Approximate Analytical Solutions for Large Flexural and Shear Deformations of Uniformly Loaded Simply Supported Bimodular Beam

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Abstract

In this paper an analytical approximate solution for large flexural deformations, shear deformations and shear stresses of a bimodular uniformly loaded simply supported beam has been developed. Verification for the solution has been performed using FEM analysis with ANSYS. The results of the program were very close the results of the analytical solution presented in this paper.

تحليل تقريبي للانحرافات الكبيرة للانحناء وانحرافات القص للعتبات مزدوجة معامل المرونه والمحملة بصورة منتظمة وبسيطة الاسناد

ظافر خليفة جدعان

مدرس مساعد في قسم الهندسة المدنية/ جامعة الانبار

الخلاصة

في هذا البحث تم تطوير تحليل تقريبي للانحرافات الكبيرة الناتجة عن الانحناء والقص بالاضافة لاجهادات القص لعتبة مزدوجة معامل المرونة ومحمله بصورة منتظمة وبسيطة الاسناد. تم تدقيق الحل بالمقارنة مع طريقة العناصر المحددة باستخدام برنامج ANSYS 11 حيث اعطى البرنامج نتائج قريبه جدا من النتائج المستحصلة من الحل المطروح في هذا البحث.

1. Introduction

Many researchers have studied the problem of large flexural and shear deformations, but up to the knowledge of the author no one has studied the analytical solution of large flexural and shear deformations of a uniformly loaded bimodular beam. The problems of large flexural and shear deformations of unimodular beams have been investigated by many researchers. The large deflection of beams has been investigated by Bisshopp and Drucker (1945) for a point load on a cantilever beam. Timoshenko and Gere (1961) developed the solution for axial load on a beam. Rohde (1953) developed the solution for uniform load on a cantilever beam. Law (1981) solved the problem for a point load at the tip of the beam and a uniform load combined.

Most of materials exhibit different tensile and compressive strains for the same stress applied in tension or compression. Classical theory of elasticity assumes that materials have the same elastic properties in tension and compression, but this is only a simplified model, and does not account for material nonlinearities. Many studies have indicated that most materials, including concrete, ceramics, graphite, and some composites, exhibit different tensile and compressive strains given the same stress applied in tension or compression (Jun-yi Sun *et al*. (2010)).

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Those materials exhibit different elastic moduli in tension and compression. Those materials are known as bimodular materials.

The elastic theory of bimodular solids was first proposed by Ambartsumyan (1982), and Medri (1982). By conducting experiments, Medri (1982) reached the conclusion that the curve of stress-strain ($\sigma-\epsilon$) at the point of origin for materials with different moduli is nonlinear. Doong and Chen (1984) developed a method for the analysis of different modulus problems based on an approximate trigonometric series. Zhang and Wang (1989) proposed the finite element method for different moduli structure. Srinivansan and Ramachandra (1989) applied a bimodulus finite element method to the calculation of large deflection of plates. Yang *et al.* (1992) presented a method using the initial stress-finite element method for the analysis of bimodular structures. Tseng and Lee (1995) used a finite strip method for the analysis of bimodular laminates. Ye (1997) and Ye *et al.* (2001) developed a finite element method in which variations of elastic modulus are different from that of Poissiôn's ratio. Tseng and Jiang (1998) used the bimodulus theory to analyse the stress of laminated structure. Liu and Zhang (2000) adopted the method of accelerating the convergence factor to increase the rate of convergence.

In this research, an approximate analytical approach has been adopted to solve the problem of large flexural and shear deformations of a uniformly loaded simply supported bimodular beam and the results obtained from this approximate analytical solution have been compared with numerical results obtained from FEM analysis using ANSYS, and the results found to be very close.

2. Problem Formulation

The shape of the deformed uniformly loaded simply supported beam inspires the following Equation for the deformed shape:

$$v_b(x) = a \sin \frac{\pi x}{l} \tag{1}$$

where, a :a constant.

$v_b(x)$: the deflection of the beam due to bending stresses only.

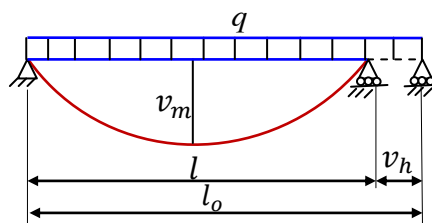l: the length of the projection of the deformed beam on the *x*-axis, as shown in Figure 1.

Figure 1. A Simply supported uniformly loaded beam.

Applying the B.Cs.:

$v_b(x=0)=0 \,\&\, M(x=0)=0$

$v_b(x=l)=0 \,\&\, M(x=l)=0$

and these conditions are already satisfied since $v_b(x=0)=0$, and $v_b(x=l)=0$

and from the symmetry of the deformed shape and the equation, one can conclude that:

$$v_b(x=l/2)=v_m \rightarrow a=v_m \rightarrow v_b(x)=v_m \sin \frac{\pi x}{l} \tag{2}$$

where, v_m: the maximum deflection of the beam due to bending stresses.

To simplify the calculations of the problem, an approximate formula for the radius of curvature instead of the exact radius of curvature would be used. The exact radius of curvature is

$$\frac{1}{\rho} = \frac{\dfrac{d^2 v_b}{dx^2}}{[1 + (\dfrac{d^2 v_b}{dx^2})^2]^{3/2}} \tag{3}$$

where, EI : flexure rigidity of the beam in which:
E: modulus of elasticity and,
I : moment of inertia of the cross section about the N.A.

Using an approximate formula for the curvature by applying Maclaurin series, as follows Equation (3) can be written in the form

$$\frac{1}{\rho} = \frac{d^2 v_b}{dx^2} \kappa 1(v_b') \tag{4}$$

where $\kappa 1(v_b') = \dfrac{1}{[1 + (\dfrac{d^2 v_b}{dx^2})^2]^{3/2}}$ $\quad and \quad v_b' = \dfrac{dv_b}{dx}$

Now using Maclaurin series for approximating $k1(v_b')$

$$\kappa 1(v_b') = \kappa 1(v_b' = 0) + \kappa 1'(v_b' = 0)v_b' + \kappa 1''(v_b' = 0)v_b'^2 / 2! + ... \tag{5}$$

It is easy to get the first two derivatives of $\kappa 1(v_b')$ and then substitute them into Equation (5) which yields the following approximate expression for degree of curvature

$$\frac{1}{\rho} = v_b'' \left(1 - \frac{3}{2} v_b'^2 \right) \tag{6}$$

Only the first two terms of Maclaurin series have been taken in the approximation of the function $k1(v_b')$ because the author found that the contribution of the third term was negligible in comparison to the tedious work added when the third term was included in the series. This will be made clear in the comparison with the FEM analysis of the same problem. The derivatives of $v_b(x)$ can be found using Equation (1) as follows:

$$v_b'(x) = \frac{\pi}{l} v_m \cos \frac{\pi x}{l} \quad and \quad v_b''(x) = -(\frac{\pi}{l})^2 v_m \sin \frac{\pi x}{l} \tag{7}$$

On the other side, the beam flexural formula is

$$\frac{1}{\rho} = \frac{-M(x)}{EI} \tag{8}$$

Substitute the two derivatives from Equation (7) into Equation (8) to get an expression for the bending moment $M(x)$ in terms of (v_m)

$$M(x) = EI(\frac{\pi}{l})^2 v_m \sin \frac{\pi x}{l} \left(1 - \frac{3}{2}(\frac{\pi}{l})^2 v_m^2 \cos^2 \frac{\pi x}{l} \right) \tag{9}$$

At $x = l/2$, $M(x)$ will equal to:

$$M(x = \frac{l}{2}) = EI(\frac{\pi}{l})^2 v_m \tag{10}$$

But for a uniformly loaded simply supported beam the bending moment after a horizontal displacement occurs can be calculated as follows
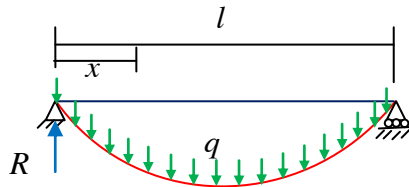
Figure 2. The reaction R after the in the deformed beam and a section (x) from the left end.

The reaction R will not be changed after the horizontal displacement occurs due to the assumption that the curved beam has the same length as that of the undeformed beam and to being the applied uniform load still vertical after deformation occurs. So this reaction will equal to

$$R = \frac{ql_o}{2} \text{ and the shear force : } V(x) = \frac{ql_o}{2} - ql_o\frac{x}{l} \tag{11}$$

where $x=0$, $V(x) = \frac{ql_o}{2}$ and when $x=l$, $V(x) = -\frac{ql_o}{2}$ and since the shear force relation is a linear function along the beam, so Equation (11) above will be suitable to take the effect of the deformed shape on the shear and hence on the bending moment values. So the bending moment at distance (x) from the left end equals

$$M(x) = \int_0^x V(t)dt = \frac{ql_o}{2}x - \frac{qx^2}{2}\frac{l_o}{l} \tag{12}$$

Hence the maximum bending moment will equal to

$$M(x = \frac{l}{2}) = \frac{ql_ol}{8} \tag{13}$$

Now equating the bending moment at the mid-span in Equations (10 and 13) yields,

$$\frac{ql_ol}{8} = EI(\frac{\pi}{l})^2 v_m \rightarrow v_m = \frac{ql_ol^3}{8\pi^2 EI} \tag{14}$$

Therefore the Equation of the deformed shape becomes

$$v_b(x) = \frac{ql_ol^3}{8\pi^2 EI}\sin\frac{\pi x}{l} \tag{15}$$

This equation uses the length (l) which needs the horizontal displacement, v_h, to be calculated first then

$$l = l_o - v_h \tag{16}$$

and, v_h, can be calculated from the following relation : The difference between the length of the curve of the deformed shape and the horizontal projection of the curve is equal to the horizontal displacement. Hence

$$v_h = \int_0^l ds - \int_0^l dx \tag{17}$$

and one could imagine this ,as the roller end moves toward its original position before the deformations by the loading occur. In calculus, the first term in Equation (17) which is the length of the curve can be written in the form

$$\int_0^l ds = \int_0^l \left(1 + (\frac{dv_b}{dx})^2\right)^{1/2} dx \tag{18}$$

Then Equation (17) takes the form

$$v_h = \int_0^l \left(1 + (\frac{dv_b}{dx})^2\right)^{1/2} dx - \int_0^l dx \tag{19}$$

The integrand in the right hand side of Equation (19) can be approximated using Maclaurin series in the same way used to approximate Equation (6) and the result is

$$v_h = \int_0^l \left(\frac{1}{2}(\frac{dv_b}{dx})^2\right) dx \tag{20}$$

Again the approximation used truncated Maclaurin series with the first two terms only.

Part of Equation (7) is restated, now, for convenience

$$v_b^{'}(x) = \frac{\pi}{l} v_m \cos\frac{\pi x}{l}$$

Substituting this derivative into Equation (20), then integrating, yields

$$v_h = \frac{(v_m \pi)^2}{4l} \tag{21}$$

Now substitute, v_m, from Equation (14), and recall that ($l=l_o-v_h$),yields

$$\left(\frac{ql_o}{16\pi EI}\right)^2 l^5 + l = l_0 \tag{22}$$

The above equation for a particular problem can be solved by trial and error to find the value of, *l*, and then the values of the vertical and horizontal displacements can be evaluated using the Equations (14) and (21) respectively.

2.1 Bimodularity of the Beam

In order to take the bimodularity of the beam into consideration, the following assumptions must be stated first

1- The material of the beam is homogeneous anisotropic.
2- The region of the cross sections subjected to compression stress has a modulus of elasticity called E_n and the region subjected to a tensile stress has a different modulus of elasticity called E_p as shown in Figure 3.
3- Straight planes of the cross sections of the beam before application the loads, remain plane after that application. Hence no shear deformations are assumed till now, but will be treated separately in the next phase in this paper).
4- The stress-strain relationship is bilinear as shown in Figure 3.
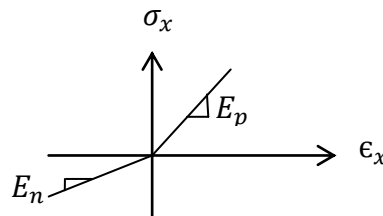

Figure 3. The stress strain-curve for a bimodular material

2.2 Locating the Neutral Axis

From reviewing literature regarding bimodular beams, the issue of locating the neutral axis (from now on written N.A.) was invariably determined by assuming that the summation of the axial forces on the cross sections of the beam equals zero. The author found it is convenient to review this issue

(also the section of calculating bending stresses) in here. And this is due to its importance in understanding its relation to the large deformations as well as the shear deformations of a bimodular beam.
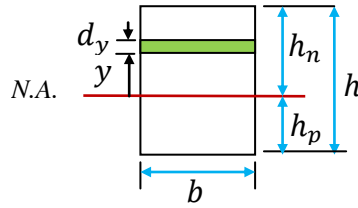

Figure 4. A cross section in a bimodular beam.

For the section of the beam shown in Figure 4

$$\sum F_x = 0 \rightarrow \int_{-h_p}^{h_n} \sigma_x b dy = \int_{-h_p}^{h_n} E\varepsilon_x b dy = \int_{-h_p}^{h_n} E\frac{y}{\rho} b dy = 0 \quad , \left(\varepsilon_x = \frac{y}{\rho} \right)$$

Remembering to set ($E=E_p$ when $y=h_p$ and $E=E_n$ when $y=h_n$) and combining this equation with the equation ($h_p + h_n = h$) and solving those two algebraic equations simultaneously give

$$h_p = \frac{\sqrt{E_n}}{\sqrt{E_n} + \sqrt{E_p}} h \quad and \quad h_n = \frac{\sqrt{E_p}}{\sqrt{E_n} + \sqrt{E_p}} h \tag{23}$$

2.3 Calculating Bending Stresses

Taking the sum of moments about N.A. equals zero yields:

$$\sum M_{N.A.} = M(x) \rightarrow \int_{-h_p}^{h_n} \sigma_x b y dy = \int_{-h_p}^{h_n} E\varepsilon_x b y dy = \int_{-h_p}^{h_n} E\frac{y^2}{\rho} b dy = M(x) \tag{24}$$

Also here it's needed to set ($E=E_p$ when $y=h_p$ and $E=E_n$ when $y=h_n$).Then rearranging this equation and making use of Equation (23) yield,

$$\frac{1}{\rho} = \frac{-M(x)}{E_r I} \tag{25}$$

where:

$I = \dfrac{bh^3}{12}$: moment of inertia of the cross section of the beam ,and

E_r: reduced modulus of elasticity for the bimodular beams and equals to

$$E_r = \frac{4E_p E_n}{(\sqrt{E_p} + \sqrt{E_p})^2} \tag{26}$$

 For a unimodular section ($E_n=E_p=E$) and by direct substitution in Equation (26) gives $E_r=E$.
The normal bending stress at any fibre within the cross section can be found using the flexural formula

$$\sigma_{xp} = E_p \varepsilon_x = E_p \frac{y}{\rho} = E_p \frac{yM}{E_r I} = \frac{My}{I} \cdot \frac{E_p}{E_r} \qquad \text{for tensile stress}$$

$$\sigma_{xn} = E_n \varepsilon_x = E_n \frac{y}{\rho} = E_n \frac{yM}{E_r I} = \frac{My}{I} \cdot \frac{E_n}{E_r} \qquad \text{for compressive stress}$$

$$\tag{27}$$

Now for the problem of large deformations discussed in the previous section, the expression of the curvature, $\dfrac{1}{\rho}$, is simply replaced with the new value stated in Equations (6) and (8), so all that one needs is just to replace the modulus of elasticity of the unimodular beam with the reduced modulus of elasticity E_r. Hence Equations (15) & (22) become, respectively

$$v_b(x) = \frac{q l_o l^3}{8\pi^2 E_r I} \sin\frac{\pi x}{l} \tag{28}$$

$$\left(\frac{q l_o}{16\pi E_r I}\right)^2 l^5 + l = l_0 \tag{29}$$

2.4 Shear Deformations and Shear Stresses

Referring to Figure 5 below, F_1 and F_2 are forces due to bending stresses where F_1 acts at a distance, x, from one end of the beam and F_2 acts at a distance, $x+dx$, from the same end. The symbol, τ, stands for shear stresses at a distance, x, from that end of the beam. Horizontal equilibrium in Figure 5 implies
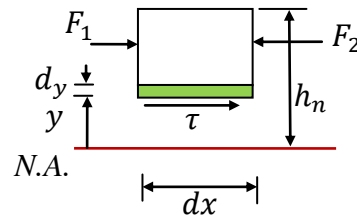

Figure 5. A cross section in the beam showing shear and bending stresses.

$$\sum F_x = 0 \rightarrow F_1 + \tau b\,dx - F_2 = 0$$

$$\rightarrow \int_y^{h_n} \sigma(x)b\,dy + \tau b\,dx - \int_y^{h_n} \sigma(x+dx)b\,dy = 0$$

$$\rightarrow \int_y^{h_n} \frac{M(x)y}{I}\cdot\frac{E_n}{E_r}b\,dy + \tau b\,dx - \int_y^{h_n} \frac{M(x+dx)y}{I}\cdot\frac{E_n}{E_r}b\,dy = 0$$

knowing that $M(x+dx) = M(x) + \dfrac{dM(x)}{dx}dx$ and $V(x) = \dfrac{dM(x)}{dx}$ is the shear force and substituting these values in the integration above , integrating and simplifying gives

$$\tau b\,dx = \int_y^{h_n} \frac{dM(x)y}{I}\cdot\frac{E_n}{E_r}b\,dy = dM(x)\int_y^{h_n} \frac{y}{I}\cdot\frac{E_n}{E_r}b\,dy$$

$$\tau b I = \frac{dM(x)}{dx}\frac{E_n}{E_r}\int_y^{h_n} y\,b\,dy = V(x)\frac{E_n}{E_r}\int_y^{h_n} y\,b\,dy$$

Now the quantity $\displaystyle\int_y^{h_n} y\,b\,dy = \frac{b}{2}(h_n^2 - y^2) = Q_n$ is the first moment of the area above, y, to the top surface of the beam at $(y=h_n)$. Hence the final formula for the shear stress at a point, y, above N.A.(which will be denoted now by $\tau_n(x)$ is

$$\tau_n(x) = \frac{VQ_n}{Ib}\frac{E_n}{E_r} = \frac{V}{2I}\frac{E_n}{E_r}(h_n^{\ 2} - y^2) \tag{30}$$

It's clear that maximum shear stress occurs at, $y = 0$, or at the N.A., denoting this maximum shear

stress by $\tau_{n\max}$, using Equation(23) to substitute the value of h_n, recalling that $E_r = \dfrac{4E_pE_n}{(\sqrt{E_p} + \sqrt{E_p})^2}$

, $\tau_{n\max}$ is found to be

$$\tau_{n\max}(x) = \frac{V}{2I}\frac{E_n}{E_r}\left(\frac{\sqrt{E_p}}{\sqrt{E_p} + \sqrt{En}}h\right)^2 \tag{31}$$

After some simplifications the above equation takes the form

$$\tau_{n\max}(x) = \frac{3}{2}\frac{V}{A} = \frac{3}{2}\tau_{avg} \tag{32}$$

where, *A=bh* : is the cross sectional area. It could be seen that this value is the same as in the unimodular beam. But one has to be aware that the general distribution of shear stresses is not the same as in unimodular beams, and this is due to the fact that the location of the neutral axis is different as well as the shear stress values other than that at the N.A. is different also, as it is apparent from Equation(30).

Now it's possible in a similar fashion to derive a formula for the shear stress at any point under the N.A.. This formula is

$$\tau_p(x) = \frac{VQ_p}{Ib}\frac{E_p}{E_r} = \frac{V}{2I}\frac{E_p}{E_r}(h_p^{\ 2} - y^2) \tag{33}$$

$$\tau_{p\max}(x) = \frac{3}{2}\frac{V}{A} = \frac{3}{2}\tau_{avg} \tag{34}$$

Since Equations (30) and (33) are parabolas and their values at the ends of the cross section are zeros, Figure 6 below shows sketches representing the results in those equations.
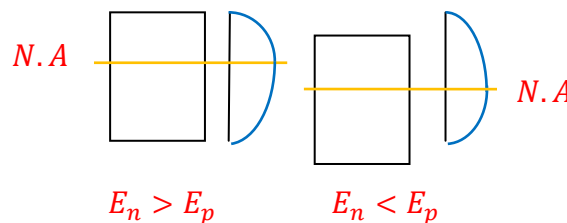

$$N.A \qquad\qquad\qquad\qquad\qquad\qquad N.A$$

$$E_n > E_p \qquad\qquad E_n < E_p$$

Figure 6. Shear stress distribution across the section of the beam.

2.5 Shear Deformations of a Bimodular Beam

According to Timoshenko's beam theory, the transverse deformations are not only bending deformations (those discussed in the preceding sections) but also shear deformations and the latter will be discussed in this section for bimodular beams. According to Timoshenko's beam theory, plane sections in beams will no longer remain plane sections when shear deformations are considered, but they will be curved as shown in Figure 7 which shows a deformed element in a beam due to shear stresses only. If the vertical sides of the elements at the neutral axis are assumed to remain vertical after deformations occur, then the slope of the deflection curve of the beam due to shear alone is approximately equal to the shear strain at the neutral axis γ_c. This is depicted in Figure 7. Hence the following relation can be deduced
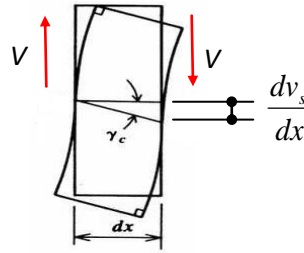
Figure 7. Shear stress deformations in the section of the beam.

$$\frac{dv_s}{dx} = \gamma_c = \frac{\tau_c}{G} \tag{35}$$

where,

v_s: transverse deflection due to shear stresses only.

G : shear modulus of elasticity.

τ_c: shear stress at the N.A. . In addition to that, the shear modulus of elasticity could be based on the modulus of elasticity E_n or based on E_p since the value of G is E-dependent according to Poisson's relation: $G = \dfrac{E}{2(1+\upsilon)}$.

The author suggests the following value for G:

$$G_r = \frac{E_r}{2(1+\upsilon)} \tag{36}$$

where:

G_r : the reduced shear modulus of elasticity (reduced because its value is always less than G for unimodular beam).

The author adopted Equation (36) because the flexural analysis of the bimodular beam led to the use of the traditional flexural formula, Equation (25), but this time with the reduced modulus of elasticity E_r a fact that inspired the use of Poisson's relation with E_r. Later this will show reasonable results when compared with FEM analysis. In addition, when the section is unimodular section then E_r and the value of $E_p = E_n = E$ in Equation (36) leads to the value G which is the value of shear modulus for a unimodular beam. Now substitute the maximum value of shear stress, Equation (32) or (34), into Equation (36), then

$$\frac{dv_s}{dx} = \frac{\tau_c}{G_r} = \frac{3}{2}\frac{V}{AG_r} \tag{37}$$

Using the expression for the shear force $V(x)$ from Equation (12), the following expression can be written for the differential shear deflection

$$dv_s(x) = \frac{3}{2}\frac{V}{AG_r}\left(0.5ql_o - qx\frac{l_o}{l}\right)dx \tag{38}$$

Substituting the expression of $V(x)$ from Equation (12), then

$$v_s(x) = \int_0^x dv_s(t)dt = \frac{3}{4}\frac{ql_o}{AG_r}\left(x - \frac{x^2}{l}\right) \tag{39}$$

The total transverse deflection $v_t(x)$ is obtained by adding the shear deformation using Equation (39) to the flexural deformation using Equation (15), then

$$v_t(x) = \frac{ql_o l^3}{8\pi^2 E_r I} \sin\frac{\pi x}{l} + \frac{3}{4}\frac{ql_o}{AG_r}\left(x - \frac{x^2}{l}\right) \tag{40}$$

At $x = l/2$ the value of v_b and v_s and hence the value of v_t will be maximum and equal to

$$v_{t\,max} = \frac{ql_o l^3}{8\pi^2 E_r I} + \frac{3}{16}\frac{ql_o l}{AG_r} \tag{41}$$

Knowing that $\dfrac{I}{A} = \dfrac{h^2}{12}$ and $G_r = \dfrac{E_r}{2}$ after taking (v) or Poisson's ratio equals to zero, with some rearrangement in Equation (41), then

$$v_{t\,max} = \frac{ql_o l^3}{8\pi^2 E_r I}\left(1 + (\frac{\pi h}{2l})^2\right) \tag{42}$$

The second term inside the parentheses is the contribution of the shear deflection to the total deflection.

3. Numerical Results

A simply supported beam with a span of 5.0 meters was taken as an example. The beam is uniformly loaded by 100 *kN/m*. The cross section of the beam is a rectangular one with width and total depth of (*b=150 mm and h=250 mm*), respectively. To study the effect of the variation of the modular ratio E_p/E_n on the stresses and on the deflection of the beam, different values of this ratio have been taken as follows: 1/2.5, 1/2.0, 1/1.5, 1 , 1.5/1, 2.0/1 and 2.5/1.

The results obtained using the formulas presented in this paper are listed in the following tables.

Table 1. Bending and shear deflections along the beam with different modular ratios.

	v_b (mm)				v_s (mm)			
x/l	$\frac{E_p}{E_n} = 1/2.5$ and 2.5/1	$\frac{E_p}{E_n} = 1/2.0$ and 2.0/1	$\frac{E_p}{E_n} = 1/1.5$ and 1.5/1	$\frac{E_p}{E_n} = 1$	$\frac{E_p}{E_n} = 1/2.5$ and 2.5/1	$\frac{E_p}{E_n} = 1/2.0$ and 2.0/1	$\frac{E_p}{E_n} = 1/1.5$ and 1.5/1	$\frac{E_p}{E_n} = 1$
0	0	0	0	0	0	0	0	0
1/4	187.1	137.6	117.4	95.0	1.23	0.90	0.77	0.62
3/8	244.5	180.2	153.5	124.2	1.54	1.13	0.96	0.78
1/2	265.7	194.6	166.1	134.4	1.65	1.22	1.04	0.83
5/8	244.5	180.2	153.5	124.2	1.54	1.13	0.96	0.78
3/4	187.1	137.6	117.4	95.02	1.23	0.90	0.77	0.62
1	0	0	0	0	0	0	0	0

Table 2. Total deflection and maximum shear stresses along the beam with different modular ratios.

	v_t (mm)				τ_{max} (MPa)			
x/l	$\frac{E_p}{E_n} = 1/2.5$ and 2.5/1	$\frac{E_p}{E_n} = 1/2.0$ and 2.0/1	$\frac{E_p}{E_n} = 1/1.5$ and 1.5/1	$\frac{E_p}{E_n} = 1$	$\frac{E_p}{E_n} = 1/2.5$ and 2.5/1	$\frac{E_p}{E_n} = 1/2.0$ and 2.0/1	$\frac{E_p}{E_n} = 1/1.5$ and 1.5/1	$\frac{E_p}{E_n} = 1$
0	0	0	0	0	9.93	9.96	9.97	9.98
1/4	188.3	138.5	118.2	95.7	4.97	4.98	4.99	4.99
3/8	246.1	181.4	154.5	125.0	2.48	2.49	2.49	2.50
1/2	267.35	195.8	167.1	135.2	0	0	0	0
5/8	246.1	181.4	154.5	125.0	2.48	2.49	2.49	2.50
3/4	188.3	138.5	118.2	95.7	4.97	4.98	4.99	4.99
1	0	0	0	0	9.93	9.96	9.97	9.98

Table 3. Tensile bending stresses along the beam with different modular ratios.

x/l	$\sigma_{pmax}(MPa)$						
	$\frac{E_p}{E_n}=1/2.5$	$\frac{E_p}{E_n}=1/2.0$	$\frac{E_p}{E_n}=1/1.5$	$\frac{E_p}{E_n}=1$	$\frac{E_p}{E_n}=1.5/1$	$\frac{E_p}{E_n}=2.0/1$	$\frac{E_p}{E_n}=2.5/1$
0	0	0	0	0	0	0	0
1/4	122.4	127.6	136.9	150.0	166.6	181.8	194.0
3/8	153.0	159.5	171.1	187.5	208.3	227.2	242.5
1/2	163.2	170.1	182.5	200.0	222.2	242.3	258.7
5/8	153.0	159.5	171.1	187.5	208.3	227.2	242.5
3/4	122.4	127.6	136.9	150.0	166.6	181.8	194.0
1	0	0	0	0	0	0	0

To verify the results obtained from the formulation presented in this paper, a finite element analysis using the program ANSYS has been performed for the same beam with the same values of E_p and E_n .After locating the N.A. using Equation (23) the upper part of the beam is modelled using modulus of elasticity equals E_n while the lower part of the beam is modelled using modulus of elasticity that equal E_p.

The results are listed in the tables below.

Table 4. Comparison between paper and FEM results.

$\frac{E_p}{E_n}$	Paper Results			FEM Results		
	$v_{tmax}(mm)$	$\tau_{max}(MPa)$	$\sigma_{pmax}(MPa)$	$v_{tmax}(mm)$	$\tau_{max}(MPa)$	$\sigma_{pmax}(MPa)$
1/2.5	267.35	9.93	163.2	279.2	9.878	160.82
1/2	195.8	9.962	170.1	204.64	10.20	169.59
1/1.5	167.13	9.93	182.48	172.04	10.01	180.79
1	135.2	10.84	200.0	140.13	11.00	199.49
1.5/1	167.13	10.46	222.15	172.36	10.30	220.82
2/1	195.8	10.3	242.33	201.77	10.20	238.52
2.5/1	267.35	9.93	258.67	272.86	10.17	251.77

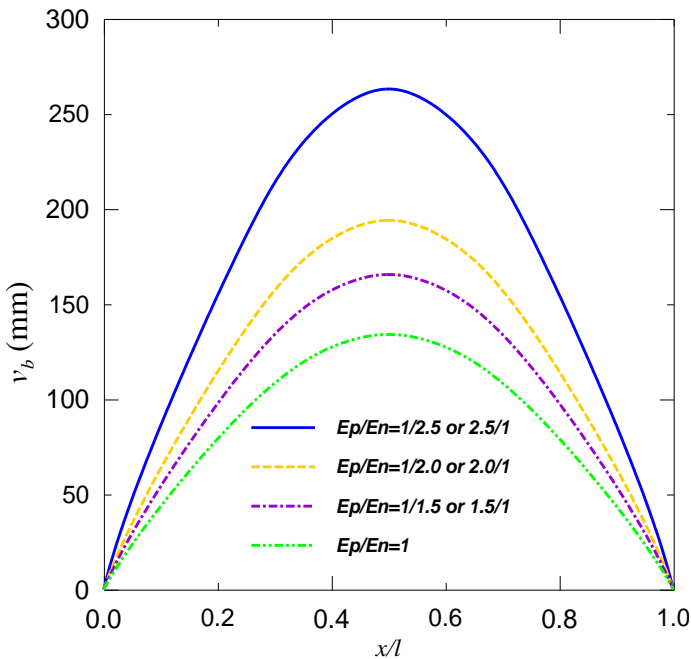*Table 5. Comparison between Bimodular and Unimodular (FEM) results.*

$\frac{E_p}{E_n}$	Bimodular Formulation			Unimodular Formulation (with $E=\frac{E_p+E_n}{2}$) using FEM		
	$v_{tmax}(mm)$	$\tau_{max}(MPa)$	$\sigma_{pmax}(MPa)$	$v_{tmax}(mm)$	$\tau_{max}(MPa)$	$\sigma_{pmax}(MPa)$
1/2.5	267.35	9.93	163.2	237.86	9.878	160.82
1/2	195.8	9.96	170.1	186.14	10.20	169.59
1/1.5	167.13	9.93	182.48	167.81	10.01	180.79
1	135.2	10.84	200.0	140.13	11.00	199.49
1.5/1	167.13	10.46	222.15	167.81	10.01	180.79
2/1	195.8	10.3	242.33	186.14	10.20	169.59
2.5/1	267.35	9.93	258.67	237.86	9.878	160.82

Table 6. Error (%) of paper results in comparison with FEM results.

$\dfrac{E_p}{E_n}$	Error (%) of paper results in comparison with FEM results.		
	v_{tmax}(mm)	τ_{max}(MPa)	σ_{pmax}(MPa)
1/2.5	4	0.5	1
1/2	4	2	0.3
1/1.5	2	0.8	0.9
1	3	1	0.2
1.5/1	3	1	0.6
2/1	2	0.9	1
2.5/1	2	2	2

Table 7. Error (%) of Unimodular results in comparison with Bimodular results.

$\dfrac{E_p}{E_n}$	Error (%) of Unimodular results in comparison with Bimodular results.		
	v_{tmax}(mm)	τ_{max}(MPa)	σ_{pmax}(MPa)
1/2.5	12	0.5	1
1/2	5	2	0.3
1/1.5	0.4	0.7	0.9
1	3	1	0.2
1.5/1	0.4	4	22
2/1	5	0.9	42
2.5/1	12	0.5	60

Figure 8. Bending deflection along the beam.
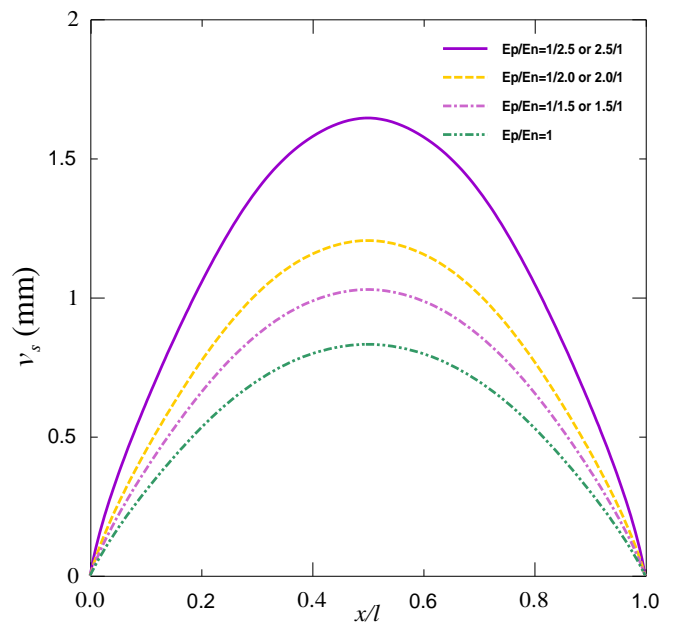
Figure 9. Shear deflection along the beam.
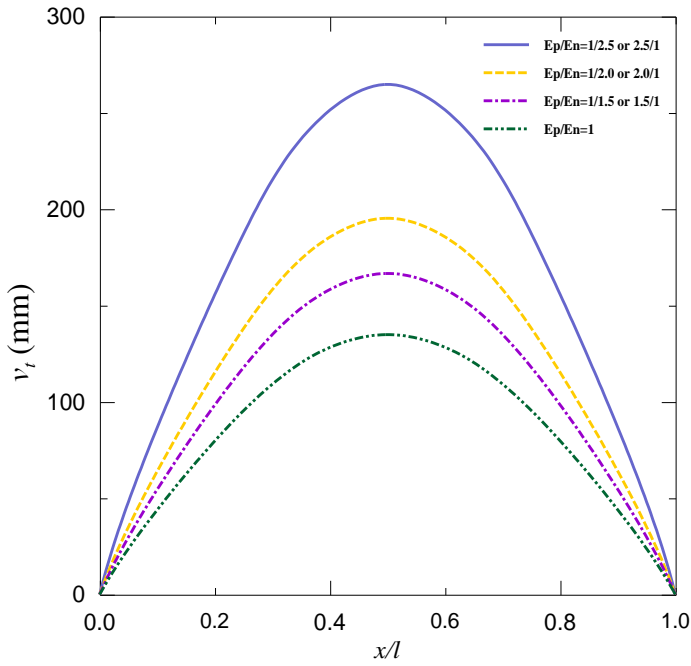
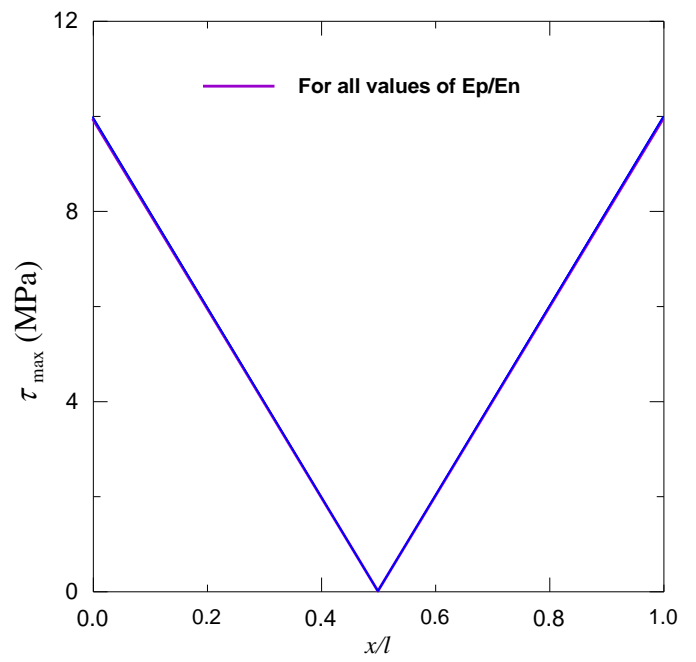**Figure 10. Total deflection along the beam.**
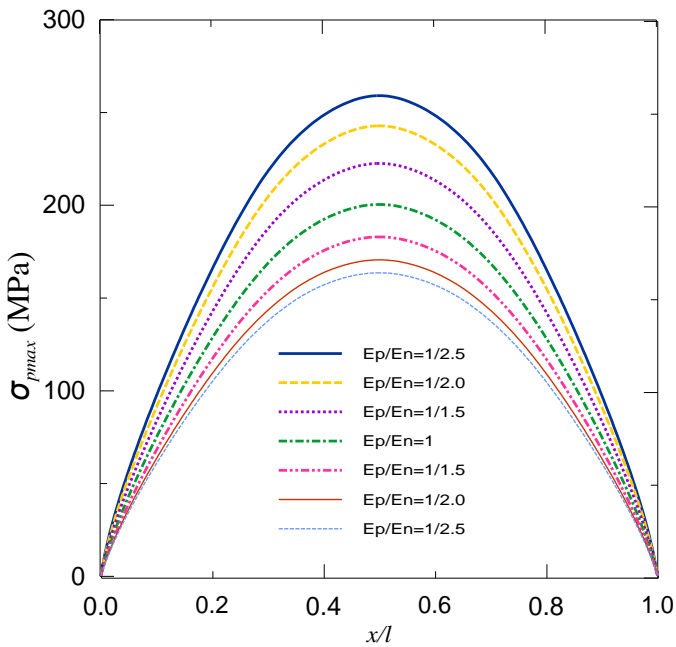

Figure 11. Max. Shear stress along the beam.

Figure 12. Tensile bending stress along the beam.
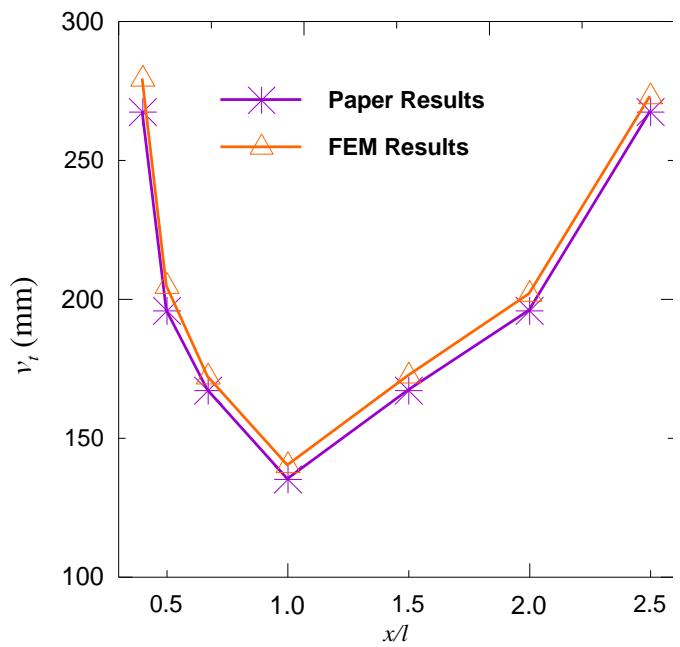

Figure 13. Paper and FEM results for total deflection
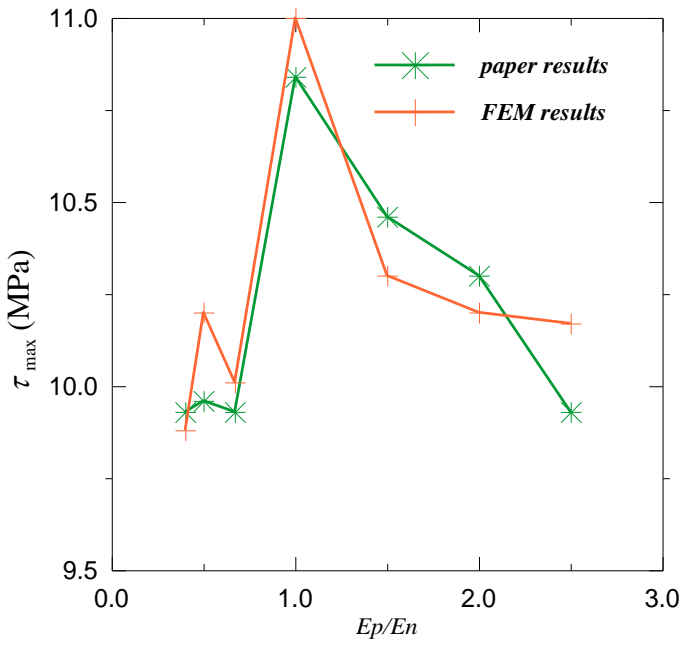
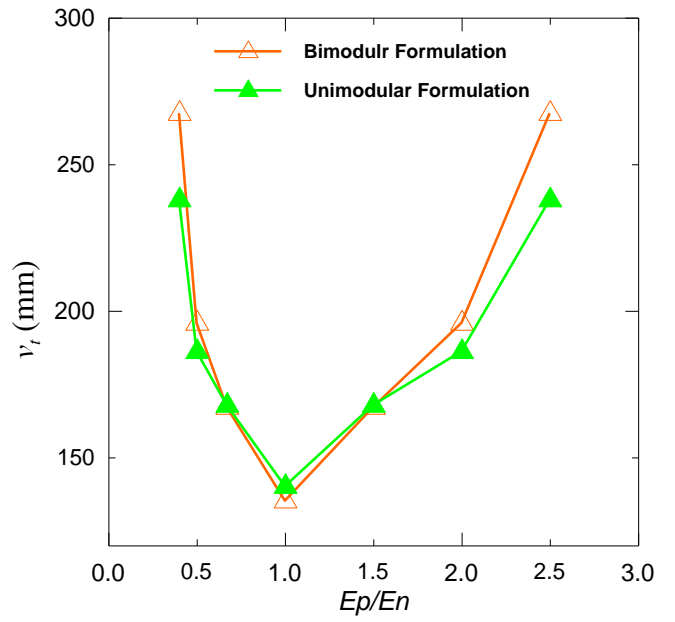**Figure 14. Paper and FEM results for shear stresses.**

Figure 15. Bimodular and Unimodular formulation in total deflection.
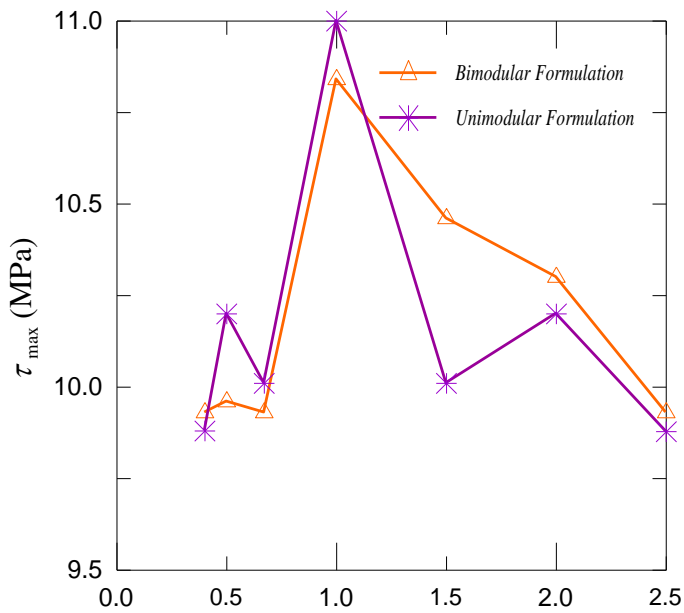

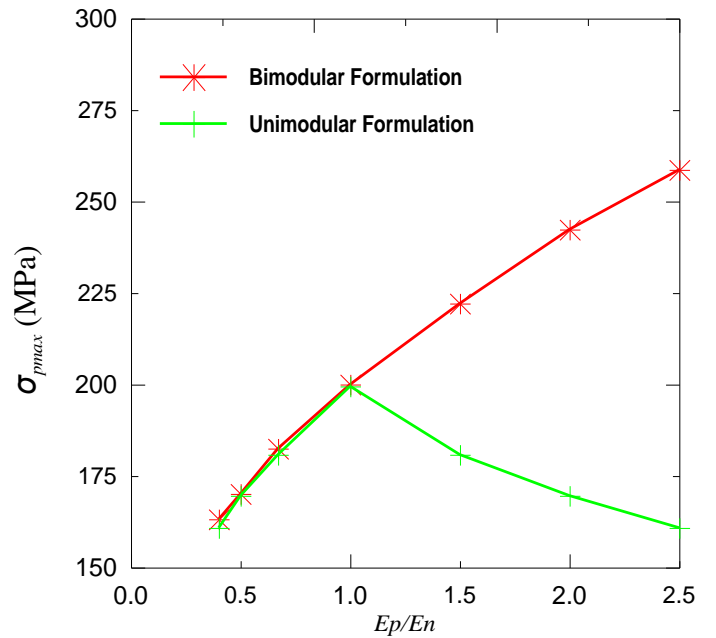**Figure 16. Bimodular and Unimodular formulation in shear stresses.**

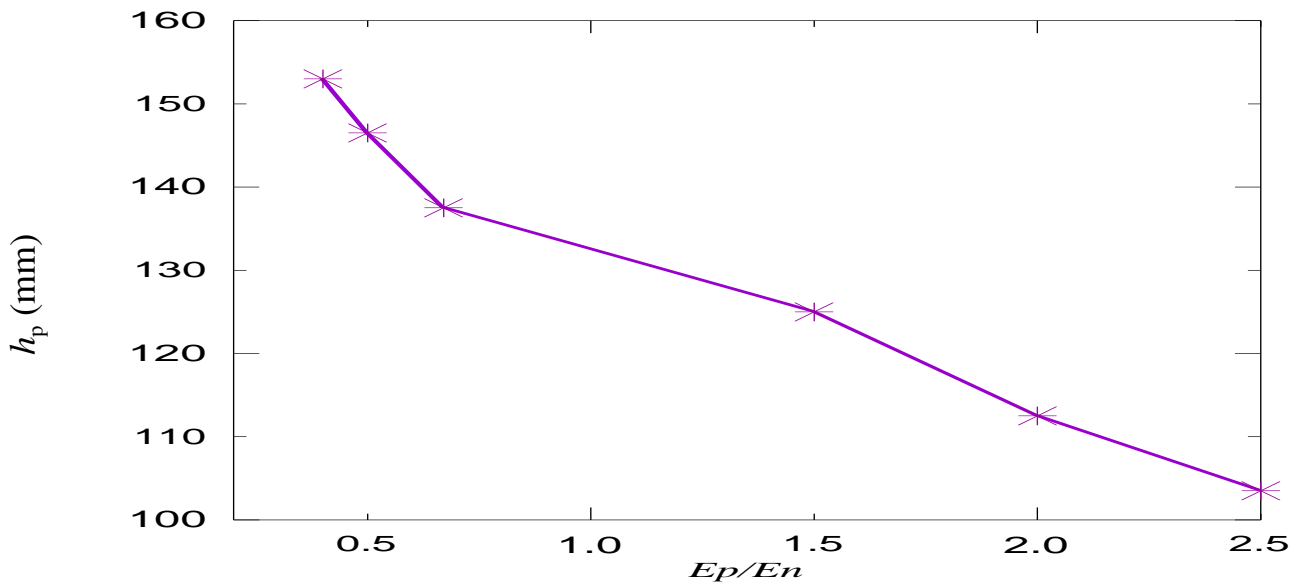**Figure 17. Bimodular and Unimodular formulation in bending stresses.**

Figure 18. The depth of the N.A. from the bottom face.(Total depth of the beam=250mm)

4. Concluding Remarks

It's noticeable from Figures 8, 9 and 10 that the deflection of the beam is not influenced by the modular ratio (E_p/E_n) but with the reduced modulus of elasticity E_r with a linear proportional relation as it appears from Equation (40). In addition, it's clear from Figure 9 that the contribution of the shear deformation to the total deflection is very small and depends upon the ratio of the depth of the section to the length of the beam. This contribution is about 2.5% from the total deflection if the ratio $h/l = 10\%$. Hence the shear deflection increases with small values of the ratio h/l, but less than $h/l = 10\%$,the shear deflection is negligable. Also one could see from Figure 11 that maximum shear stresses are not influenced by the modular ratio nor with the reduced modulus of elasticity E_r and this is clear from Equations (32 and 34) which is the same case in the unimodular beam. But at locations other than N.A. the shear stresses are different from those in the unimodular beam, since they depend on the modular ratio as in Equations (30 and 33). Figure 12 shows that the tensile bending stress increases with the increase of the modular ratio. Hence if it's required to decrease the tensile bending stress (as it is the case in the concrete structures), it's only needed to decrease the modular ratio instead of decreasing the material modulus of elasticity E. Figures 13 and 14 show that the formulation presented in this paper is close to those gained from an FEM analysis using the commercial package ANSYS. As shown from Table 6 the maximum error between the results obtained from the paper formulation and the FEM analysis is (4%) in the total deflection and the shear stresses which are very small and accepted for an approximate analysis. The error in the deflection increases with the decrease in the modular ratio and due to the fact that the smaller the modular ratio the smaller E_p (compared to E_n) and so the higher extension in the fibres below the N.A. and finally increases deflection even if the fibres above the N.A. undergoes small deformations. On the other hand, the error in the deflection decreases with the increase in the modular ratio and this is because the larger the modular ratio, the larger E_p (compared to E_n) and hence the smaller the extension in the fibres below the N.A. and finally decreases deflection even if the fibres above the N.A. undergoes large deformations. Regarding the tensile bending stress, the error increases with the increase in the modular ratio, since the increase in the modular ratio means

an increase in E_p in comparison to E_n, while the analytical equation of the bimodular tensile stress deals with E_r which does not differ if the modular ratio was (1/2.5) or (2.5/1), for example.

The error in the bending deflection in the approximate expression for deflection in Equation (1) is due to relating the deflection in all nodes in the beam with the maximum deflection with a sine curve. Another expression would be more accurate if the same expression had two degrees of freedom representing two deflections, but the solution would be more difficult then.

On the other hand to show the importance of the bimodular analysis, the same example has been solved using a unimodular analysis with an average modulus of elasticity where the error was very large and sometimes reaches 60% as shown in Table 7 and Figures 15, 16 and 17.

It's noticeable that the error in the shear stress values whether between the bimodular and FEM or between the bimodular and unimodular analysis was very small. The reason behind that is that the maximum shear stress was constant along the beam and has the same value derived in strength of material for a bimodular beam.

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