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The Number of Idempotent Elements in Symmetric Semigroup T_n

عدد العناصر المتساوية القوى في شبه الزمرة التناظرية T_n

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Abstract:

The main purpose of this paper to find the number of the idempotent elements in sym T_n Semigroup .

The problem of finding the number of idempotent elements in the symmetric semigroup is solved by using the partitions of an *N*-set ;we have found that :

$$U_n = \sum_{b_1+2b_2+\ldots+nb_n=n} P(b_1,\ldots,b_n)\,\Pi\ parts$$

Such that $P(b_1, \ldots ,b_n)$ is the number of partition for a given partition and by *parts* we mean the parts for the given partition .Some other results concerning U_n have been established .

الخلاصة :

الهدف الرئيسي من هذا البحث هو ايجاد عدد العناصر المتساوية القوى في شبه الزمرة التناظرية T_n .

مشكلة ايجاد عدد العناصر المتساوية القوى في شبه الزمرة التناظرية تم حلها بأستحدام التجزئة على المجموعة التي تحوي n من العناصر حيث وجدنا ان

$$U_n = \sum_{b_1+2b_2+\ldots+nb_n=n} P(b_1,\ldots,b_n)\,\Pi\ parts$$

حيث ان $P(b_1, \ldots ,b_n)$ هو العدد الكلي للتجزئة المعطاة للمجموعة N التي تحتوي على n من العناصر ونعني بـــ *parts* اجزاء التجزئة المعطاة . ثم برهنا بعض النتائج الأخرى المتعلقة بـــ U_n .

تم بعد ذلك تقديم طرق أخرى لحل المشكلة أعلاه وبأستخدام العلاقة المرتدة ودالة التوليد

1.Introduction :

Let T_n be the semigroup of all mappings of the set $N=\{1, \ldots ,n\}$ into itself under the operation of composition of mappings ,inside this semigroup we have the group S_n of all one to one and onto mappings of the set *N* onto itself. We have in T_n idempotent elements that is mappings f with the property $fof=f$, it is our aim in this work is to find the number of idempotent elements in T_n for $n \in N$.

Let U_n be the number of idempotent elements in T_n .It is shown that:

$$U_n = \sum_{b_1+2b_2+\ldots+nb_n=n} P(b_1,\ldots,b_n)\,\Pi\ parts$$

Then we prove that if n is even number then U_n is odd number and if n is odd number then U_n is even number .

2.Defintions and Notations:

Definition 2.1 [1],[2] **:**

Let X be a finite set and T_X be a set of all mapping from X into X ,then T_X with binary operation the composition of mapping form a semigroup called the *Symmetric Semigroup* denoted by T_X .If $|X|=n$, then we can identify X with the set $\{1, \ldots ,n\}$ and write T_n.

Definition2.2 [1],[2],[3] :
Let *(S,*)* be a semigroup and let $y \in S$, then we say that *y* is an idempotent element in *S* if *y*y=y* .
Definition 2.3 [4],[5],[6] :
Partition of a set is decomposition of the set into cells such that every element of the set exactly in one of the cells.

The number of Partitions of the integer n into exactly m classes is denoted by P_n^m . Hence the total number of Partitions of *n* into *m* or fewer parts: $P_n^1 + \ldots + P_n^m$.

If *m=n* , this number is denoted by *P(n),* the number of all Partitions of the integer *n*.

Example 2.1:

The partition of	are	whence
2	2,11	$P_2^1=1=P_2^2$,
3	3,21,111	$P_3^1=P_3^2=P_3^3=1$,

Thus, P(2)=2,P(3)=3.

If $\partial=(\partial_1, \ldots, \partial_P)$ is a partition of *n* , ∂ may also can be written as $1^{r_1}2^{r_2}\ldots n^{r_n}$.
Where r_i is the number of parts equal to i in ∂ , i varying in *{1,2, ... }*.For example ,all the following denote the same partition of 17:
4+3+3+2+2+2+1 , 4332221 (condensed as $4\ 3^2\ 2^3\ 1$;
in the other relation, it is ($1\ 2^3\ 3^2\ 4$), and we say that ∂ is a partition of type $1^{r_1}2^{r_2}\ldots n^{r_n}$.
Theorem 2.1 [7] :
The number of mapping $f : N \to R$ is :

$$|Map(N,R)|=r^n \ , \text{ such that } |N|=n\ ,\ |R|=r\ .$$

Proposition 2.1 [3] :
Denote $Per(b_1, \ldots, b_n)$ the number of permutation of an *n*-set *N* of type $1^{b_1}\ldots n^{b_n}$ and by $P(b_1, \ldots, b_n)$ the number of Partitions of *N* of type $1^{b_1}\ldots n^{b_n}$ .Then

$$i)\ Per(b_1, \ldots, b_n) = \begin{cases} \dfrac{n!}{b_1!\ \ldots\ b_n!\ 2^{b_2}\ldots n^{b_n}} & if\ n=\sum_{i=1}^{n} ib_i \\ 0 & otherwis \end{cases}$$

$$ii)\ P(b_1, \ldots, b_n) = \begin{cases} \dfrac{n!}{b_1!\ \ldots\ b_n!\ (2!)^{b_2}\ldots(n!)^{b_n}} & if\ n=\sum_{i=1}^{n} ib_i\ . \\ 0 & otherwise \end{cases}$$

3.The Main result

The subset S_n of T_n consisting of all one to one and onto mappings form a subsemigroup and infact it is a group of all permutations of *n* letters laying inside T_n which is called *Symmetric group*.
Now inside T_n we have idempotent elements for example T_3:
The set of all mapping is $\{f_1,f_2, \ldots, f_{27}\}$ where:

$f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$, $f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}$, $f_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix}$, $f_4 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix}$,$f_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix}$,

$f_6 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \end{pmatrix}$, $f_7 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}$, $f_8 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}$,$f_9 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 1 \end{pmatrix}$, $f_{10} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{pmatrix}$,

$f_{11} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$,$f_{12} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$,$f_{13} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \end{pmatrix}$,$f_{14} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \end{pmatrix}$,$f_{15} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \end{pmatrix}$,

$f_{16} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 3 \end{pmatrix}$,$f_{17} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 3 \end{pmatrix}$,$f_{18} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix}$,$f_{19} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \end{pmatrix}$,$f_{20} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \end{pmatrix}$,

$f_{21} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 3 \end{pmatrix}$,$f_{22} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$,$f_{23} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$,$f_{24} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$,$f_{25} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$,

$f_{26} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$, $f_{27} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$.

The idempotent elements for T_3 are :

$f_{22} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$,$f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$, $f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}$,$f_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix}$,$f_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix}$,

$f_{12} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$, $f_{16} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 3 \end{pmatrix}$, $f_{18} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix}$,$f_{20} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \end{pmatrix}$, $f_4 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix}$

In this section we are interested finding the number of the idempotent elements in the T_n semigroup ,a few example show that :

N	Number of idempotent
1	1
2	3
3	10
4	41
5	196
6	1057
.	.
.	.
.	.

For example let us take T_4 we can make partition for it as follows :

1111 , 2 11 , 2 2 , 1 3 , 4 ;

211 connecting two elements {1,2,3,4}=partitioning the set {1,2,3,4} into cell contining two elements and two cells each containg one element= number of partition of 4 of type $1^2 2^1$.

Remark: Let $f \in T_n$,now to be f an idempotent element in T_n we have either f(i)=i for some $i \in N$ or if f(i)=j ,then we must have f(j)=j ,since otherwise if f(j)=k ,where $k \neq j$ then (fof)(i)=f(f(i))=f(j)=k$\neq$f(i) whence $f^2 \neq f$, so the set N can be partitioned into subsets either singleton subsets {i} if f(i)=i or to subsets S_k consisting k elements in the other cases.

For example in T_3

f_{22}={{1},{2},{3}}, f_1={1,2,3}, f_2={1,2,3}, f_4={1,2,3},f_5={{1,2},{3}},
f_{12}={{1,3},{2}},f_{16}={{1,3},{2}},f_{18}={2,3},{1}},f_{20}={{2,3},{1}}, ,f_4={{1,2},{3}}

consider the three partitions of *3* which are:
$1+1+1=1^3$
1+2=12
3=3

So we have the number of idempotent elements in T_3 equal to

$$\frac{3!}{3!}\times 1+\frac{3!}{1!2!}\times 1\times 2+\frac{3!}{3!}\times 3$$
$$=1+6+3=10$$

Theorem 3.1: The number of idempotent elements in T_n semigroup is

$$U_n=\sum_{b_1+2b_2+\ldots+nb_n=n} P(b_1,\ldots,\mathrm{b_n})\,\Pi\ parts$$

Proof*:* The idempotent elements in T_n are mappings sending the elements of the partitioned subsets $S_1,S_2,\ldots,S_k$ into one of the elements in the subset S_i such that S_k is a subset of the partition contains k elements ,so we have for each partition of n of type $1^{b_1}\ \ldots\ n^{b_n}$ idempotent elements as many as the number of $|S_1|\ldots|S_k|$ which implies that as many as the product of the parts of the partition and since by proposition 2.1. the number of partition of type $1^{b_1}\ \ldots\ n^{b_n}$ is $p(b_1,\ldots,b_n)$, therefore :

$$U_n=\sum_{b_1+2b_2+\ldots+nb_n=n} P(b_1,\ldots,\mathrm{b_n})\,\Pi\ parts$$

4. Another Way to Find The Number of Idempotent Elements of Symmetric Semigroup.

Definition 4.1 [8] *:*

For a sequence of number $(a_0,\ \ldots\ ,a_n,\ \ldots\ )$ an equation relating a number a_n to some of its predecessors in the sequence , for any *n*, is called a recurrence relation .

Example 4.1:

The sequence $\{a_n\}=\{1,2,3,5,8,\ldots\}$ which is called Fibonacci sequence given by the boundary conditions $a_1=1$,$a_2=2$
and the recurrence relation $a_n=a_{n-1}+a_{n-2}$,$n\geq 3$.
Solving this recurrence relation means obtaining a formula for the *n*th term a_n as a function of n .

By using recurrence relation we have the number of idempotent elements [8], given by :

$$U_{m+1}=\sum_{j=0}^{m}\binom{m}{j}(\mathrm{j}+1)\,\mathrm{U_{m-j}}$$ with boundary condition :$U_0=1,U_1=1$.

For example:

$$U_{3+1}=U_4=\sum_{j=0}^{3}\binom{3}{j}(j+1)\,U_{3\text{-}j}=\binom{3}{0}.1.\ U_3+\binom{3}{1}.\ 2.U_2+\binom{3}{2}.3.U_1+\binom{3}{3}.4.U_0$$
$$=\ 10+3.2.3+3.3.1+1.4.1$$
$$=41$$

Definition 4.2 [8] **:**

Let $(a_0,\ \ldots\ ,a_r,\ \ldots\ )$ be a sequence of numbers,
the function $F(x)=a_0M_0(x)+\ldots+a_rM_r(x)+\ldots$; is called the ordinary generating function of the sequence $(a_0,\ \ldots\ ,a_r,\ \ldots\ )$, where $a_0M_0(x),\ldots,\ a_rM_r(x)\ldots$ is a sequence of function of x that are used as indicators . If $M_x=x^r$, in this case for the sequence $(a_0,\ \ldots\ ,a_r,\ \ldots\ )$, we have

$F(x)=a_0+a_1x+\ldots+a_rx^r+\ldots.$

By using generating functions we have the number of idempotent elements in[8]given by :

$$\sum_{n=0}^{\infty}U_n\frac{Z^n}{n!}=\exp(Ze^Z)$$

$$\sum_{n=0}^{\infty} U_n \frac{Z^n}{n!} = U_0 + U_1 \frac{Z_1}{1!} + U2\frac{Z^2}{2!} + U_3 \frac{Z^3}{3!}.+ \ \ . \ .$$

$$\exp(Ze^Z) = (1 + Z + \frac{Z^2}{2!} + \frac{Z^3}{3!} + \ \ldots)(1 + Z^2 + \frac{Z^4}{2!} + \frac{Z^6}{3!} + \ldots)(1 + \frac{Z^3}{2!} + \frac{Z^6}{(2!)^2 3!} + \ldots)$$

So

$U_1Z=Z$ then $U_1=1$;

$U_2 \frac{Z^2}{2} = (1 + \frac{1}{2!})Z^2$ then $U_2=3$;

$\frac{1}{3!}U_3 Z^3 = (\frac{1}{2!} + 1 + \frac{1}{3!})Z^3 = \frac{10}{6} Z^3$ then $U_3=10$.

We will use the formula of U_n that is given by using recurrence relation to determined that U_n is even or odd .

Proposition 4.1:

If n is even in T_n then the number of idempotent is odd and if n is odd then the number of idempotent is even .

Proof: since $U_n = U_{m+1} = \sum_{j=0}^{m} \binom{m}{j} (j+1)\, U_{m-j}$ [8], (1)

if m+1=2 then $U_2 = 3$ note that it is odd, if m+1=3 then $U_3 = 10$ is even ,suppose that the statement of Proposition true for each positive integer less than m+1 ,we want to prove that it is true for m+1,note that in (1)each term contains one of U_{m-j} which is less than U_{m+1} and so the statement is true.

Now, let m+1 is even , if j is even then m-j is odd contrary, therfore if $\binom{m}{j}$ is even so $\binom{m}{j}(j+1)\, U_{m-j}$ is even and if it is odd then since $\binom{m}{r} = \binom{m}{m-r}, 1 \le r \le m$, [5] (2)

we have $\binom{m}{j}$ twice , first product by $U_{m-j}(j+1)$ and second by $U_j(m-j+1)$ which are both even except $\binom{m}{m-j} mU_1$ which is always odd, the sum of terms above is odd, add the terms contain $\binom{m}{j}$ even .Hence, U_{m+1} is odd.

Let m+1 is odd , if j is even(odd)then m-j is even (odd) therefore if $\binom{m}{j}$ is even so $\binom{m}{j}(j+1)\, U_{m-j}$ is even and if it is odd then by(2) we have $\binom{m}{j}$ twice , first product by $U_{m-j}(j+1)$ and second by $U_j(m-j+1)$ which are both odd(even),the number of this terms is even by(2), so the sum of it will be even , add the terms $\binom{m}{m/2}(m/2\ +1)\, U_{m/2}$ which is even where it is clear that $\binom{m}{m/2}$ always even when m even and m/2 positive integer .Hence, U_{m+1} is even.

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