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On the Symmetric Inverse Semigroup
حول شبه الزمرة التناظرية العكسية

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Abstract:

In this paper we obtain a formula for the order of the I_n semigroup (The symmetric Inverse Semigroup) and a formula to find the number of idempotent elements in it also we prove that this number always even.

الخلاصة:

تناولنا في هذا البحث شبه الزمرة التناظرية العكسية حيث اوجدنا صيغة لحساب عدد العناصر فيها ثم اوجدنا صيغة لحساب عدد العناصر المتساوية القوىفيها كما اثبتنا ان هذا العدد دائما عدد زوجي.

1. Introduction:

Let $X_n=\{1,2,...,n\}$ then a partial transformation $\alpha: Dom\,\alpha \subseteq X_n \to \mathrm{Im}\alpha \subseteq X_n$ is said to be *a full or total transformation* if $Dom\,\alpha = X_n$, otherwise it is called *strictly partial* .Three fundamental semigroups of transformations under the usual composite that have been extensively studied are : T_n *the full transformation semigroup* (or *the symmetric semigroup*); I_n *,the semigroup of partial one-one mappings* (or *the Symmetric inverse semigroup*); and P_n *the semigroup of partial transformations*(or the *Partial symmetric semigroup*), [1],[2] .

In *a* semigroup *S*,an element *a* in *S* is called *idempotent* if $a^2=a$,[3],[4].

Definition1.1: Let S be a non empty set and * the associative binary operation on it then (S,*) is called semigroup ,[3],[4].

Definition1.2: Let T_n be the set of all partial transformation $\alpha : X_n = \{1,\dots,n\} \to X_n = \{1,\dots,n\}$ and ○ the usual composite then (T_n, ○) is called the full transformation semigroup (or the symmetric Semigroup),[1],[2].

Definition1.3: Let I_n be the set of all partial one-one transformation $\alpha : \mathrm{Dom}\,\alpha \subseteq X_n \to \mathrm{Im}\alpha \subseteq X_n$ and ○ the usual composite then (I_n, ○) is called the semigroup of partial one-one mappings(or the symmetric inverse semigroup),[1].[2].

Definition1.4:Consider $X_n=\{1,2,\dots,n\}$ and let $\alpha \in I_n$,the height of α is $|I\mathrm{m}\alpha|$,[1],[2].

Definition 1.5: The order of a semigroup S is the number of its element if S is finite, otherwise S is of infinite order .[3] .

Remark: Let $\alpha \in I_n$ we will use $\alpha(x) = -$ if $x \notin \mathrm{Dom}\,\alpha$ and $\alpha(x_1)=\alpha(x_2)=...=\alpha(x_n)=-$ to denote the zero element of the semigroup I_n if $|Dom\alpha|=0$.

2. The Main Result

Example 2.1: I_3 is the set of all mapping is $\alpha_1, \alpha_2,\dots\dots, \alpha_{34}$,where

if $Dom\,\alpha = x_3$ and contains three elements

$\alpha_1(1)=1,\ \alpha_1(2)=2,\ \alpha_1(3)=3$
$\alpha_2(1)=1, \alpha_2(2)=3, \alpha_2(3)=2$
$\alpha_3(1)=3,\ \alpha_3(2)=1,\ \alpha_3(3)=2$
$\alpha_4(1)=2,\ \alpha_4(2)=1,\ \alpha_4(3)=3$
$\alpha_5(1)=3,\ \alpha_5(2)=2,\ \alpha_5(3)=1$

$\alpha_6(1)=2,\ \alpha_6(2)=3,\ \alpha_6(3)=1$

if $Dom\,\alpha \subset x_3$ and contains two elements

$\alpha_7(1)=1,\ \alpha_7(2)=2$
$\alpha_8(1)=2,\ \alpha_8(2)=1$
$\alpha_9(1)=1,\ \alpha_9(2)=3$
$\alpha_{10}(1)=3,\ \alpha_{10}(2)=1$
$\alpha_{11}(1)=2,\ \alpha_{11}(2)=3$
$\alpha_{12}(1)=3,\ \alpha_{12}(2)=2$
$\alpha_{13}(1)=1,\ \alpha_{13}(3)=3$
$\alpha_{14}(1)=3,\ \alpha_{14}(3)=1$
$\alpha_{15}(1)=1,\ \alpha_{15}(3)=2$
$\alpha_{16}(1)=2,\ \alpha_{16}(3)=1$
$\alpha_{17}(1)=2,\ \alpha_{17}(3)=3$
$\alpha_{18}(1)=3,\ \alpha_{18}(3)=2$
$\alpha_{19}(2)=2,\ \alpha_{19}(3)=3$
$\alpha_{20}(2)=3,\ \alpha_{20}(3)=2$
$\alpha_{21}(2)=1,\ \alpha_{21}(3)=2$
$\alpha_{22}(2)=2,\ \alpha_{22}(3)=1$
$\alpha_{23}(2)=1,\ \alpha_{23}(3)=3$
$\alpha_{24}(2)=3,\ \alpha_{24}(3)=1$

if $Dom\,\alpha \subset x_3$ and contains one element

$\alpha_{25}(1)=1,\ \alpha_{26}(1)=2,\ \alpha_{27}(2)=1,\ \alpha_{28}(1)=3,$,f $\alpha_{29}(2)=2,$
$\alpha_{30}(2)=3,\ \alpha_{31}(3)=1,\ \alpha_{32}(3)=2,\ \alpha_{33}(3)=3$ and $\alpha_{34}=$ the zero element .

because the work is easy with matrices more than mapping we can write the mapping above as matrices.for the mapping sending i to j we put one in ij-th position and zero's elsewhere for instance the matrices of the mappings above are:

$$\alpha_1=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\quad \alpha_2=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix},\quad \alpha_3=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$$

$$\alpha_4=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix},\quad \alpha_5=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix},\quad \alpha_6=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$$

$$\alpha_7=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix},\quad \alpha_8=\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix},\quad \alpha_9=\begin{bmatrix}1&0&0\\0&0&1\\0&0&0\end{bmatrix}$$

$$\alpha_{10}=\begin{bmatrix}0&0&1\\1&0&0\\0&0&0\end{bmatrix},\quad \alpha_{11}=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix},\quad \alpha_{12}=\begin{bmatrix}0&0&1\\0&1&0\\0&0&0\end{bmatrix}$$

$$\alpha_{13}=\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix},\quad \alpha_{14}=\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix},\quad \alpha_{15}=\begin{bmatrix}1&0&0\\0&0&0\\0&1&0\end{bmatrix}$$

$\alpha_{16}=\begin{bmatrix}010\\000\\100\end{bmatrix}$, $\alpha_{17}=\begin{bmatrix}010\\000\\001\end{bmatrix}$, $\alpha_{18}=\begin{bmatrix}001\\000\\010\end{bmatrix}$

$\alpha_{19}=\begin{bmatrix}000\\010\\001\end{bmatrix}$, $\alpha_{20}=\begin{bmatrix}000\\001\\010\end{bmatrix}$, $\alpha_{21}=\begin{bmatrix}000\\100\\010\end{bmatrix}$

$\alpha_{22}=\begin{bmatrix}000\\010\\100\end{bmatrix}$, $\alpha_{23}=\begin{bmatrix}000\\100\\001\end{bmatrix}$, $\alpha_{24}=\begin{bmatrix}000\\001\\100\end{bmatrix}$

$\alpha_{25}=\begin{bmatrix}100\\000\\000\end{bmatrix}$, $\alpha_{26}=\begin{bmatrix}010\\000\\000\end{bmatrix}$, $\alpha_{27}=\begin{bmatrix}000\\100\\000\end{bmatrix}$

$\alpha_{28}=\begin{bmatrix}001\\000\\000\end{bmatrix}$, $\alpha_{29}=\begin{bmatrix}000\\010\\000\end{bmatrix}$, $\alpha_{30}=\begin{bmatrix}000\\001\\000\end{bmatrix}$

$\alpha_{31}=\begin{bmatrix}000\\000\\100\end{bmatrix}$, $\alpha_{32}=\begin{bmatrix}000\\000\\010\end{bmatrix}$, $\alpha_{33}=\begin{bmatrix}000\\000\\001\end{bmatrix}$, $\alpha_{34}=\begin{bmatrix}000\\000\\000\end{bmatrix}$

The idempotent elements for I_3 are

$\alpha_{1}=\begin{bmatrix}100\\010\\001\end{bmatrix}$, $\alpha_{7}=\begin{bmatrix}100\\010\\000\end{bmatrix}$, $\alpha_{13}=\begin{bmatrix}100\\000\\001\end{bmatrix}$

$\alpha_{19}=\begin{bmatrix}000\\010\\001\end{bmatrix}$, $\alpha_{55}=\begin{bmatrix}100\\000\\000\end{bmatrix}$, $\alpha_{29}=\begin{bmatrix}000\\010\\000\end{bmatrix}$, $\alpha_{33}=\begin{bmatrix}000\\000\\001\end{bmatrix}$, $\alpha_{34}=\begin{bmatrix}000\\000\\000\end{bmatrix}$.

This can be easily checked by multiplying each matrix above by itself , we get the same matrix.

In this paper we are interested in finding the order of the Symmetric inverse semigroup and the number of the idempotent elements in it .

propostion 2.1: The order of the Symmetric inverse semigroup is

$$|I_n| = \sum_{0}^{n}\binom{n}{r}^2 r!.$$

Proof: First, the elements of I_n are one -one elements of P_n since $I_n \subseteq P_n$ therefore $|Dom\,\alpha| = |\mathrm{Im}\,\alpha|\ \forall\ \alpha \in I_n$,

For if,for example $|Dom\,\alpha|. > |\mathrm{Im}\,\alpha|$, it would follow that $\alpha \notin I_n$ since it will be not one-one.

Second,we can choose the domain of α in $\binom{n}{r}$ ways ,

where $r=0, \ldots, n$ and to determine the number of elements of I_r for each choice of $Dom\,\alpha$ since $|Dom\,\alpha| = |\mathrm{Im}\,\alpha|$ we note that there are r choices for the image of the first element in the Dom α , there $r-1$ choices for the image of the second element in the Dom α , there $r-2$ choices for the image of the third element in the Dom α ,etc,thus there is

$r(r\text{-}1)(r\text{-}2)$ ... 2 1=$r!$ for each $r=0$, ... , n , so there are $\binom{n}{r} r!$ elements $\forall\, r = 0$, ... ,n.

Finally we can choose the image of α in $\binom{n}{r}$ ways since $|Dom\,\alpha| = |\mathrm{Im}\,\alpha|$ in I_n ,so there are $\binom{n}{r}\binom{n}{r} r!$ elements for each $r=0$, ... , n ; so the order of I_n is given by

$$\sum_{0}^{n}\binom{n}{r}^2 \text{ r!}.$$

Theorem 2.1: The number of idempotent element in I_n semigroup is given by

$$H_n = \sum_{r=0}^{n}\binom{n}{r}.$$

Proof: The semigroup I_n contains all one-one mapping with

Dom $\{x_1, x_2, ..., x_r\} \subseteq X_n$ and of height s , we can choose the domain of α in $\binom{n}{r}$ ways ,we choose the elements of $\mathrm{Im}\,\alpha \subseteq \{x_1, x_2, ..., x_s\}$ where $s = |\mathrm{Im}\,\alpha|$, $0 \le s = r$.

Let $\alpha \in I_n$, if $\alpha\,(i)=j$ and $j \notin Dom\ \alpha$ then $\alpha\,(\alpha\,(i))= \alpha\,(j)=-$,so $\alpha^2 \ne \alpha$ so we must cancel each mapping with image have element different from the element in the $Dom\ \alpha$. Therefore we have the mapping such that $Im\,\alpha = Dom\ \alpha = \{x_1, x_2, ..., x_r\} \subseteq X_n$, and α is one-one that is mean we deal with the Symmetric group (S_r) for each r , $r=0$, ... , n,where $S_r \subset I_r\ \forall\, r = 0$, ... ,n.

so the only one idempotent exist in I_r which is the identity element in S_r .Now since $r=0$, ... , n; and we can choose the Domain of α in $\binom{n}{r}$ ways so there exist $\binom{n}{r}$.1 idempotent elements $\forall\, r = 0$, ... ,n,i.e.; there exist

$\binom{n}{1}+\binom{n}{2}+\ldots+\binom{n}{n} = \sum_{r=0}^{n}\binom{n}{r}$ idempotent element in I_n that is mean

$$H_n = \sum_{r=0}^{n}\binom{n}{r}.$$

Corollary2.1: The number of idempotent in I_n is even .

Proof: since $H_n = \sum_{r=0}^{n}\binom{n}{r}$ and $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\ldots+\binom{n}{n}=2^n$,[5], so this number always even.

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