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On the Probability Density Function of the Non-Central χ^2 Distribution

حول دالة الكثافة الاحتمالية لتوزيع مربع كاي اللامركزي

Jinan Hamzah Farhood
Department of Mathematics - College of Education
University of Babylon , 2012

Abstract

In this paper, we consider the probability density function (pdf) of a non-central χ^2 distribution with odd number of degrees of freedom *n*. This pdf is represented in the literature as an infinite sum. Kettani [10] presented two alternative expressions to this pdf. The first expression is in terms of the partial derivative of the hyperbolic cosine function and the second expression, on the other hand, is a finite sum representation of $(n+1)/2$ terms only instead of the infinite sum.

In this paper, we prove Theorem 3.4 that implies if $m = 0$ which introduced by Kettani [10].

Also, before the end section of this paper, we present four different theorems includes the pdf for the non-central χ^2 distribution which are general recurrence relation for such pdf.

الخلاصة

إن دالة الكثافة*n*. في هذا البحث, نعتبر دالة الكثافة الاحتمالية لتوزيع مربع كاي اللامركزي ذات درجة حرية فردية وهي الاحتمالية هذه تمثل جمع لانهائي من الحدود. في عام 2006 قدم العالم كيتاني تعبيرين بديلين لدالة الكثافة الاحتمالية هذه إن التعبير الأول من ناحية الاشتقاق الجزئي لدالة الجيب تمام والتعبير الثاني من الناحية الأخرى يمثل الجمع النهائي ل $(n+1)/2$ من الحدود
بدلا من الجمع اللانهائي.
تساوي صفر. نقدم قبل m في هذا البحث, نبرهن نظرية 4.3 التي قدمت من قبل العالم كيتاني والتي تحققت إذا كانت قيمة الجزء الأخير من هذا البحث (الجزء الرابع) أربع نظريات تتضمن دالة الكثافة الاحتمالية لتوزيع مربع كاي اللامركزي والتي مثلت علاقة عامة لهذه الدالة.

1. Introduction

The non-central χ^2 distribution is studied by other researchers whose worked in our field by taking different subjects of this distribution which as the following:

The computation of the non-central chi-square distribution studied by Robertson [12] but the noncentral chi-square distribution in misspecified structural equation models: finite sample results from a monte carlo simulation introduced by Curran, Bollen, Paxton, Kirby & Chen [3] . χ^2 and non-central χ^2 distributions defined by Ben-Haim [2] and calculations for the noncentral chi-distribution studied by Kent [9] while hand-book on statistical distributions for experimentalists presented by Walck [14]. On the trivariate non-central chi-squared distribution stated by the Dharmawansa, Rajatheva and Tellambura [4] while the new series representation for the trivariate non-central chi-squared distribution derived by Dharmawansa, Rajatheva and Tellambura [5].

The non-central chi-square distribution used with other distributions by stating different subjects of these distributions such as:

Krishnamoorthy [11] introduced the computing discrete mixtures of continuous distributions: noncentral chisquare, noncentral *t* and the distribution of the square of the sample multiple correlation coefficient while Hélie [7] presented the understanding statistical power using noncentral probability distributions: Chi-squared, G-squared, and ANOVA.

The non-central χ^2 distribution is very close to the normal distribution and consequently frequently occurs in finance, estimation theory, decision theory and time series analysis (see [13] for examples and details). In addition, and for numerical evaluation purpose, the infinite sum is the pdf of non-central chi-squared distribution tends to be approximated by a finite sum.

This paper was divided in five sections which are the following.

In section 2, we introduce the partial derivative and probability density function of the non - central χ^2 distribution with odd degree of freedom.

In section 3, Kettani [10] presented two different representations of the pdf. The first is in terms of the partial derivative of the hyperbolic cosine function and the second is a finite sum representation instead of the infinite sum in equation (1).

In this section, Theorem 3.4 was stated by Kettani [10] and we will prove this theorem by using Theorem 3.1 which presented in the same section.

In section 4, we consider the case when the degree of freedom n is odd, and we will introduce four different Theorems (4.1, 4.2, 4.3 and 4.4) of the pdf for the non-central χ^2 distribution which are general recurrence relations for the pdf.

Concluding remarks was presented in section 5.

2. Partial Derivative and Probability Density Function of the Non-Central χ^2 Distribution with Odd Degree of Freedom [1], [8], [10]

A non-central χ^2 distribution with *n* degrees of freedom is the distribution of the sum of the squares of n random variables that are normally distributed with unit variance and nonzero means.

In other words, let $x_i \sim N(\mu_i,1)$, and $y=\sum_{i=1}^{n} x_i^2$. Then the distribution of *y* is a non-central χ^2 with *n* degrees of freedom and non-centrality parameter $\lambda=\sum_{i=1}^{n}\mu_i^2$.

The probability density function (pdf) of such distribution is expressed as

$$f_n^{\lambda}(y)=2^{-\frac{n}{2}}\exp\left[-\frac{1}{2}(y+\lambda)\right]\sum_{i=0}^{\infty}\frac{y^{(n/2)+i-1}\lambda^i}{\Gamma\left(\frac{n}{2}+i\right)2^{2i}i!} \tag{1}$$

where $\Gamma(.)$ is the Gamma function.

When $\lambda=0$, (equivalently, the means μ_i are zero), then this distribution is reduced to the central χ^2 or simply χ^2. The pdf of this distribution is given by

$$f_n^0(y)=\frac{y^{(n/2)-1}\exp(-y/2)}{2^{n/2}\Gamma(n/2)}$$

The pdf of non-central χ^2 distribution can be expressed as an infinite weighted sum of central χ^2 pdf as follows:

$f_n^{\lambda}(y)=\sum_{i=0}^{\infty}p^{\lambda/2}(i)f_{n+2i}^{0}(y)$, where

$p^{\theta}(i)=e^{-\theta}\theta^i/i!$ is the Poisson pdf with parameter θ see [8].

The pdf in equation (1) can also be expressed as

$$f_n^{\lambda}(y)=\exp\left[-\frac{1}{2}(\lambda+y)\right]\frac{1}{2}\left(\frac{y}{\lambda}\right)^{(n-2)/4} I_{(n-2)/2}\left(\sqrt{\lambda y}\right) \tag{2}$$

where $I_\alpha(.)$ is the modified Bessel function of the first kind of degree α and nonetheless is given by the following infinite sum

$$I_\alpha(y) = (y/2)^\alpha \sum_{i=0}^{\infty} \frac{(y/2)^{2i}}{i!\Gamma(\alpha+i+1)} . \qquad (3)$$

See [6, pp.900-932] for more information on various kinds of Bessel functions and some of the identities and approximations associated with them. See also chapter 29 of [8] for detailed discussion on non-central χ^2 distribution .

3. Alternative Expressions

To prove our theorems that introduced in section 4 we need the following theorems which presented by Kettani [10] which included that the set forth alternative expressions to the pdf of a non-central χ^2 distribution when the number of degrees of freedom is odd. Accordingly, Theorem 3.1 expresses the pdf in terms of the m^{th} partial derivative of the hyperbolic cosine function. Theorem 3.2 presents a finite sum representation of the modifield Bessel function instead of the infinite sum in equation (3). This result is needed to prove Theorem 3.3 which presents a finite sum representation of the pdf that consists of $(n+1)/2$ terms only instead of the infinite sum in equation (1) .

3.1 Theorem (partial Derivative Theorem) [1], [6], [10]

For $n = 2m+1, m \in \mathrm{N},$ the pdf of a non-central χ^2 distribution is given by

$$f(y) = \frac{\exp\left[-\frac{1}{2}(y+\lambda)\right]}{\sqrt{2\pi y}} (2y/\lambda)^m \frac{\partial^m \cosh\left(\sqrt{\lambda y}\right)}{\partial y^m} \qquad (4)$$

Proof:

We first write

$$f(y) = \exp\left[-\frac{1}{2}(y+\lambda)\right] g(y)$$

where ,

$$g(y) = 2^{-\frac{n}{2}} \sum_{j=0}^{\infty} \frac{y^{(n/2)+j-1}\lambda^j}{\Gamma\left(\frac{n}{2}+j\right) 2^{2j}\, j!}$$

We next substitute $n = 2m+1$ and get

$$g(y) = 2^{-m-\frac{1}{2}} \sum_{j=0}^{\infty} \frac{y^{m+j-\frac{1}{2}}\lambda^j}{\Gamma\left(m+j+\frac{1}{2}\right) 2^{2j}\, j!}$$

Next we use the identity(see [6, p.888])

$$\Gamma\left(m+\frac{1}{2}\right) = \frac{(2m)!\sqrt{\pi}}{2^{2m}\, m!}$$

$$g(y) = \left(1/\sqrt{\pi}\right)(2y)^{m-\frac{1}{2}} \sum_{j=0}^{\infty} \frac{(y\lambda)^j (m+j)!}{[2(m+j)]!\, j!}$$

We then make the following change of variables $t = m+j$, and get

$$g(y)=\left(1/\sqrt{\pi}\right)(2y)^{m-\frac{1}{2}}\sum_{t=m}^{\infty}\frac{(y\lambda)^{t-m}t!}{(2t)!(t-m)!}$$

Note now that the Maclaurin series expansion of the hyperbolic cosine function is given by (see [6, p.41])

$$\cosh(y)=\sum_{j=0}^{\infty}\frac{y^{2j}}{(2j)!}$$

Thus, it can be shown by induction that

$$\frac{\partial^m \cosh\left(\sqrt{\lambda y}\right)}{\partial y^m}=\sum_{t=m}^{\infty}\frac{t!\lambda^t y^{t-m}}{(2t)!(t-m)!}$$

Therefore,

$$g(y)=\frac{(2y/\lambda)^m}{\sqrt{2\pi y}}\frac{\partial^m \cosh\left(\sqrt{\lambda y}\right)}{\partial y^m}$$

and this concludes the proof.

In the next theorem, we present a new expression for the modified Bessel function of the first kind when the degree $\alpha=(-1)^j\left(m+\frac{1}{2}\right)$

3.2 Theorem [6], [10]

Let $\alpha=(-1)^j\left(m+\frac{1}{2}\right)$, where *m* is a non-negative integer, the modified Bessel function of the first kind is given by

$$I_{(-1)^j\left(m+\frac{1}{2}\right)}(z)=\sqrt{\frac{2}{\pi z}}\sum_{i=0}^{m}(-1)^i(2i-1)!!C_{m-i}^{m+i}z^{-i}\cosh_{m-i+j+1}(z) \qquad (5)$$

Proof:

A finite sum representation of the modified Bessel function of the first kind is presented in [6] as follows:

$$I_{\pm\left(m+\frac{1}{2}\right)}(z)=\frac{1}{\sqrt{2\pi z}}\left(e^z\sum_{i=0}^{m}\frac{(-1)^i(m+i)!}{i!(m-i)!(2z)^i}\pm(-1)^{m+1}e^{-z}\sum_{i=0}^{m}\frac{(m+i)!}{i!(m-i)!(2z)^i}\right)$$

Now by substituting $(-1)^j$ for $\pm$ and rearranging the terms, we get

$$I_{(-1)^j\left(m+\frac{1}{2}\right)}(z)=\frac{1}{\sqrt{2\pi z}}\sum_{i=0}^{m}\frac{(-1)^i(m+i)!}{i!(m-i)!(2z)^i}\left(e^z+(-1)^{m-i+j+1}e^{-z}\right)$$

$$=\sqrt{\frac{2}{\pi z}}\sum_{i=0}^{m}(-1)^i(2i-1)!!C_{m-i}^{m+i}z^{-i}\cosh_{m-i+j+1}z$$

The next theorem presents a finite sum expression of the non-central χ^2 distribution when the number of degree of freedom is odd.

3.3 Theorem [10]

For $n = 2m+1, m \in \mathrm{N},$ the non-central χ^2 distribution is given by

$$f_{2m+1}^{\lambda}(y) = \frac{\exp\left[-\frac{1}{2}(y+\lambda)\right]}{\sqrt{2\pi y}}\left(\frac{y}{\lambda}\right)^{\frac{m}{2}}\sum_{i=0}^{m-1}(-1)^{i}(\lambda y)^{-\frac{i}{2}}(2i-1)!!C_{m-1-i}^{m-1+i}\cosh_{m-i}\left(\sqrt{\lambda y}\right)$$

if $m \geq 1.$ (6)

Proof:

When $m \geq 1,$ we substitute $n = 2m+1$ in equation (2) and get

$$f_{2m+1}^{\lambda}(y) = \exp\left[-\frac{1}{2}(\lambda+y)\right]\frac{1}{2}\left(\frac{y}{\lambda}\right)^{(2m-1)/4} I_{m-\frac{1}{2}}\left(\sqrt{\lambda y}\right)$$

This, the result follows from Theorem 3.2, and this concludes the proof.

The next theorem presented by Kettani [10] and we prove this theorem by using Theorem 3.1 which given by:

3.4 Theorem

For $n = 2m+1, m \in \mathrm{N},$ the non-central χ^2 distribution is given by

$$f_{2m+1}^{\lambda}(y) = \frac{\exp\left[-\frac{1}{2}(y+\lambda)\right]}{\sqrt{2\pi y}}\cosh\left(\sqrt{\lambda y}\right) \qquad \text{if } m=0. \qquad (7)$$

Proof:

We first write

$$f(y) = \exp\left[-\frac{1}{2}(y+\lambda)\right]g(y)$$

where ,

$$g(y) = 2^{-\frac{n}{2}}\sum_{j=0}^{\infty}\frac{y^{(n/2)+j-1}\lambda^{j}}{\Gamma\left(\frac{n}{2}+j\right)2^{2j}\, j!}$$

Since, $n = 2m+1,$ we get

if m = 0 then $n = 1,$ we next substitute $n = 1$ and get

$$g(y) = 2^{-\frac{1}{2}}\sum_{j=0}^{\infty}\frac{y^{(1/2)+j-1}\lambda^{j}}{\Gamma\left(\frac{1}{2}+j\right)2^{2j}\, j!} = 2^{-\frac{1}{2}}\sum_{j=0}^{\infty}\frac{y^{j-\frac{1}{2}}\lambda^{j}}{\Gamma\left(j+\frac{1}{2}\right)2^{2j}\, j!}$$

Next we use the identity , $\Gamma\left(j+\frac{1}{2}\right) = \frac{(2j)!\sqrt{\pi}}{2^{2j}\, j!}$

$$g(y) = 2^{-\frac{1}{2}}\sum_{j=0}^{\infty}\frac{y^{j-\frac{1}{2}}\lambda^{j}}{\frac{(2j)!\sqrt{\pi}}{2^{2j}\, j!}2^{2j}\, j!} = 2^{-\frac{1}{2}}y^{-\frac{1}{2}}\sum_{j=0}^{\infty}\frac{(\lambda y)^{j}}{\sqrt{\pi}(2j)!} = \frac{1}{\sqrt{2\pi y}}\sum_{j=0}^{\infty}\frac{(\lambda y)^{j}}{(2j)!}$$

Note now that the Maclaurin series expansion of the hyperbolic cosine function is given

$$\cosh(y) = \sum_{j=0}^{\infty}\frac{y^{2j}}{(2j)!}$$

Thus, it can be shown by induction that

$$g(y)=\frac{1}{\sqrt{2\pi y}}\sum_{j=0}^{\infty}\frac{\left(\sqrt{\lambda y}\right)^{2j}}{(2j)!}=\frac{1}{\sqrt{2\pi y}}\cosh\left(\sqrt{\lambda y}\right)$$

$$f(y)=\exp\left[-\frac{1}{2}(y+\lambda)\right]g(y)=\frac{\exp\left[-\frac{1}{2}(y+\lambda)\right]}{\sqrt{2\pi y}}\cosh\left(\sqrt{\lambda y}\right)$$

$$\therefore\ f_1^{\lambda}(y)=\frac{\exp\left[-\frac{1}{2}(y+\lambda)\right]}{\sqrt{2\pi y}}\cosh\left(\sqrt{\lambda y}\right)$$

and this concludes the proof.

Notations [10]

(1) $n!!$ is the double factorial function defined as

$n!!=n(n-2)$.......3.1 if n > 0 is odd

$=n(n-2)$.......4.2 if n > 0 is even

$=1$ if n = -1,0.

(2) C_i^j where i and j non-negative integers with $j\geq i$ denotes the binomial coefficient defined as

$$C_i^j=\frac{j!}{i!(j-i)!}$$

(3) $\cosh_i(z)$ is alternate hyperbolic cosine/sine function defined as

$\cosh_i(z)=\cosh(z)$ if i is even

$=\sinh(z)$ if i is odd

i.e. $\cosh_i(z)=\dfrac{e^z+(-1)^i e^{-z}}{2}$.

4. Main Result

In this section, we introduce and prove several theorems that set alternative expressions to the pdf of a non-central χ^2 distribution when the number of degrees of freedom is odd.

The next four theorems present a general recurrence relation for the pdf of the non-central χ^2 distribution with odd number of degrees of freedom. Writing $n=2m+1$, first and third theorems presents such recurrence in the case when m is even such that $(m=2v+2)$ and $(m=4v)$, respectively, while second and fourth theorems presents such recurrence in the case when m is odd such that $(m=2v+3)$ and $(m=4v+1)$, respectively.

Using Theorems 3.3 and 3.4 we obtain the following theorems:

4.1 Theorem

For $n=4v+5$, v is an integer with $v\geq 1$, the following recurrence holds true for the non-central χ^2 distribution

$$f_{4v+5}^{\lambda}(y)=\sum_{t}\lambda^{-v-t-2}y^{v-t}(4t+3)!!C_{2(v-t)-1}^{2(v+t)+3}f_{1}^{\lambda}(y)-\sum_{t}\lambda^{-v-t-2}y^{v-t-1}(4t+5)!!C_{2(v-t-1)}^{2(v+t+2)}f_{3}^{\lambda}(y)$$

Proof:

First note from Theorem 3.4 that

$$f_{1}^{\lambda}(y)=\frac{\exp\left[-\frac{1}{2}(y+\lambda)\right]}{\sqrt{2\pi y}}\cosh\left(\sqrt{\lambda y}\right),\text{ and}$$

$$f_{3}^{\lambda}(y)=\frac{\exp\left[-\frac{1}{2}(y+\lambda)\right]}{\sqrt{2\pi\lambda}}\sinh\left(\sqrt{\lambda y}\right)$$

Next, for $m=2v+2$ with $v\geq 1,$ we have from Theorem 3.3

$$f_{4v+5}^{\lambda}(y)=\frac{\exp\left[-\frac{1}{2}(y+\lambda)\right]}{\sqrt{2\pi y}}\left(\frac{y}{\lambda}\right)^{v+1}h(y),\text{ where}$$

$$h(y)=\sum_{i=0}^{2v+1}(-1)^{i}(\lambda y)^{\frac{-i}{2}}(2i-1)!!C_{2v+1-i}^{2v+1+i}\cosh_{2v+2-i}\sqrt{\lambda y}$$

$$=\sum_{\substack{i=0\\ ieven}}^{2v+1}(\lambda y)^{\frac{-i}{2}}(2i-1)!!C_{2v+1-i}^{2v+1+i}\cosh_{2v+2-i}\sqrt{\lambda y}-\sum_{\substack{i=0\\ iodd}}^{2v+1}(\lambda y)^{\frac{-i}{2}}(2i-1)!!C_{2v+1-i}^{2v+1+i}\cosh_{2v+2-i}\sqrt{\lambda y}$$

Now, substitute $i=2t+2$, and $i=2t+3$ when i is even and odd, respectively, and get

$$h(y)=\cosh\sqrt{\lambda y}\sum_{t}(\lambda y)^{-t-1}(4t+3)!!C_{2(v-t)-1}^{2(v+t)+3}-\sinh\sqrt{\lambda y}\sum_{t}(\lambda y)^{-t-\frac{3}{2}}(4t+5)!!C_{2(v-t-1)}^{2(v+t+2)}$$

$$\because\ f_{4v+5}^{\lambda}(y)=\frac{\exp\left[-\frac{1}{2}(y+\lambda)\right]}{\sqrt{2\pi y}}\left(\frac{y}{\lambda}\right)^{v+1}h(y)$$

$$f_{4v+5}^{\lambda}(y)=\sum_{t}\lambda^{-v-t-2}y^{v-t}(4t+3)!!C_{2(v-t)-1}^{2(v+t)+3}\frac{\exp\left[-\frac{1}{2}(y+\lambda)\right]}{\sqrt{2\pi y}}\cosh\sqrt{\lambda y}$$

$$-\sum_{t}\lambda^{-v-t-2}y^{v-t-1}(4t+5)!!C_{2(v-t-1)}^{2(v+t+2)}\frac{\exp\left[-\frac{1}{2}(y+\lambda)\right]}{\sqrt{2\pi\lambda}}\sinh\sqrt{\lambda y}$$

$$f_{4v+5}^{\lambda}(y)=\sum_{t}\lambda^{-v-t-2}y^{v-t}(4t+3)!!C_{2(v-t)-1}^{2(v+t)+3}f_{1}^{\lambda}(y)-\sum_{t}\lambda^{-v-t-2}y^{v-t-1}(4t+5)!!C_{2(v-t-1)}^{2(v+t+2)}f_{3}^{\lambda}(y)$$

4.2 Theorem

For $n = 4v + 7,$ v is an integer with $v \geq 1,$ the following recurrence holds true for the non-central χ^2 distribution

$$f_{4v+7}^{\lambda}(y) = \sum_t \lambda^{-v-t-2} y^{v-t} (4t+3)!!C_{2(v-t)}^{2(v+t+2)} f_3^{\lambda}(y) - \sum_t \lambda^{-v-t-3} y^{v-t} (4t+5)!!C_{2(v-t)-1}^{2(v+t)+5} f_1^{\lambda}(y)$$

Proof:
First note from Theorem 3.4 that

$$f_1^{\lambda}(y) = \frac{\exp\left[-\frac{1}{2}(y+\lambda)\right]}{\sqrt{2\pi y}} \cosh\left(\sqrt{\lambda y}\right), \text{ and}$$

$$f_3^{\lambda}(y) = \frac{\exp\left[-\frac{1}{2}(y+\lambda)\right]}{\sqrt{2\pi\lambda}} \sinh\left(\sqrt{\lambda y}\right)$$

Next, for $m = 2v + 3$ with $v \geq 1,$ we have from Theorem 3.3

$$f_{4v+7}^{\lambda}(y) = \frac{\exp\left[-\frac{1}{2}(y+\lambda)\right]}{\sqrt{2\pi y}} \left(\frac{y}{\lambda}\right)^{v+\frac{3}{2}} h(y), \text{ where}$$

$$h(y) = \sum_{i=0}^{2v+2} (-1)^i (\lambda y)^{\frac{-i}{2}} (2i-1)!!C_{2(v+1)-i}^{2(v+1)+i} \cosh_{2v+3-i} \sqrt{\lambda y}$$

$$= \sum_{\substack{i=0 \\ i\,even}}^{2v+2} (\lambda y)^{\frac{-i}{2}} (2i-1)!!C_{2(v+1)-i}^{2(v+1)+i} \cosh_{2v+3-i} \sqrt{\lambda y} - \sum_{\substack{i=0 \\ i\,odd}}^{2v+1} (\lambda y)^{\frac{-i}{2}} (2i-1)!!C_{2(v+1)-i}^{2(v+1)+i} \cosh_{2v+3-i} \sqrt{\lambda y}$$

Now, substitute $i = 2t + 2$, and $i = 2t + 3$ when i is even and odd, respectively, and get

$$h(y) = \sinh\sqrt{\lambda y} \sum_t (\lambda y)^{-t-1} (4t+3)!!C_{2(v-t)}^{2(v+t+2)} - \cosh\sqrt{\lambda y} \sum_t (\lambda y)^{-t-\frac{3}{2}} (4t+5)!!C_{2(v-t)-1}^{2(v+t)+5}$$

$$\because\ f_{4v+7}^{\lambda}(y) = \frac{\exp\left[-\frac{1}{2}(y+\lambda)\right]}{\sqrt{2\pi y}} \left(\frac{y}{\lambda}\right)^{v+\frac{3}{2}} h(y)$$

$$f_{4v+7}^{\lambda}(y) = \sum_t \lambda^{-v-t-2} y^{v-t} (4t+3)!!C_{2(v-t)}^{2(v+t+2)} \frac{\exp\left[-\frac{1}{2}(y+\lambda)\right]}{\sqrt{2\pi\lambda}} \sinh\sqrt{\lambda y}$$

$$- \sum_t \lambda^{-v-t-3} y^{v-t} (4t+5)!!C_{2(v-t)-1}^{2(v+t)+5} \frac{\exp\left[-\frac{1}{2}(y+\lambda)\right]}{\sqrt{2\pi y}} \cosh\sqrt{\lambda y}$$

$$f_{4v+7}^{\lambda}(y) = \sum_t \lambda^{-v-t-2} y^{v-t} (4t+3)!!C_{2(v-t)}^{2(v+t+2)} f_3^{\lambda}(y) - \sum_t \lambda^{-v-t-3} y^{v-t} (4t+5)!!C_{2(v-t)-1}^{2(v+t)+5} f_1^{\lambda}(y)$$

4.3 Theorem

For $n = 8v + 1,$ v is an integer with $v \geq 1,$ the following recurrence holds true for the non-central χ^2 distribution

$$f_{8v+1}^{\lambda}(y) = \sum_t \lambda^{-2(v+t)} y^{2(v-t)} (8t-1)!!C_{4(v-t)-1}^{4(v+t)-1} f_1^{\lambda}(y) - \sum_t \lambda^{-2(v+t)} y^{2(v-t)-1} (8t+1)!!C_{4(v-t)-2}^{4(v+t)} f_3^{\lambda}(y)$$

Proof:

First note from Theorem 3.4 that

$$f_1^{\lambda}(y) = \frac{\exp\left[-\frac{1}{2}(y+\lambda)\right]}{\sqrt{2\pi y}} \cosh\left(\sqrt{\lambda y}\right), \text{ and}$$

$$f_3^{\lambda}(y) = \frac{\exp\left[-\frac{1}{2}(y+\lambda)\right]}{\sqrt{2\pi\lambda}} \sinh\left(\sqrt{\lambda y}\right)$$

Next, for $m = 4v$ with $v \geq 1,$ we have from Theorem 3.3

$$f_{8v+1}^{\lambda}(y) = \frac{\exp\left[-\frac{1}{2}(y+\lambda)\right]}{\sqrt{2\pi y}} \left(\frac{y}{\lambda}\right)^{2v} h(y), \text{ where}$$

$$h(y) = \sum_{i=0}^{4v-1} (-1)^i (\lambda y)^{\frac{-i}{2}} (2i-1)!!C_{4v-1-i}^{4v-1+i} \cosh_{4v-i} \sqrt{\lambda y}$$

$$= \sum_{\substack{i=0 \\ ieven}}^{4v-1} (\lambda y)^{\frac{-i}{2}} (2i-1)!!C_{4v-1-i}^{4v-1+i} \cosh_{4v-i} \sqrt{\lambda y} - \sum_{\substack{i=0 \\ iodd}}^{4v-1} (\lambda y)^{\frac{-i}{2}} (2i-1)!!C_{4v-1-i}^{4v-1+i} \cosh_{4v-i} \sqrt{\lambda y}$$

Now, substitute $i = 4t$, and $i = 4t+1$ when i is even and odd, respectively, and get

$$h(y) = \cosh\sqrt{\lambda y} \sum_t (\lambda y)^{-2t} (8t-1)!!C_{4(v-t)-1}^{4(v+t)-1} - \sinh\sqrt{\lambda y} \sum_t (\lambda y)^{-2t-\frac{1}{2}} (8t+1)!!C_{4(v-t)-2}^{4(v+t)}$$

$$\because \; f_{8v+1}^{\lambda}(y) = \frac{\exp\left[-\frac{1}{2}(y+\lambda)\right]}{\sqrt{2\pi y}} \left(\frac{y}{\lambda}\right)^{2v} h(y)$$

$$f_{8v+1}^{\lambda}(y) = \sum_t \lambda^{-2(v+t)} y^{2(v-t)} (8t-1)!!C_{4(v-t)-1}^{4(v+t)-1} \frac{\exp\left[-\frac{1}{2}(y+\lambda)\right]}{\sqrt{2\pi y}} \cosh\sqrt{\lambda y}$$

$$- \sum_t \lambda^{-2(v+t)} y^{2(v-t)-1} (8t+1)!!C_{4(v-t)-2}^{4(v+t)} \frac{\exp\left[-\frac{1}{2}(y+\lambda)\right]}{\sqrt{2\pi\lambda}} \sinh\sqrt{\lambda y}$$

$$f_{8v+1}^{\lambda}(y) = \sum_t \lambda^{-2(v+t)} y^{2(v-t)} (8t-1)!!C_{4(v-t)-1}^{4(v+t)-1} f_1^{\lambda}(y) - \sum_t \lambda^{-2(v+t)} y^{2(v-t)-1} (8t+1)!!C_{4(v-t)-2}^{4(v+t)} f_3^{\lambda}(y)$$

4.4 Theorem

For $n = 8v + 3$, v is an integer with $v \geq 1$, the following recurrence holds true for the non-central χ^2 distribution

$$f_{8v+3}^{\lambda}(y) = \sum_t \lambda^{-2(v+t)} y^{2(v-t)} (8t-1)!!C_{4(v-t)}^{4(v+t)} f_3^{\lambda}(y) - \sum_t \lambda^{-2(v+t)-1} y^{2(v-t)} (8t+1)!!C_{4(v-t)-1}^{4(v+t)+1} f_1^{\lambda}(y)$$

Proof:

First note from Theorem 3.4 that

$$f_1^{\lambda}(y) = \frac{\exp\left[-\frac{1}{2}(y+\lambda)\right]}{\sqrt{2\pi y}} \cosh\left(\sqrt{\lambda y}\right), \text{ and}$$

$$f_3^{\lambda}(y) = \frac{\exp\left[-\frac{1}{2}(y+\lambda)\right]}{\sqrt{2\pi\lambda}} \sinh\left(\sqrt{\lambda y}\right)$$

Next, for $m = 4v + 1$ with $v \geq 1$, we have from Theorem 3.3

$$f_{8v+3}^{\lambda}(y) = \frac{\exp\left[-\frac{1}{2}(y+\lambda)\right]}{\sqrt{2\pi y}} \left(\frac{y}{\lambda}\right)^{2v+\frac{1}{2}} h(y), \text{ where}$$

$$h(y) = \sum_{i=0}^{4v} (-1)^i (\lambda y)^{\frac{-i}{2}} (2i-1)!!C_{4v-i}^{4v+i} \cosh_{4v+1-i} \sqrt{\lambda y}$$

$$= \sum_{\substack{i=0 \\ ieven}}^{4v} (\lambda y)^{\frac{-i}{2}} (2i-1)!!C_{4v-i}^{4v+i} \cosh_{4v+1-i} \sqrt{\lambda y} - \sum_{\substack{i=0 \\ iodd}}^{4v} (\lambda y)^{\frac{-i}{2}} (2i-1)!!C_{4v-i}^{4v+i} \cosh_{4v+1-i} \sqrt{\lambda y}$$

Now, substitute $i = 4t$, and $i = 4t + 1$ when i is even and odd, respectively, and get

$$h(y) = \sinh\sqrt{\lambda y} \sum_t (\lambda y)^{-2t} (8t-1)!!C_{4(v-t)}^{4(v+t)} - \cosh\sqrt{\lambda y} \sum_t (\lambda y)^{-2t-\frac{1}{2}} (8t+1)!!C_{4(v-t)-1}^{4(v+t)+1}$$

$$\because \; f_{8v+3}^{\lambda}(y) = \frac{\exp\left[-\frac{1}{2}(y+\lambda)\right]}{\sqrt{2\pi y}} \left(\frac{y}{\lambda}\right)^{2v+\frac{1}{2}} h(y)$$

$$f_{8v+3}^{\lambda}(y) = \sum_t \lambda^{-2(v+t)} y^{2(v-t)} (8t-1)!!C_{4(v-t)}^{4(v+t)} \frac{\exp\left[-\frac{1}{2}(y+\lambda)\right]}{\sqrt{2\pi\lambda}} \sinh\sqrt{\lambda y}$$

$$- \sum_t \lambda^{-2(v+t)-1} y^{2(v-t)} (8t+1)!!C_{4(v-t)-1}^{4(v+t)+1} \frac{\exp\left[-\frac{1}{2}(y+\lambda)\right]}{\sqrt{2\pi y}} \cosh\sqrt{\lambda y}$$

$$f_{8v+3}^{\lambda}(y) = \sum_t \lambda^{-2(v+t)} y^{2(v-t)} (8t-1)!!C_{4(v-t)}^{4(v+t)} f_3^{\lambda}(y) - \sum_t \lambda^{-2(v+t)-1} y^{2(v-t)} (8t+1)!!C_{4(v-t)-1}^{4(v+t)+1} f_1^{\lambda}(y)$$

5. Conclusions

We presented four theorems that set alternative expressions to the pdf of a non-central χ^2 distribution and these four theorems presented a general recurrence relation for the pdf of the non-central χ^2 distribution when the number of degrees of freedom is odd. First and third theorems presented such recurrence in the case when m is even while second and fourth theorems presented such recurrence in the case when m is odd.

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