

Semi - α - $T_{1/2}$ - Space in Bitopological Spaces

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Abstract :

The objective of this research is to study some properties of ij - semi - α - $T_{1/2}$ - spaces in bitopological spaces and their relationship with ij - semi - α - T_0 - spaces and ij - semi - α - T_1 - spaces .

Key words : Bitopological space , ij - semi - α - open set , ij - semi - α - $T_{1/2}$ - space .

الخلاصة :

الهدف من هذا البحث هو دراسة بعض خصائص فضاءات ij -شبه ألفا $T_{1/2}$ في الفضاءات ثنائية التبولوجي وعلاقتها مع فضاءات ij -شبه ألفا T_0 و فضاءات ij -شبه ألفا T_1 .

1. Introduction :

In 1963 Kelly [2] defined : a set equipped with two topologies is called a bitopological space , denoted by (X, τ_1, τ_2) where (X, τ_1) and (X, τ_2) are two topological spaces . This notion was studied in different senses , one of these is the ij - semi - α - open sets , that suggested by Qays , 2012 [4] . In 1997 , Kumar Sampath , S ., [3] introduced the concept of ij - α -open sets in bitopological spaces . In 1981 , Bose , S. , [1] introduced the notion of ij -semi-open sets in bitopological spaces . In this paper we first (in section 2) ij - semi - α - open sets , and then (in section 3) we study especial case of ij - semi - α - $T_{1/2}$ - space in bitopological spaces in the sense of ij - semi - α - open sets .

2. Preliminaries :

Throughout the paper , spaces always mean a bitopological spaces , the closure and the interior of any subset A of X with respect to τ_i , will be denoted by $\tau_i - cl(A)$, and $\tau_i - int(A)$ respectively, for $i = 1, 2$.

Definition 2.1 : [4]

Let (X, τ_1, τ_2) be a bitopological space , $A \subseteq X$. Then A is said to be ij - semi - α - open set if there exists an ij - α - open set U in X , such that $U \subseteq A \subseteq j - cl(U)$. The family of all ij - semi - α - open sets of X is denoted by $ij - S_\alpha O(X)$, where $i \neq j; i, j = 1, 2$.

The following proposition will give an equivalent definition of ij - semi - α - open sets .

Proposition 2.2 : [4]

Let (X, τ_1, τ_2) be a bitopological space , $A \subseteq X$. Then A is an ij - semi - α - open set if and only if $A \subseteq j - cl(i - int(j - cl(i - int(A))))$.

Remark 2.3 : [4]

The intersection of any two ij - semi - α - open sets is not necessary ij - semi - α - open set as in the following example .

Example 2.4 :

Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$, and $\tau_2 = \{X, \phi, \{a\}\}$. The family of all 12 - semi - α - open sets of X is : $12 - S_\alpha O(X) = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. Hence $\{a, c\}$ and $\{b, c\}$ are two 12 - semi - α - open sets , but $\{a, c\} \cap \{b, c\} = \{c\}$ is not 12 - semi - α - open set .

Proposition 2.5 : [4]

The union of any family of ij -semi- α -open sets is ij -semi- α -open set .

Remark 2.6 : [4]

- (i) Every τ_i -open set is ij -semi- α -open set ,but the converse need not be true .
- (ii) If every τ_i -open set is τ_i -closed and every nowhere τ_i -dense set is τ_i -closed in any bitopological space , then every ij -semi- α -open set is an τ_i -open set .

Definition 2.7 :

A set A is said to be an ij -semi- α -neighborhood of a point x if there exists an ij -semi- α -open set U such that $x \in U \subseteq A$.

Definition 2.8 : [4]

The complement of ij -semi- α -open set is called ij -semi- α -closed set . Then family of all ij -semi- α -closed sets of X is denoted by $ij-S_\alpha C(X)$, where $i \neq j; i, j = 1,2$.

Remark 2.9 : [4]

The intersection of any family of ij -semi- α -closed sets is ij -semi- α -closed set .

Definition 2.10 : [4]

Let (X, τ_1, τ_2) be a bitopological space and $A \subseteq X$, the intersection of all ij -semi- α -closed sets containing A is called ij -semi- α -closure of A , and is denoted by $ij-S_\alpha-cl(A)$; i.e $ij-S_\alpha-cl(A) = \bigcap \{B \subseteq X : B \text{ is } ij\text{-semi-}\alpha\text{-closed set, } A \subseteq B\}$.

Theorem 2.11 : [4]

Let (X, τ_1, τ_2) be a bitopological space , and let $A \subseteq X$, then :

- (i) $ij-S_\alpha-cl(A)$ is the smallest ij -semi- α -closed set containing A .
- (ii) A is ij -semi- α -closed set if and only if $ij-S_\alpha-cl(A) = A$.

Remark 2.12 : [4]

Let (X, τ_1, τ_2) be a bitopological space , and let A, B be any subsets of X , then :

- (i) $A \subseteq ij-S_\alpha-cl(A)$.
- (ii) If $A \subseteq B$, then $ij-S_\alpha-cl(A) \subseteq ij-S_\alpha-cl(B)$.
- (iii) $ij-S_\alpha-cl(ij-S_\alpha-cl(A)) = ij-S_\alpha-cl(A)$.

Lemma 2.13 :

If $ij-S_\alpha-cl(\{x\}) \cap A \neq \phi$, for each $x \in ij-S_\alpha-cl(A)$, then $(ij-S_\alpha-cl(A)) - A$ does not contain a non empty ij -semi- α -closed set .

Proof :

Suppose $(ij-S_\alpha-cl(A)) - A$ contains a non empty ij -semi- α -closed set , say B , then $x \in B$ implies $ij-S_\alpha-cl(\{x\}) \subseteq B \subseteq (ij-S_\alpha-cl(A)) - A$, and $ij-S_\alpha-cl(\{x\}) \cap A = \phi$, which contradicts the hypothesis . ■

Definition 2.14 :

A subset A of a bitopological space (X, τ_1, τ_2) is said to be ij -semi- α -generalized closed set (briefly ij -semi- α -g-closed set) if $ij-S_\alpha-cl(A) \subseteq U$ whenever $A \subseteq U$ and U is ij -semi- α -open set in (X, τ_1, τ_2) .

Lemma 2.15 :

A subset A of a bitopological space (X, τ_1, τ_2) is ij -semi- α -g-closed set if and only if $ij-S_\alpha-cl(\{x\}) \cap A \neq \phi$ for each $x \in ij-S_\alpha-cl(A)$.

Proof :

Suppose that A is ij -semi- α -g-closed set, and for some $x \in ij-S_\alpha-cl(A)$, $ij-S_\alpha-cl(\{x\}) \cap A = \phi$, then $A \subseteq X - (ij-S_\alpha-cl(\{x\}))$, where $X - (ij-S_\alpha-cl(\{x\}))$ is a

ij -semi- α -open set , so by definition (2.14) $ij - S_\alpha - cl(A) \subseteq X - (ij - S_\alpha - cl(\{x\}))$, hence $x \in X - (ij - S_\alpha - cl(\{x\}))$ i.e $x \notin X - (ij - S_\alpha - cl(\{x\}))$ which is a contradiction .

Conversely assume that for each $x \in ij - S_\alpha - cl(A)$, $ij - S_\alpha - cl(\{x\}) \cap A \neq \emptyset$; if there is a ij -semi- α -open set U such that $A \subseteq U$ but $ij - S_\alpha - cl(A) \not\subseteq U$ then there exists $x \in ij - S_\alpha - cl(A)$ and $x \notin U$, so $x \in X - U$ which implies $ij - S_\alpha - cl(\{x\}) \subseteq X - U$ (since $X - U$ is ij -semi- α -closed) i.e $ij - S_\alpha - cl(\{x\}) \cap A = \emptyset$, a contradiction . Therefore A is ij -semi- α -g-closed set . ■

3. Semi - α - $T_{1/2}$ - space in Bitopological Spaces :

In this section the notion of ij -semi- α - $T_{1/2}$ -space is introduced in bitopological spaces and their relationships with ij -semi- α - T_0 -space and ij -semi- α - T_1 -space are studied .

Definition 3.1 :

Let (X, τ_1, τ_2) be a bitopological space ;two subsets A and B of X are ij -semi- α -separated if each is disjoint from the other's ij -semi- α -closure . (i.e $A \cap ij - S_\alpha - cl(B) = \emptyset$ and $ij - S_\alpha - cl(A) \cap B = \emptyset$) . Two points x and y in X are ij -semi- α -distinguishable if they do not have exactly the same ij -semi- α -neighborhoods (i.e there exists a ij -semi- α -open set containing x but not containing y or containing y but not containing x) . Two points x and y are ij -semi- α -separated if the singletons $\{x\}$ and $\{y\}$ are ij -semi- α -separated .

Definition 3.2 :

A bitopological space (X, τ_1, τ_2) is called ij -semi- α - T_0 -space if any two distinct points are ij -semi- α -distinguishable .

Remark 3.3 :

Every $i - T_0$ -space is ij -semi- α - T_0 -space.

Proof :

Follows from remark (2.6) (i) . ■

Remark 3.4 :

The converse of remark (3.3) is not true ,see the following example .

Example 3.5 :

Let $X = \{a, b, c, d\}$, $\tau_1 = \{X, \emptyset, \{a\}, \{a, d\}, \{b, c\}, \{a, b, c\}\}$, $\tau_2 = \{X, \emptyset, \{a\}\}$. Then $12 - S_\alpha O(X) = \tau_1 \cup \{\{a, b\}, \{a, b, d\}, \{a, c\}, \{a, c, d\}\}$ and it is clear that (X, τ_1, τ_2) is 12 -semi- α - T_0 -space but (X, τ_1) is not $1 - T_0$ -space since the points b and c are not distinguishable .

Theorem 3.6 :

A bitopological space (X, τ_1, τ_2) is ij -semi- α - T_0 -space iff for each distinct points x, y in X , $ij - S_\alpha - cl(\{x\}) \neq ij - S_\alpha - cl(\{y\})$.

Proof :

Suppose that (X, τ_1, τ_2) is ij -semi- α - T_0 -space and let $x \neq y$ be two points of X such that $ij - S_\alpha - cl(\{x\}) = ij - S_\alpha - cl(\{y\})$, therefore $x \in ij - S_\alpha - cl(\{y\})$ and $y \in ij - S_\alpha - cl(\{x\})$. If U is ij -semi- α -open set such that $x \in U$ and $y \notin U$, then $y \in X - U$ (ij -semi- α -closed set) so $ij - S_\alpha - cl(\{y\}) \subseteq X - U$ which means $ij - S_\alpha - cl(\{x\}) \subseteq X - U$ and so $x \in X - U$ i.e $x \notin U$ a contradiction . Similarly the assumption that $x \notin V$ and $y \in V$ (for some ij -semi- α -open set V) leads to a contradiction , that is (X, τ_1, τ_2) is not an ij -semi- α - T_0 -space .

On the other hand, suppose that for each $x, y \in X$ and $x \neq y$ we have $ij - S_\alpha - cl(\{x\}) \neq ij - S_\alpha - cl(\{y\})$,

therefore either $x \notin ij - S_\alpha - cl(\{y\})$ and so $x \in X - (ij - S_\alpha - cl(\{y\}))$ but $y \notin X - (ij - S_\alpha - cl(\{y\}))$, or $y \notin ij - S_\alpha - cl(\{x\})$ and so $y \in X - (ij - S_\alpha - cl(\{x\}))$ but $x \notin X - (ij - S_\alpha - cl(\{x\}))$ where $X - (ij - S_\alpha - cl(\{x\}))$ and $X - (ij - S_\alpha - cl(\{y\}))$ are ij -semi- α -open sets in (X, τ_1, τ_2) i.e x and y are ij -semi- α -distinguishable, hence (X, τ_1, τ_2) is ij -semi- α - T_0 -space. ■

Theorem 3.7 :

Every subspace of ij -semi- α - T_0 -space is ij -semi- α - T_0 -space .

Proof :

Let $(Y, \tau_{1Y}, \tau_{2Y})$ be a subspace of a bitopological space (X, τ_1, τ_2) and let X is ij -semi- α - T_0 -space . To prove that Y is ij -semi- α - T_0 -space, let $y_1 \neq y_2 \in Y$. Since $Y \subseteq X$, then $y_1 \neq y_2 \in X$ and X is ij -semi- α - T_0 -space . There exists ij -semi- α -open set G in X , such that $y_1 \in G$ and $y_2 \notin G$. So $G \cap Y$ is ij -semi- α -open set in Y and $y_1 \in G \cap Y$, $y_2 \notin G \cap Y$.

Hence $(Y, \tau_{1Y}, \tau_{2Y})$ is ij -semi- α - T_0 -space. ■

Definition 3.8 :

A bitopological space (X, τ_1, τ_2) is said to be ij -semi- α - $T_{1/2}$ -space if every ij -semi- α - g -closed set in (X, τ_1, τ_2) is ij -semi- α -closed .

Theorem 3.9 :

A bitopological space (X, τ_1, τ_2) is ij -semi- α - $T_{1/2}$ -space if and only if, for each $x \in X$, $\{x\}$ is ij -semi- α -closed or ij -semi- α -open .

Proof :

Assume that (X, τ_1, τ_2) is ij -semi- α - $T_{1/2}$ -space and $\{x\}$ is neither ij -semi- α -closed nor ij -semi- α -open then $X - \{x\}$ is not ij -semi- α -closed so $ij - S_\alpha - cl(X - \{x\}) = X \subseteq X$ i.e $X - \{x\}$ is ij -semi- α - g -closed set, by definition of ij -semi- α - $T_{1/2}$ -space, $X - \{x\}$ must be ij -semi- α -closed, a contradiction with the assumption .

On the other hand, suppose that for each x in (X, τ_1, τ_2) , $\{x\}$ is ij -semi- α -closed or ij -semi- α -open. Let A be ij -semi- α - g -closed set in (X, τ_1, τ_2) , then by lemma (2.13) and lemma (2.15) $(ij - S_\alpha - cl(A)) - A$ does not contain a non empty ij -semi- α -closed set, so if $x \in (ij - S_\alpha - cl(A)) - A$ then $ij - S_\alpha - cl(\{x\}) \not\subseteq (ij - S_\alpha - cl(A)) - A$ i.e $ij - S_\alpha - cl(\{x\}) \neq \{x\}$ which means $\{x\}$ is not ij -semi- α -closed, so it must be ij -semi- α -open, but $\{x\} \cap A = \emptyset$ implies $x \notin ij - S_\alpha - cl(A)$,

a contradiction, hence $(ij - S_\alpha - cl(A)) - A = \emptyset$. Therefore A is ij -semi- α -closed, and so (X, τ_1, τ_2) is ij -semi- α - $T_{1/2}$ -space. ■

Theorem 3.10 :

If (X, τ_1, τ_2) is ij -semi- α - $T_{1/2}$ -space then it is ij -semi- α - T_0 -space .

Proof :

Suppose (X, τ_1, τ_2) is ij -semi- α - $T_{1/2}$ -space by theorem (3.9) every singleton is either ij -semi- α -closed or ij -semi- α -open. Let $x \neq y$ (in X), if $\{x\}$ is ij -semi- α -closed then $X - \{x\}$ is ij -semi- α -open set containing y but not containing x ; and if $\{x\}$ ij -semi- α -open then it is containing x but not containing y . So (X, τ_1, τ_2) is ij -semi- α - T_0 -space. ■

Remark 3.11 :

The converse of theorem (3.10) is not true , see the following example .

Example 3.12 :

Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}, \{a, b\}\}$, $\tau_2 = D$ (the discrete topology on X) then $12 - S_\alpha O(X) = \tau_1$ and (X, τ_1, τ_2) is a $12 - \text{semi} - \alpha - T_0$ - space but not $12 - \text{semi} - \alpha - T_{\frac{1}{2}}$ - space (since $\{b\}$ is neither $12 - \text{semi} - \alpha$ - closed nor $12 - \text{semi} - \alpha$ - open) .

Theorem 3.13 :

Every $i - T_{\frac{1}{2}}$ - space is $ij - \text{semi} - \alpha - T_{\frac{1}{2}}$ - space .

Proof :

It follows the fact that $\tau_i \subseteq ij - S_\alpha O(X)$ and theorem (3.9) . ■

Remark 3.14 :

The converse of theorem (3.13) is not true , see the following example .

Example 3.15 :

Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}, \{a, b\}\}$, $\tau_2 = I$ (the indiscrete topology on X) then $12 - S_\alpha O(X) = \tau_1 \cup \{\{a, c\}\}$ and (X, τ_1) is not $1 - T_{\frac{1}{2}}$ - space but (X, τ_1, τ_2) is $12 - \text{semi} - \alpha - T_{\frac{1}{2}}$ - space .

Definition 3.16 :

A bitopological space (X, τ_1, τ_2) is said to be $ij - \text{semi} - \alpha - T_1$ - space if any two distinct points in X are $ij - \text{semi} - \alpha$ - separated .

Proposition 3.17 :

A bitopological space (X, τ_1, τ_2) is $ij - \text{semi} - \alpha - T_1$ - space iff every singleton subset $\{x\}$ of X is $ij - \text{semi} - \alpha$ - closed .

Proof :

Suppose (X, τ_1, τ_2) is $ij - \text{semi} - \alpha - T_1$ - space , and $x \in X$, if $y \in ij - S_\alpha - cl(\{x\})$ but $x \neq y$ then $ij - S_\alpha - cl(\{y\}) \subseteq ij - S_\alpha - cl(\{x\})$. On the other hand by definition (3.16) we have $\{y\} \cap ij - S_\alpha - cl(\{x\}) = \phi$ which is a contradiction ,so $ij - S_\alpha - cl(\{x\}) = \{x\}$ i.e $\{x\}$ is $ij - \text{semi} - \alpha$ - closed set .

Conversely if for each x , $\{x\}$ is $ij - \text{semi} - \alpha$ - closed then $ij - S_\alpha - cl(\{x\}) = \{x\}$ and any two distinct points of X are $ij - \text{semi} - \alpha$ - separated i.e (X, τ_1, τ_2) is $ij - \text{semi} - \alpha - T_1$ - space . ■

Remark 3.18 :

Every $i - T_1$ - space is $ij - \text{semi} - \alpha - T_1$ - space.

Proof :

It follows from the fact that in $i - T_1$ - space every singleton is an i - closed set in (X, τ_1) also any i - closed set in (X, τ_1) is $ij - \text{semi} - \alpha$ - closed set in (X, τ_1, τ_2) . ■

Remark 3.19 :

The converse of remark (3.18) is not true , see the following example .

Example 3.20 :

Let $X = \{a, b, c, d\}$, $\tau_1 = \{X, \phi, \{a\}, \{d\}, \{a, d\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}\}$, and $\tau_2 = \{X, \phi, \{a\}, \{d\}, \{a, d\}\}$ then $12 - S_\alpha O(X) = \tau_1 \cup \{\{a, b\}, \{a, c\}, \{b, d\}, \{c, d\}, \{a, c, d\}, \{a, b, d\}\}$. So (X, τ_1, τ_2) is $12 - \text{semi} - \alpha - T_1$ - space since all singletons are $12 - \text{semi} - \alpha$ - closed sets but (X, τ_1) is not a $1 - T_1$ - space since $\{b\}$ and $\{c\}$ are not 1 - closed sets .

Theorem 3.21 :

Every subspace of $ij - \text{semi} - \alpha - T_1$ - space is $ij - \text{semi} - \alpha - T_1$ - space .

Proof :

Let $(Y, \tau_{1Y}, \tau_{2Y})$ be a subspace of a bitopological space (X, τ_1, τ_2) and let (X, τ_1, τ_2) is ij -semi- α - T_1 -space . To prove that Y is ij -semi- α - T_1 -space , let $y_1 \neq y_2 \in Y$. Since $Y \subseteq X$, then $y_1 \neq y_2 \in X$ and since X is ij -semi- α - T_1 -space . Then there exist two ij -semi- α -open sets G, H in X , such that $y_1 \in G, \text{but } y_2 \notin G$; and $y_2 \in H, \text{but } y_1 \notin H$. Then we obtain two sets $G_1 = G \cap Y, H_1 = H \cap Y$ are ij -semi- α -open sets in Y , we have $y_1 \in G_1, \text{but } y_2 \notin G_1$; and $y_2 \in H_1, \text{but } y_1 \notin H_1$. Hence $(Y, \tau_{1Y}, \tau_{2Y})$ is ij -semi- α - T_1 -space . ■

Remark 3.22 :

Every ij -semi- α - T_1 -space is ij -semi- α - T_0 -space , but the converse is not true .

Proof :

This follows directly from the definitions (3.2) and (3.16) . ■

But the converse is not true as the following example .

Example 3.23 :

Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}\}$, and $\tau_2 = \{X, \phi, \{a\}, \{a, b\}\}$, the family of all 12-semi- α -open sets of X is : $12-S_\alpha O(X) = \{X, \phi, \{a\}, \{a, b\}, \{a, c\}\}$. If we take a and $b, a \neq b$, then we can not find two 12-semi- α -open sets , such that one of them contains a but not b and the other contains b but not a . Therefore (X, τ_1, τ_2) is not 12-semi- α - T_1 -space , but it is clear that (X, τ_1, τ_2) is ij -semi- α - T_0 -space .

Theorem 3.24 :

If (X, τ_1, τ_2) is ij -semi- α - T_1 -space then it is ij -semi- α - $T_{1/2}$ -space .

Proof :

It follows from theorem (3.9) and proposition (3.17) . ■

Remark 3.25 :

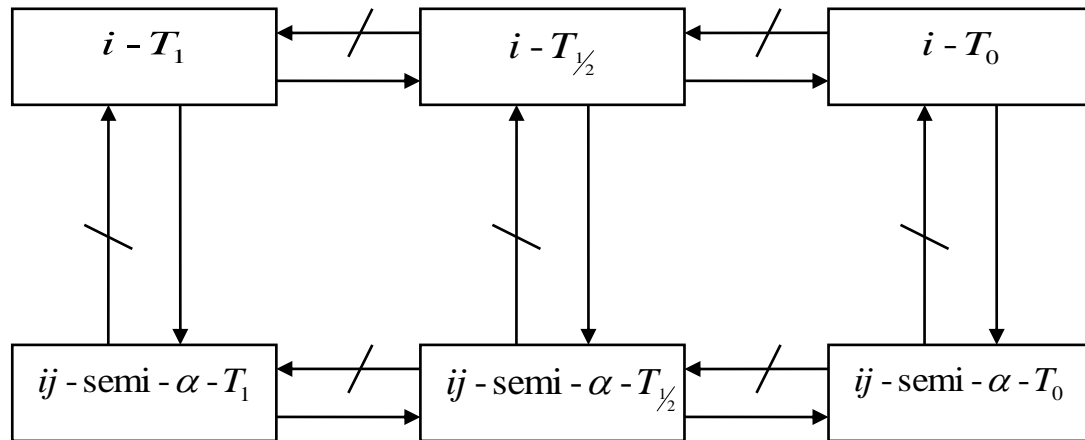
The converse of theorem (3.24) is not true , see the following example .

Example 3.26 :

Let $X = \{a, b\}$, $\tau_1 = \{X, \phi, \{a\}\}$, $\tau_2 = D$ (the discrete topology on X) , then $12-S_\alpha O(X) = \tau_1$, and (X, τ_1, τ_2) is 12-semi- α - $T_{1/2}$ -space (since $\{a\}$ is 12-semi- α -open and $\{b\}$ is 12-semi- α -closed) , but not 12-semi- α - T_1 -space since $\{a\}$ is not 12-semi- α -closed .

Remark 3.27 :

The following diagram shows the relations between $i - T_k$ and $ij - \text{semi} - \alpha - T_k$ spaces , where $k = 0, \frac{1}{2}, 1$:



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