]:**

Let A be a fuzzy subset of S, for all $t \in [0, 1]$, the set $A_t = \{x \in S, A(x) \geq t\}$ is called a level subset of A.

Note that, $A_t$ is a subset of S in the ordinary sense.

**Definition 1.9[8],[6]**

Let M be an R-module. A fuzzy subset X of M is called fuzzy module of an R-module M if:

**1.** $X(x-y) \geq \min\{X(x), X(y)\}$, for all $x, y \in M$.
**2.** $X(rx) \geq X(x)$, for all $x \in M$ and $r \in R$.
**3.** $X(0) = 1$.

**Definition 1.10[9],[8]**

Let A and B be two fuzzy modules of an R-module M . B is called a fuzzy submodule of A, if $B \subseteq A$.

**Definition (1.11) [10]**

Let $A_1$ and $A_2$ be two fuzzy modules of R-modules $M_1$ and $M_2$ respectively . $f : A_1 \rightarrow A_2$ is called ***a fuzzy homomorphism*** if $f : M_1 \rightarrow M_2$ is R-homomorphism and $A_2(f(x)) = A_1(x)$, for each $x \in M_1$.

**Proposition 1.12[10]**

Let $A_1$ and $A_2$ be two fuzzy modules of R-modules $M_1$ and $M_2$ respectively. Let $f: A_1 \rightarrow A_2$ be a fuzzy homomorphism.

If $N_1$ and $N_2$ are two fuzzy submodules of $A_1$ and $A_2$ respectively, then :

1. $f(N_1)$ is a fuzzy submodule of $A_2$ .
2. $f^{-1}(N_2)$ is a fuzzy submodule of $A_1$.
3. $f(N \cap N_1) = f(N) \cap f(N_1)$ is a fuzzy submodule of $A_2$ , N is a fuzzy submodule of $A_1$ .

**Proposition 1.13[9]**

Let A be a fuzzy module of an R-module M. Let $\{N\alpha : \alpha \in \Lambda\}$ be a family of fuzzy submodules of A , then :-

1- $\left(\bigcap_{\alpha \in \Lambda} N_\alpha\right)$ is a fuzzy submodule of A.

2- If $\{N\alpha : \alpha \in \Lambda\}$ is a chain , then $\left(\bigcup_{\alpha \in \Lambda} A_\alpha\right)$ is a fuzzy submodule of A .

**Proposition 1.14[9]:**

Let A be a fuzzy set of an R-module M. Then the level subset $A_t$, $t \in [0, 1]$ is a submodule of M if and only if A is a fuzzy submodule of X where X is a fuzzy module of an R-module M.

**Definition 1.15[9]:**

If A is a fuzzy module of an R-module M, then the submodule $A_t$ of M is called the level submodule of M where $t \in [0, 1]$.

**Definition 1.16[11]**

Let A be a fuzzy module of an R-module M and let N be a fuzzy submodule of A then N is called **an essential fuzzy submodule** of A if $N \cap U = 0_1$ implies to $U = 0_1$ for all fuzzy submodule U of A .

**Proposition 1.17[11]**

Let A be a fuzzy module of an R-module M . A fuzzy submodule N of A is an essential fuzzy submodule of A if and only if $N_t$ is an essential submodule of M for all $N_t \neq \{0\}$ where $t \in (0,1]$ .

**Definition 1.18[11]**

Let A be a fuzzy module of an R-module M then A is called **a uniform fuzzy module** if each non-zero fuzzy submodule of A is an essential fuzzy submodule of A .

**For example** a fuzzy module A of an R-module $Z_{14}$ that definied by A(x)=1 for all $x \in Z_{14}$ .

(2) ,(7) and $Z_{14}$ represented all non-zero submodule of $Z_{14}$ .

Every fuzzy submodule N of A must $N_t$ =(2) or (7) or $Z_{14}$.

Since each of (2) ,(7) and $Z_{14}$ are essential submodule of $Z_{14}$

Therefore N is an essential fuzzy submodule of A.

**2 THE MAIN RESELT**

This section is devoted to study the concept of uniform fuzzy module and some important properties which are connected with concept of essential fuzzy submodule .

**Proposition 2.1**

Let A be a fuzzy module of an R-module M if $N_1$ and $N_2$ are two fuzzy essential submodules of A then $N_1 \cap N_2$ is an essential fuzzy module of A.

**Proof**

$N_1 \cap N_2$ is fuzzy submodule of A (By proposition (1.13)).

Suppose that U is a fuzzy submodule of A and $(N_1 \cap N_2) \cap U = 0_1$ .

$\Longrightarrow$ $N_1 \cap (N_2 \cap U) = 0_1$ (Since the intersection satisfies associative law).

$\Longrightarrow (N_2 \cap U) = 0_1$ (Since $N_1$ is fuzzy essential submodule of A).

$\Longrightarrow$ $U = 0_1$ (Since $N_2$ is fuzzy essential submodule of A).

Thus $N_1 \cap N_2$ is an essential fuzzy module of A ■

**Proposition 2.2**

Let f be an epimorphisem from a fuzzy module $A_1$ into a fuzzy module $A_2$ and $A_1$ is an f-invariant . If N is an essential fuzzy submodule of $A_1$ then f(N) is an essential fuzzy submodule of $A_2$ .

**Proof**

To prove f(N) isan essential fuzzysubmodule of $A_2$.

f(N) is a fuzzy submodule of $A_2$ (By proposition (1.12) ) .

Now, Suppose that U is a fuzzy submodule of $A_2$ such that $f(N) \cap U = 0_1$

Therefore, $f^{-1}(f(N) \cap U) = f^{-1}(0_1)$.

$\Longrightarrow f^{-1}(f(N)) \cap f^{-1}(U) = 0_1$ (By proposition (1.7) ) .

$\Longrightarrow N \cap f^{-1}(U) = 0_1$ (Since $f^{-1}(f(N)) = N$ by proposition (1.6) ) .

$\Longrightarrow$ $f^{-1}(U) = 0_1$(Since N is an essential fuzzy submodule and $f^{-1}(U)$ is fuzzy submodule of A by proposition (1.6)).

$f(f^{-1}(U)) = f(0_1) \Longrightarrow U = 0_1$ (By proposition (1.6) ) .

Then f(N) is an essential fuzzy submodule of $A_2$■

**Proposition 2.3**

Let f is an epimorphism from a fuzzy module $A_1$ into a fuzzy module $A_2$ . If N is an essential fuzzysubmodule of $A_2$ then $f^{-1}(N)$ is an essential fuzzy submodule of $A_1$ whenever $A_1$ is f-invariant

**Proof**

To prove $f^{-1}(N)$ is an essential fuzzy submodule of $A_1$.

$f^{-1}(N)$ is a fuzzy submodule of $A_1$ (By proposition( 1.12)) .

Now, Suppose U is a fuzzy submodule of A such that $f^{-1}(N) \cap U = 0_1$

$\Longrightarrow f(f^{-1}(N) \cap U) = f(0_1)$

$\Longrightarrow f(f^{-1}(N)) \cap f(U) = f(0_1)$ (By proposition (1.7) ).

$\Longrightarrow N \cap f(U) = 0_1$ (Since $f(f^{-1}(N)) = N$ by proposition (1.6) and f is an epimorphism ) .

$\Longrightarrow f(U) = 0_1$ ( Since N is an essential fuzzy submodule of A ) .

$\Longrightarrow f^{-1}(f(U)) = f^{-1}(0_1)$

$\Longrightarrow U = 0_1$ (Since $f^{-1}(f(U)) = U$ by proposition (1.6) and f is an epimorphism ) .

Then $f^{-1}(N)$ is an essential fuzzy submodule of $A_1$ ■

**Proposition 2.4**

Let A be a fuzzy module of an R-module M and let $N_1$ and $N_2$be two fuzzy submodules of A such that $N_1$is a fuzzy submodule of $N_2$ then $N_1$ is an essential fuzzy submodule of A if and only if $N_1$ is an essential fuzzy submodule of $N_2$ and $N_2$ is an essential fuzzy submodule of A.

Proof

To prove $N_2$ is an essential fuzzy submodule of A suppose U be a fuzzy submodule of A such that $N_2 \cap U = 0_1$

$\Longrightarrow N_1 \cap U = 0_1$ (Since $N_1 \subseteq N_2$).

$\Longrightarrow U = 0_1$ (Since $N_1$ isan essential fuzzy submodule of A).

$\Longrightarrow N_2$ is an essential fuzzy submodule of A .

To prove $N_1$ is an essential fuzzy submodule of $N_2$, suppose that U is a fuzzy submodule of $N_2$ such that $N_1 \cap U = 0_1$

Since $U \subseteq N_2 \subseteq A \Longrightarrow$ U is a fuzzy submodule of A .

$\Longrightarrow U = 0_1$ (Since $N_1$ is an essential fuzzy submodule of A ) .

Thus $N_1$ is an essential fuzzy submodule of $N_2$ .

Conversely;

To prove that $N_1$ is an essential fuzzy submodule of A, let U be a fuzzy submodule of A such that $N_1 \cap U = 0_1$

The fuzzy submodule$N_2 \cap U$ is a fuzzysubmodule of U and also of $N_2$ therefore,

$N_1 \cap (N_2 \cap U) = 0_1$ (Since$N_2 \cap U \subseteq U$).

$N_2 \cap U = 0_1$ (Since $N_1$ is an essential fuzzy submodule of $N_2$).

$U = 0_1$ (Since $N_2$ is an essential fuzzy submodule of A).

Therefore $N_1$ is an essential fuzzy submodule of A ■

**Proposition 2.5**

Let A be a uniform fuzzy module of an R-module M and N be a fuzzy submodule of A , then N is a uniform fuzzy module of M .

**Proof**

Suppose that U is a fuzzy submodule of N .

Then U is a fuzzy submodule of A (Since $U \subseteq N \subseteq A$ ) .

$\Longrightarrow$ U is an essential fuzzy submodule of A (Since A is a uniform fuzzy module ) .

Consequently; U is an essential fuzzy submodule of N (By proposition (2.4)).

Therefore N is a uniform fuzzy module ■

**Proposition 2.6**

Let $A_1$ be a uniform fuzzy module and $A_2$ be a fuzzy module of an R-module M, then $A_1 \cap A_2$ is a uniform fuzzy module of M.

**Proof**

Let N be a fuzzy submodule of $A_1 \cap A_2$.

Then N is a fuzzy submodule of $A_1$.

Therefore; N is an essential fuzzy submodule of $A_1$

(Since $A_1$ is a uniform fuzzy module) .

Consequently; N is an essential fuzzy submodule of $A_1 \cap A_2$

Thus $A_1 \cap A_2$ is a uniform fuzzy module ■

**Corollary 2.7**

If $A_1$ and $A_2$ are two uniform fuzzy modules of an R-module M, then $A_1 \cap A_2$ is a uniform fuzzy module .

**Proof**

Since every a uniform fuzzy module is a fuzzy module,

Then by proposition (2.6) the proof is completed ■

Now, we give a generalization of Corollary (2.4) by the following:

**Corollary 2.8**

Let $\{A_\alpha : \alpha \in \Lambda\}$ be a family of uniform fuzzy module of an R-module M .Then

$\bigcap_{\alpha\in\Lambda} A_\alpha$ is a uniform fuzzy module .

**Proof**

Since $\bigcap_{\alpha\in\Lambda} A_\alpha$ is a fuzzy submodule of $A_\alpha$ for all $\alpha \in \Lambda$ (By Proposition ( 1.13)).

Then; $\bigcap_{\alpha\in\Lambda} A_\alpha$ is a uniform fuzzy module (By proposition (2.5))■

**Proposition 2.9**

Let N be an essential fuzzy submodule of a fuzzy module A of an R-module M , If N is a uniform fuzzy module of an R-module M then A is a uniform module of an R-module M .

**Proof**

Let U be a fuzzy submodule of A, to prove that U is an essential fuzzy submodule.We have U∩N is a fuzzy submodule of N ( By proposition (1.13) ).

So, U∩N is an essential fuzzy submodule of N ( Since N is a uniform fuzzy module ).

U∩N is an essential fuzzy submodule of A ( By proposition (2.4) ) .

U is an essential fuzzy submodule of N ( By proposition (2.4) ).

Thus U is an essential fuzzy submodule of A (By proposition (2.4)).

Therefore, A is a uniform fuzzy module ■

**Corollary 2.10**

Let A be a fuzzy module of an R-module M such that $N_1$ and $N_2$ are two essential fuzzy submodules of A, then $N_1$ and $N_2$ are fuzzy uniform modules of A if and only if $N_1 \cap N_2$ is a fuzzy uniform module of A.

**Proof**

Let $N_1$ and $N_2$be two fuzzy uniform modules of an R-module M.

Then $N_1 \cap N_2$ is a fuzzy submodule of $N_1$ and $N_2$ (By proposition (1.13)).

So, $N_1 \cap N_2$ is a fuzzy uniform module (By proposition (2.5) ) .

Conversely;Let $N_1 \cap N_2$ be a fuzzy uniform module of A.

Since $N_1 \cap N_2$ is an essential fuzzy submodule of $N_1$ and $N_2$ (By proposition (2.1)).

Then by proposition (2.9) the proof is completed ■

**Proposition 2.11**

Let $N_1$ and $N_2$ be two fuzzy submodules of a fuzzy module A of an R-module M such that $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$ then $N_1$ and $N_2$ are uniform fuzzy module if and only if $N_1 \cup N_2$ is a uniform fuzzy module of an R-module M .

**Proof**

To prove the case where $N_1 \subseteq N_2$

Let $N_1$ and $N_2$ be a uniform fuzzy module ,we have

$N_1 \cup N_2 = N_2$ (Since $N_1 \subseteq N_2$ ) .

Then $N_1 \cup N_2$ is a uniform fuzzy module .

Similarly if $N_2 \subseteq N_1$ .

Conversely;

Let $N_1 \cup N_2$ be a uniform fuzzy module of an R-module M .

Since $N_1$ and $N_2$ are fuzzy submodules of $N_1 \cup N_2$ .

Then by proposition (2.5) the proof is completed ■

**Proposition 2.12**

Let $A_1$ and $A_2$ be two fuzzy modules of an R-module $M_1$ and $M_2$ respectively if $f: A_1 \rightarrow A_2$ is a fuzzy epimorphism and $A_1$ is a uniform fuzzy module, then $A_2$ is a uniform fuzzy module .

**Proof**

Let U be a fuzzy submodule of $A_2$, to prove U is an essential fuzzy submodule of $A_2$.

We have $f^{-1}(U)$ is a fuzzy submodule of $A_1$ (By proposition (1.12).

But $A_1$ is a Uniform fuzzy module.

Then $f^{-1}(U)$ is an essential fuzzy submodule of $A_1$.

Now,f $(f^{-1}(U))=U$ is an essential fuzzy submodule of $A_2$( By proposition (1.6) and proposition ( 2.2)).
Therefore $A_2$ is a uniform fuzzy module ■

**Proposition 2.13**

Let $A_1$ and $A_2$ be two fuzzy modules of an R-module $M_1$, $M_2$ respectively. If f: $A_1 \to A_2$ is a fuzzyepimomorphism , $A_1$ is an f-invariant and $A_2$ is a uniform fuzzy module, then $A_1$ is a uniform fuzzy module .
**Proof**

Let U be a fuzzy submodule of $A_1$. To prove U is an essential fuzzy submodule of $A_1$ .

We have f(U) is a fuzzy submodule of $A_2$ ( By proposition (1.12) .

But $A_2$ is a Uniform fuzzy module.

Then f(U) is an essential fuzzy submodule of $A_2$.

Now $f^{-1}(f(U))=U$ is an essential fuzzy submodule of $A_1$ ( By proposition (1.6) and proposition (2.3) ) .

Therefore $A_1$ is a Uniform fuzzy module ■

## References

[1] Zadeh L. A., "Fuzzy Sets, Information and control" , Vol. 8, PP. 338-353, (1965) .
[2] Rosenfeld .A., "Fuzzy groups," J. Math. Anal. Appl., vol. 35, pp. 512-517, 1971
[3] Kumar R., "Fuzzy Cosets and Some Fuzzy Radicals", Fuzzy Sets and Systems, Vol. 46, PP. 261-265, (1992).
[4] Hamil. M. A,"F-regular fuzzy module", M.Sc . Thesis, University of Baghdad, (2002).
[5] Zahedi M. M., "A characterization of L-Fuzzy Prime Ideals", Fuzzy Sets and Systems, Vol. 44, 147-160, (1991).
[6] Zahedi M. M., "On L-Fuzzy Residual Quotient Modules and P. Primary Submodules", Fuzzy Sets and Systems, Vol. 51, PP. 33-344, (1992).
[7] Kumar R., "Fuzzy Semiprimary Ideals of Rings", Fuzzy Sets and Systems, Vol. 42, PP. 263-272, (1991).
[8]Mashinchi M. and Zahedi M. M.," On L-Fuzzy Primary Submodules ", Fuzzy Sets and Systems, Vol. 49, PP. 231-236, (1992).
[9] Martinez L., "Fuzzy Modules Over Fuzzy Rings in Connection with Fuzzy Ideal of Ring", J. Fuzzy Math. Vol. 4, PP. 843-857, (1996).
[10] Zahedi M. M.., "Some Results on L- Fuzzy Modules", Fuzzy Sets and Systems, Vol. 55, PP. 355-363, (1993).
[11]Hadi I.M," on some special fuzzy ideal of a fuzzy ring " Accepted in I. Soc. OfPhy. And Math., (2000) .