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The Solution of Special kind of Linear Second Order Partial Differential Equations and Solution The Heat Equation as Physical Application in Spherical Coordinates

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Abstract

The main aim of this search is to solve kind of partial differential equations of second order with variable coefficients, which have the general form

$$A(x,y)Z_{xx} + B(x,y)Z_{xy} + C(x,y)Z_{yy} + D(x,y)Z_x + E(x,y)Z_y + F(x,y)Z = 0,$$

where some of $A(x,y)$, $B(x,y)$, $C(x,y)$, $D(x,y)$, $E(x,y)$ and $F(x,y)$ are functions of x or y or both x and y.

For this purpose, we will use one kind of the partial differential equations its formula

$$\mathrm{A}_1 x^2 Z_{xx} + \mathrm{D}_1 x Z_x + \mathrm{C} Z_{yy} + \mathrm{E} Z_y + \mathrm{F} Z = 0$$

where $\mathrm{A}_1, \mathrm{D}_1, \mathrm{C}, \mathrm{E}$ and F are real constants.

We found the following substitution

$$Z(x,y) = \mathrm{e}^{\int \frac{U(x)}{x} dx + \int V(y)\, dy}.$$

Transforms the above kind to the first order ordinary differential equation with two independent functions $U(x)$ and $V(y)$, which have the general form

$$\mathrm{A}_1 \left(xU'(x) + U^2(x) - U(x) \right) + \mathrm{D}_1 U(x) + \mathrm{C}\left(V'(y) + V^2(y)\right) + \mathrm{E}V(y) + \mathrm{F} = 0.$$

We found the general form of its complete solution and we applied this form for solving the heat equation as physical application in spherical coordinates

المستخلص

الهدف الرئيسي من هذا البحث هو إيجاد حل نوع من المعادلات التفاضلية الجزئية من الرتبة الثانية ذات المعاملات المتغيرة والتي صيغتها العامة

$$A(x,y)Z_{xx} + B(x,y)Z_{xy} + C(x,y)Z_{yy} + D(x,y)Z_x + E(x,y)Z_y + F(x,y)Z = 0,$$

حيث بعض $F(x,y)$, $E(x,y)$, $D(x,y)$, $C(x,y)$, $B(x,y)$, $A(x,y)$ تكون دوال إلى x أو y أوكلاهما.

سوف نتناول نوع واحد من المعادلات التفاضلية الجزئية والتي تعتمد على صيغ الدوال $F(x,y)$, $E(x,y)$, $D(x,y)$, $C(x,y)$, $B(x,y)$, $A(x,y)$ وصيغتها العامة

$$\mathrm{A}_1 x^2 Z_{xx} + \mathrm{D}_1 x Z_x + \mathrm{C} Z_{yy} + \mathrm{E} Z_y + \mathrm{F} Z = 0$$

حيث F, E, C, D_1, A_1 ثوابت حقيقية.

فوجدنا التعويض

$$Z(x,y) = \mathrm{e}^{\int \frac{U(x)}{x} dx + \int V(y)\, dy}$$

يحول المعادلة أعلاه إلى معادلة تفاضلية اعتيادية من الرتبة الأولى بمتغيرين والتي صيغتها العامة

$$A_1\left(xU'(x)+U^2(x)-U(x)\right)+D_1\,U(x)+C+EV(y)+F=0,$$

كذلك وجدنا للحالة أعلاه الصيغة العامة لحلها ثم تم تطبيقها لحل معادلة الحرارة كتطبيق فيزيائي في إحداثيات كروية.

1. Introduction

Many scientists used theory of differential equations like Newton, Leibniz and others in the seventeenth century to describe many phenomena in Physics, Chemistry, Biology, and other fields. This study focuses on important types from the partial differential equations, which is linear second order with variable coefficients. The researcher **Kudaer [7]**, searched function $Z(x)$, such that the assumption $y(x)=e^{\int Z(x)\,dx}$ gives the general solution of the linear second order ordinary differential equations, which have the general form $y''+P(x)\,y'+Q(x)\,y=0$, and its solution depends on the forms of *P(x)* and *Q(x).*

The researcher **Abd Al-Sada [1]**, searched functions $U(x)$ and $V(y)$, such that the assumption $Z(x,y)=e^{\int U(x)\,dx+\int V(y)\,dy}$ gives the complete solution of the linear second order partial differential equations, which have the general form $A\,Z_{xx}+B\,Z_{xy}+C\,Z_{yy}+D\,Z_x+E\,Z_y+F\,Z=0$, where A , B , C , D , E and F are real constants.

The researcher **Hani [5**], searched functions $U(x)$, $V(y)$ and W(t), such that the assumption $Z(x,y)=e^{\int U(x)\,dx+\int V(y)\,dy+\int W(t)\,dt}$ gives the complete solution of the linear second order partial differential equations, which have the general form

$A\,Z_{xx}+B\,Z_{xy}+C\,Z_{xt}+D\,Z_{yy}+E\,Z_{yt}+F\,Z_{tt}+G\,Z_x+H\,Z_y+I\,Z_t+J\,Z=0$,

where A , B , … , and J are real constants.

These ideas made us to search functions $U(x)$ and $V(y)$, that give the complete solution of the linear second order partial differential equations with variable coefficients, which have the form

$A(x,y)Z_{xx}+B(x,y)Z_{xy}+C(x,y)Z_{yy}+D(x,y)Z_x+E(x,y)Z_y+F(x,y)\,Z=0,$

and this solution depends on the forms of the functions $A(x,y), B(x,y), C(x,y), D(x,y), E(x,y)$ and $F(x,y)$.

2. The Complete Solution of partial differential equations of Second Order with Variable Coefficients

Our aim now is to solve special kind of the linear second order of partial differential equations with variable coefficients, which have the general form

$A(x,y)\,Z_{xx}+B(x,y)\,Z_{xy}+C(x,y)\,Z_{yy}+D(x,y)\,Z_x+E(x,y)\,Z_y+F(x,y)\,Z=0$,

where some of $A(x,y)$, $B(x,y)$, $C(x,y)$, $D(x,y)$, $E(x,y)$ and $F(x,y)$ are functions of x or y or both x and y. So, for this purpose we will search functions $U(x)$ and $V(y)$ such that the assumption $Z(x,y)=e^{\int \frac{U(x)}{x}\,dx+\int V(y)\,dy}$, gives the complete solution to the above equation and to do this we will get one kind of the above equation its formula

$A_1\,x^2\,Z_{xx}+D_1\,x\,Z_x+C\,Z_{yy}+E\,Z_y+F\,Z=0$, (i.e. $B=0$)

∋ A_1, D_1, C, E and F are constants and not identically zero.

3. Description of The Method That Gives The Complete Solution

Let us consider the second order partial differential equation which has the general form

$$A_1\,x^2\,Z_{xx}+D_1\,x\,Z_x+C\,Z_{yy}+E\,Z_y+F\,Z=0 \qquad \ldots(1)$$

In order to find a complete solution of (1), we search functions $U(x)$ and $V(y)$, such that the assumption

$$Z(x,y) = e^{\int \frac{U(x)}{x}\, dx + \int V(y)\, dy} \qquad \ldots(2)$$

represents the complete solution of it.
This assumption will transform (1) to first order ordinary differential equation, its general form is given by $A_1 \left(xU'(x) + U^2(x) - U(x) \right) + D_1\, U(x) + C\left(V'(y) + V^2(y)\right) + EV(y) + F = 0 \quad \ldots(3)$

The equation (3) is of the first order ordinary differential equation and contains two independent functions $U(x)$ and $V(y)$.

4. The Complete Solution of the Linear Second Order of Partial Differential Equations
To find the complete solution of the linear second order of partial differential equations with variable coefficients, we go back to the previous equation on the beginning of this search:If B = 0, and the partial differential equation is given by :

$$A_1 x^2 Z_{xx} + D_1\, x Z_x + C\, Z_{yy} + E\, Z_y + F\, Z = 0$$

Then the complete solution is given by:

i)

$$Z(x,y) = x^{-\frac{1}{2}(\frac{D_1}{A_1}-1)}\, e^{-\frac{E}{2C}y} [\, a_1 x^{\sqrt{\frac{1}{4}(\frac{D_1}{A_1}-1)^2 + \frac{\lambda^2}{A_1}}} + a_2\, x^{-\sqrt{\frac{1}{4}(\frac{D_1}{A_1}-1)^2 + \frac{\lambda^2}{A_1}}}\,]\, (\, d_1 \cos\sqrt{\frac{F+\lambda^2}{C} - \frac{E^2}{4C^2}}\; y + d_2 \sin\sqrt{\frac{F+\lambda^2}{C} - \frac{E^2}{4C^2}}\; y\,)$$

If $\frac{1}{4}(\frac{D_1}{A_1}-1)^2 \neq \frac{-\lambda^2}{A_1}$ and $\frac{F+\lambda^2}{C} \neq \frac{E^2}{4C^2}$,
where λ, a_i and d_i ; (i = 1, 2) are arbitrary constants.

ii)

$$Z(x,y) = x^{-\frac{1}{2}(\frac{D_1}{A_1}-1)}\, e^{-\frac{E}{2C}y} [\, a_1 x^{\sqrt{\frac{1}{4}(\frac{D_1}{A_1}-1)^2 + \frac{\lambda^2}{A_1}}} + a_2\, x^{-\sqrt{\frac{1}{4}(\frac{D_1}{A_1}-1)^2 + \frac{\lambda^2}{A_1}}}\,](y - c_2\,)$$

If $\frac{1}{4}(\frac{D_1}{A_1}-1)^2 \neq \frac{-\lambda^2}{A_1}$ and $\frac{F+\lambda^2}{C} = \frac{E^2}{4C^2}$,
where λ, c_2, a_1 and a_2 are arbitrary constants.

iii)

$$Z(x,y) = x^{-\frac{1}{2}(\frac{D_1}{A_1}-1)}\, e^{-\frac{E}{2C}y} [d_1 \cos\sqrt{\frac{F+\lambda^2}{C} - \frac{E^2}{4C^2}}\; y + d_2 \sin\sqrt{\frac{F+\lambda^2}{C} - \frac{E^2}{4C^2}}\; y]\; (\ln(c_4\, x)) \;;\; (c_4\, x) \rangle 0$$

, If $\frac{1}{4}(\frac{D_1}{A_1}-1)^2 = \frac{-\lambda^2}{A_1}$ and $\frac{F+\lambda^2}{C} \neq \frac{E^2}{4C^2}$
where λ, c_4, d_1 and d_2 are arbitrary constants .

iv)

$$Z(x,y) = K_1 \, x^{-\frac{1}{2}(\frac{D_1}{A_1}-1)} \, e^{-\frac{E}{2C}y} \, (y - c_2) \, \ln(c_4 x) \quad ; \quad (c_4 x) \rangle 0$$

If $\frac{1}{4}(\frac{D_1}{A_1} - 1)^2 = \frac{-\lambda^2}{A_1}$ and $\frac{F+\lambda^2}{C} = \frac{E^2}{4C^2}$,

where $K_1 = e^g$, c_2 and c_4 are arbitrary constants.

proof : Since

$$A_1 \left(xU'(x) + U^2(x) - U(x) \right) + D_1 U(x) + C\left(V'(y) + V^2(y) \right) + EV(y) + F = 0,$$

therefore,

$$C\left(V'(y) + V^2(y)\right) + EV(y) + F = -A_1 \left(xU'(x) + U^2(x) - U(x) \right) - D_1 U(x) = -\lambda^2.$$

So $V'(y) + V^2(y) + \frac{E}{C}V(y) + \frac{F+\lambda^2}{C} = 0.$ Let $B_1 = \frac{E}{C}$ and $B_2 = \frac{F+\lambda^2}{C}$,

then the last equation becomes : $V'(y) + V^2(y) + B_1 V(y) + B_2 = 0$ … (4)

Also $$xU'(x) + U^2(x) + (\frac{D_1}{A_1} - 1)U(x) - \frac{\lambda^2}{A_1} = 0.$$

Let $A_2 = \frac{D_1}{A_1} - 1$ and $A_3 = \frac{-\lambda^2}{A_1}$, then the last equation becomes:

$$xU'(x) + U^2(x) + A_2 U(x) + A_3 = 0 \qquad \dots (5)$$

The equation (4) is variable separable equation, now

i) If $B_2 \neq \frac{B_1^2}{4}$, we get $\frac{dV}{\left(V(y) + \frac{B_1}{2}\right)^2 + b_1^2} + dy = 0$; $b_1^2 = B_2 - \frac{B_1^2}{4}$

$$\Rightarrow \frac{1}{b_1}\tan^{-1}\left(\frac{V(y) + \frac{B_1}{2}}{b_1} \right) = c_1 - y \Rightarrow V(y) = b_1 \tan(b_1 c_1 - b_1 y) - \frac{B_1}{2}$$

ii) If $B_2 = \frac{B_1^2}{4}$, we get $\frac{dV}{\left(V(y) + \frac{B_1}{2}\right)^2} + dy = 0 \Rightarrow \frac{-1}{V(y) + \frac{B_1}{2}} = c_2 - y$

$$\Rightarrow V(y) = \frac{1}{y - c_2} - \frac{B_1}{2} \quad ; \quad y \neq c_2$$

Also, equation(5) is variable separable equation, so we can solve it as follows:

$$\frac{dU}{U^2(x)+A_2U(x)+A_3}+\frac{dx}{x}=0$$

$$\frac{dU}{\left(U(x)+\frac{A_2}{2}\right)^2-b_2^2}+\frac{dx}{x}=0 \qquad ; \qquad b_2^2=\frac{A_2^2}{4}-A_3$$

i) If $\frac{A_2^2}{4}\neq A_3$, we get

$$\frac{1}{b_2}\tanh^{-1}\left(\frac{U(x)+\frac{A_2}{2}}{b_2}\right)=\ln(c_3x) \quad ; \quad -1\langle \frac{U(x)+\frac{A_2}{2}}{b_2}\langle 1 \quad \text{and} \quad (c_3x)\rangle 0$$

$$\Rightarrow \qquad U(x)=b_2\tanh(b_2\ \ln(c_3x))-\frac{A_2}{2}$$

ii) If $\frac{A_2^2}{4}=A_3$, we get $\qquad \frac{dU}{\left(U(x)+\frac{A_2}{2}\right)^2}+\frac{dx}{x}=0 \quad \Rightarrow$

$$\frac{-1}{U(x)+\frac{A_2}{2}}+\ln x=-\ln c_4 \Rightarrow U(x)=\frac{1}{\ln(c_4x)}-\frac{A_2}{2} \qquad ; \qquad c_4x\in R^+/\{1\}$$

So, the complete solution of (1), is given by:

i) $\qquad Z(x,y)= e^{\int(\frac{b_2\tanh\ (b_2\ \ln(c_3x))}{x}\ -\frac{A_2}{2x})\,dx+\int(b_1\tan(b_1c_1-b_1y)-\frac{B_1}{2})\,dy}$

If $\quad \frac{A_2^2}{4}\neq A_3 \qquad$ and $\qquad B_2\neq\frac{B_1^2}{4}$

So $\qquad Z(x,y)= e^{\ln\,[\cosh(b_2\ln(c_3x))]-\frac{A_2}{2}\ln x+\ln[\cos(b_1c_1-b_1y)]-\frac{B_1}{2}y+g} \qquad ; \ \cos(b_1c_1-b_1y)\rangle 0$

$$= x^{-\frac{A_2}{2}}\ e^{-\frac{B_1}{2}y}\ \cosh(b_2\ln(c_3x))\,(d_1\cos b_1y+d_2\sin b_1y)\,,$$

where $d_1=e^g\cos b_1c_1 \quad$ and $\quad d_2=e^g\sin b_1c_1$

$$\Rightarrow Z(x,y)= x^{-\frac{A_2}{2}}\ e^{-\frac{B_1}{2}y}\ (a_1\,x^{b_2}+a_2\,x^{-b_2})\,(d_1\cos b_1y+d_2\sin b_1y)$$

where $a_1=\frac{c_3^{b_2}}{2} \qquad$ and $\qquad a_2=\frac{c_3^{-b_2}}{2}$

So, $\qquad Z(x,y)= x^{-\frac{A_2}{2}}\ e^{-\frac{B_1}{2}y}\ [a_1\,x^{\sqrt{\frac{A_2^2}{4}-A_3}}+a_2\,x^{-\sqrt{\frac{A_2^2}{4}-A_3}}]$

$$[d_1\cos\sqrt{B_2-\frac{B_1^2}{4}}\,y+d_2\sin\sqrt{B_2-\frac{B_1^2}{4}}\,y]$$

$$= x^{-\frac{1}{2}(\frac{D_1}{A_1}-1)} e^{-\frac{E}{2C}y} [a_1 x^{\sqrt{\frac{1}{4}(\frac{D_1}{A_1}-1)^2+\frac{\lambda^2}{A_1}}} + a_2 x^{-\sqrt{\frac{1}{4}(\frac{D_1}{A_1}-1)^2+\frac{\lambda^2}{A_1}}}]$$

$$[d_1 \cos\sqrt{\frac{F+\lambda^2}{C}-\frac{E^2}{4C^2}}\, y + d_2 \sin\sqrt{\frac{F+\lambda^2}{C}-\frac{E^2}{4C^2}}\, y]$$

where λ, a_i, and d_i ; (i = 1,2) are arbitrary constants.

ii) If $\frac{A_2^2}{4} \neq A_3$ and $B_2 = \frac{B_1^2}{4}$

Therefore, $Z(x,y) = e^{\int(\frac{b_2 \tanh(b_2 \ln(c_3 x))}{x} - \frac{A_2}{2x})dx + \int(\frac{1}{y-c_2} - \frac{B_1}{2})dy}$

$$= e^{\ln[\cosh(b_2 \ln(c_3 x))] - \frac{A_2}{2}\ln x + \ln(y-c_2) - \frac{B_1}{2}y + g}$$

$$= x^{-\frac{A_2}{2}} e^{-\frac{B_1}{2}y} (a_1 x^{b_2} + a_2 x^{-b_2})(y-c_2)$$

where $a_1 = \frac{c_3^{b_2}}{2} e^g$ and $a_2 = \frac{c_3^{-b_2}}{2} e^g$.

So, the complete solution of (1), is given by:

$$Z(x,y) = x^{-\frac{1}{2}(\frac{D_1}{A_1}-1)} e^{-\frac{E}{2C}y} [a_1 x^{\sqrt{\frac{1}{4}(\frac{D_1}{A_1}-1)^2+\frac{\lambda^2}{A_1}}} + a_2 x^{-\sqrt{\frac{1}{4}(\frac{D_1}{A_1}-1)^2+\frac{\lambda^2}{A_1}}}](y-c_2)$$

where λ, c_2, a_1 and a_2 are arbitrary constants.

iii) If $\frac{A_2^2}{4} = A_3$ and $B_2 \neq \frac{B_1^2}{4}$

Therefore, $Z(x,y) = e^{\int(\frac{1}{x\ln(c_4 x)} - \frac{A_2}{2x})dx + \int(b_1 \tan(b_1 c_1 - b_1 y) - \frac{B_1}{2})dy}$

$$= e^{\ln(\ln(c_4 x)) - \frac{A_2}{2}\ln x + \ln[\cos(b_1 c_1 - b_1 y)] - \frac{B_1}{2}y + g} \quad ; \quad \cos(b_1 c_1 - b_1 y) \rangle 0$$

$$= x^{-\frac{A_2}{2}} e^{-\frac{B_1}{2}y} (d_1 \cos b_1 y + d_2 \sin b_1 y) \ln(c_4 x) \quad ; (c_4 x) \rangle 0$$

where $d_1 = e^g \cos b_1 c_1$ and $d_2 = e^g \sin b_1 c_1$.

So, the complete solution of (1), is given by :

$$Z(x,y) = x^{-\frac{A_2}{2}} e^{-\frac{B_1}{2}y} [d_1 \cos\sqrt{B_2 - \frac{B_1^2}{4}}\, y + d_2 \sin\sqrt{B_2 - \frac{B_1^2}{4}}\, y](\ln(c_4 x))$$

$$= (x^{-\frac{1}{2}(\frac{D_1}{A_1}-1)} e^{-\frac{E}{2C}y})(d_1 \cos\sqrt{\frac{F+\lambda^2}{C}-\frac{E^2}{4C^2}}\, y + d_2 \sin\sqrt{\frac{F+\lambda^2}{C}-\frac{E^2}{4C^2}}\, y)(\ln(c_4 x))$$

where λ, c_4, d_1 and d_2 are arbitrary constants.

iv) If $\frac{A_2^2}{4} = A_3$ and $B_2 = \frac{B_1^2}{4}$

Therefore, $Z(x,y) = e^{\int (\frac{1}{x\ln(c_4 x)} - \frac{A_2}{2x})\,dx + \int (\frac{1}{y-c_2} - \frac{B_1}{2})\,dy}$

$= e^{\ln(\ln(c_4 x)) - \frac{A_2}{2}\ln x + \ln(y-c_2) - \frac{B_1}{2} y + g}$

So, the complete solution of (1), is given by :

$$Z(x,y) = K_1\, x^{-\frac{A_2}{2}}\, e^{-\frac{B_1}{2}y}\, (y-c_2)(\ln(c_4 x)) \quad ; K_1 = e^g \text{ and } c_4 x > 0$$

$$= K_1\, x^{-\frac{1}{2}(\frac{D_1}{A_1}-1)}\, e^{-\frac{E}{2C}y}\, (y-c_2)(\ln(c_4 x))$$

where K_1, c_2, and c_4 are arbitrary constants.

Note **:**If we write

$$C\left(V'(y) + V^2(y)\right) + EV(y) = -A_1\left(xU'(x) + U^2(x) - U(x)\right) - D_1 U(x) - F = -\lambda^2 ,$$

then the complete solution is established by the same method , but

$$A_2 = \frac{D_1}{A_1} - 1 \;,\quad A_3 = \frac{F-\lambda^2}{A_1} \;,\quad B_1 = \frac{E}{C} \quad \text{and} \quad B_2 = \frac{\lambda^2}{C}$$

5. Solution of the Heat equation (as application) in spherical coordinates

6. Derivation of the Heat Equation

Newton articulated some principles of heat flow through solids, but it was ***Fourier*** who created the correct systematic theory. Inside a solid, there is no transfer of heat energy and little radioactive transfer, so temperature changes only by conduction, as the energy we now recognize as molecular kinetic energy flows from hotter regions to cooler regions **[4]**.

The first basic principle of heat is **[4, 9]:**

1) The heat energy contained in a material is proportional to the temperature, the density of the material ρ and a physical characteristic of the material called the specific heat capacity c (is the quantity of heat which should be flow from (to) mass unit from material to change its temperature per a degree, its measurement units is $cal/g \cdot c^o$) **[10].**

In mathematical terms, $Q = \iiint_{\Omega} \rho\, c\, U(x,t)\, d^3x$ … (6).

In other words, the rate of heat flow from one region to another is proportional to the temperature gradient between the regions, we see that the rate of heat transfer depends on the material, as measured with a physical constant as the heat conductivity K (is the time average of heat flow through the material per unit area for all gradient unit heat, its measurement units is $cal/cm \cdot sec \cdot c^o$) **[10].**

The second basic principle is :

2) The heat transfers through the boundary of a region is proportional to the heat conductivity, to the gradient of the temperature across the region, and to the area of contact, so if the boundary of the region Ω is written as $\partial\Omega$, with outward normal vector n, then

$$\frac{dQ}{dt} = \iint_{\partial\Omega} K\, n \cdot \nabla U(x,t)\, d^2x \qquad \dots (7).$$

If we differentiate equation (6),with respect to time and apply ***Gauss's divergence theorem*** equation (7), we find that $\frac{dQ}{dt}$ can be expressed in two ways as an integral :

$$\iiint_\Omega \rho \, c \, U_t(x,t)\, d^3x = \iiint_\Omega \nabla \cdot \mathrm{K} \nabla U(x,t)\, d^3x$$

Since the region Ω can be an arbitrary piece of the material under study, the integrands must be equal at almost every point. If the material under study is a slab of a homogeneous substance, then ρ, c and K are independent of the position x , and we obtain the heat equation

$$\frac{\partial U}{\partial t} = \mathrm{k}\, \nabla^2 U \qquad \dots (8)$$

where $\mathrm{k} = \mathrm{K}/\rho\, \mathrm{c}$ (diffusivity), ordinary substances have values of (k) ranging from about 5 to 9000 $\mathrm{cm}^2/\mathrm{sec}$.

***The one-dimensional heat equation* [3]**,is

$$\frac{\partial U}{\partial t} = \mathrm{k}\, \frac{\partial^2 U}{\partial x^2} \qquad \dots (9)$$

where $\frac{\partial U}{\partial t} \equiv U_t$. If U in equation (8) depends only on x , y and t , and does not depend on z because of the symmetry, then this equation will reduce to the form

$$\frac{\partial U}{\partial t} = \mathrm{k} \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) \qquad \dots (10).$$

The last equation is called ***the heat equation in two - dimensional*[10]**.

7. The Heat Equation in Different Coordinate Systems

Since we can write ***Laplacian*** ∇^2 in three – dimensional, so the heat equation **[5, 9],** as follows: $\frac{\partial U}{\partial t} = \mathrm{k}\, \nabla^2 U$, where $\nabla^2 U$ given in cartesian, cylindrical and spherical coordinates as follows**[3]**:

In cartesian coordinates:

$$\nabla^2 U \equiv \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \qquad \dots (11).$$

In cylindrical coordinates (r, θ, z):

$$\nabla^2 U \equiv \frac{1}{r} \cdot \frac{\partial}{\partial r} \left(r \frac{\partial U}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 U}{\partial \theta^2} + \frac{\partial^2 U}{\partial z^2} \qquad \dots (12).$$

In spherical coordinates (r, θ, ϕ) :

$$\nabla^2 U \equiv \frac{1}{r^2} \cdot \frac{\partial}{\partial r} \left(r^2 \frac{\partial U}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \cdot \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial U}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \cdot \frac{\partial^2 U}{\partial \phi^2} \qquad \dots (13).$$

Now, we will try to solve ***the heat equation in spherical coordinates*** by using our method.

8. The Heat Equation in Spherical Coordinates

Let us consider mineral solid sphere with radius equal to one unit, as in Fig.(1). Suppose that the temperature of the upper half surface of sphere with angle θ (where $0 \le \theta < \pi/2$) is kept fixed at $\mathrm{U}_o \neq 0$, while the temperature of the lower half (where $\pi/2 \le \theta \le \pi$) is kept at zero.

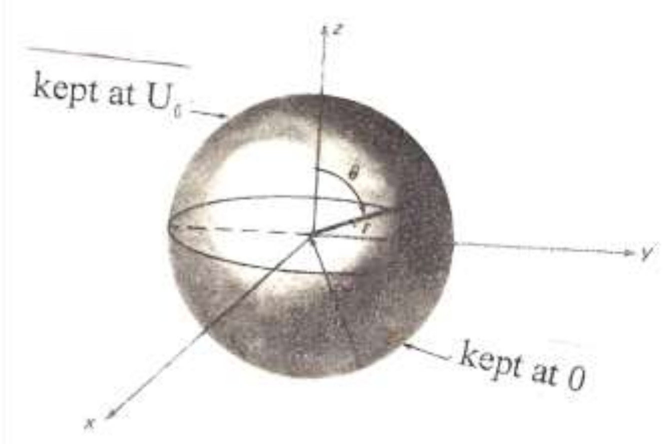

Fig. (1) [9]

The general form of the heat equation in spherical coordinates is given by:

$$U_t = \mathrm{k}\left[U_{rr} + \frac{2}{r}U_r + \frac{1}{r^2}U_{\theta\theta} + \frac{\cot\theta}{r^2}U_\theta + \frac{1}{r^2\sin^2\theta}U_{\phi\phi} \right] \qquad \ldots (14).$$

Since U depends on r and θ, and does not depend on ϕ or t (i.e $U = U(r,\theta)$), then the equation (14) becomes

$$U_{rr} + \frac{2}{r}U_r + \frac{1}{r^2}U_{\theta\theta} + \frac{\cot\theta}{r^2}U_\theta = 0 \qquad \ldots (15).$$

The initial condition is given by :

$$U(1,\theta) = f(\theta) = \begin{cases} \mathrm{U_0} & 0 \le \theta \langle \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \le \theta \le \pi \end{cases} \qquad \ldots (16).$$

In addition to that, the solution U is finite inside the sphere.
The complete solution of the (15) , is given by **[9]** , which has the form

$$U(r,\theta) = (\mathrm{c}_1 r^{\mathrm{n}} + \frac{\mathrm{c}_2}{r^{\mathrm{n}+1}})[\mathrm{c}_3 P_{\mathrm{n}}(x) + \mathrm{c}_4 Q_{\mathrm{n}}(x)]$$

Where $\lambda^2 = -\mathrm{n}(\mathrm{n}+1)$, $x = \cos\theta$, $P_{\mathrm{n}}(x)$ and $Q_{\mathrm{n}}(x)$ are ***Legendre functions***, and $\mathrm{c_i}$; $(\mathrm{i} = 1, 2, 3, 4)$ are arbitrary constants. Now, we attempt to solve this equation by our suggested method. Since $r \neq 0$ then multiplying (15) by r^2, we get

$$r^2 U_{rr} + 2r U_r + U_{\theta\theta} + \cot\theta U_\theta = 0 \qquad \ldots (17).$$

To find the complete solution of the last equation, we try to find two new functions $R(r)$ and $W(\theta)$, such that the assumption

$$U(r,\theta) = \mathrm{e}^{\int \frac{R(r)}{r}dr + \int W(\theta)\,d\theta} \qquad \ldots (18)$$

will help us to find the complete solution of the above mentioned equation.
From (18), we find U_r , U_{rr} , U_θ and $U_{\theta\theta}$, and substituting them in (17) to obtain

$$\left[r^2 \frac{[rR'(r) + R^2(r) - R(r)]}{r^2} + 2r\frac{R(r)}{r} + W'(\theta) + W^2(\theta) + \cot\theta W(\theta) \right] \mathrm{e}^{\int \frac{R(r)}{r}dr + \int W(\theta)\,d\theta} = 0$$

Since $e^{\int \frac{R(r)}{r} dr + \int W(\theta)\, d\theta} \neq 0$, which gives

$$rR'(r) + R^2(r) + R(r) + W'(\theta) + W^2(\theta) + \cot\theta\, W(\theta) = 0$$

Putting $W'(\theta) + W^2(\theta) + \cot\theta\, W(\theta) = -\left[rR'(r) + R^2(r) + R(r)\right] = -\lambda^2$

where λ is an arbitrary constant .

So $$rR'(r) + R^2(r) + R(r) - \lambda^2 = 0 \quad \ldots (19)$$

$$W'(\theta) + W^2(\theta) + \cot\theta\, W(\theta) + \lambda^2 = 0 \quad \ldots (20)$$

From equation (19), we get $\dfrac{dR}{R^2(r) + R(r) - \lambda^2} + \dfrac{dr}{r} = 0$

$$\Rightarrow \quad \frac{dR}{\left(R(r) + \frac{1}{2}\right)^2 - \mathrm{b}^2} + \frac{dr}{r} = 0 \quad ; \quad \mathrm{b}^2 = \lambda^2 + \frac{1}{4}$$

$$\Rightarrow \quad \mathrm{R(r)} = \mathrm{b} \tanh(\,\mathrm{b} \ln(\mathrm{c}_1 r)) - \frac{1}{2} \quad ; \quad (\mathrm{c}_1 r) \rangle\, 0$$

The equation (20), is different than of the previous case which appeared, but we can solve it as follows :

This equation is similar to ***Riccati equation***[8], and we will transform it to linear ordinary differential equation of second order[6], using the following assumption, keeping in mind that *V* is a function of θ

$$V = -W \quad \Rightarrow \quad V' = -W' \Rightarrow \quad V' = V^2 - \cot\theta\, V + \lambda^2$$

Let $V = -\dfrac{M'}{M} \qquad \Rightarrow \qquad V' = -\dfrac{M''}{M} + \left(\dfrac{M'}{M}\right)^2$

$$\frac{M''}{M} = \cot\theta\, V - \lambda^2 \quad \Rightarrow M'' + \cot\theta\, M' + \lambda^2 M = 0$$

$$\Rightarrow \quad \sin\theta \frac{d^2M}{d\theta^2} + \cos\theta \frac{dM}{d\theta} + \lambda^2 \sin\theta\, M(\theta) = 0$$

This equation is a linear ordinary differential equation of second order, it can be solved by using the assumption $x = \cos\theta$, and by chain rule,

$$\frac{dM}{d\theta} = \frac{dM}{dx} \cdot \frac{dx}{d\theta} = -\sin\theta \frac{dM}{dx} \Rightarrow \sin\theta \frac{dM}{d\theta} = -\sin^2\theta \frac{dM}{dx} = (x^2 - 1)\frac{dM}{dx}$$

Therefore,

$$\frac{d}{d\theta}(\sin\theta \frac{dM}{d\theta}) = \frac{d}{dx}\left[(x^2 - 1)\frac{dM}{dx}\right]\frac{dx}{d\theta} = \sin\theta \frac{d}{dx}\left[(1 - x^2)\frac{dM}{dx}\right]$$

$$\Rightarrow \quad (1 - x^2)\frac{d^2M}{dx^2} - 2x\frac{dM}{dx} + \lambda^2 M = 0 \quad \ldots (21)$$

Let $\lambda^2 = \mathrm{n}(\mathrm{n}+1)$, then (21) will take the form

$$(1 - x^2)\frac{d^2M}{dx^2} - 2x\frac{dM}{dx} + \mathrm{n}(\mathrm{n}+1)\, M = 0 \quad \ldots (22)$$

This equation is ***Legendre differential equation*****[9]**, then the general solution of (22) is given by:

$$M(x) = c_2 P_n(x) + c_3\, Q_n(x)\,; \quad n = 0,1,2,\ldots$$

where $P_n(x)$ and $Q_n(x)$ are Legendre functions such that $P_n(x)$ is given by :

If n = 0, 1, 2, 3, …, m , then $P_0(x) = 1,\, P_1(x) = x\,,\, P_2(x) = \frac{1}{2}(3x^2 - 1),$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)\,,\ \ldots\ ,P_m(x) = \frac{1}{2^m\, m!}\cdot\frac{d^m}{dx^m}(x^2-1)^m$$

The last formula is called (***Rodrigue's formula***), and

$$Q_n(x) = P_n(x)\int\frac{dx}{(x^2-1)[P_n(x)]^2}$$

$$\Rightarrow \quad W(\theta) = \frac{M'}{M} = \frac{-\sin\theta\,[c_2 P_n'(\cos\theta) + c_3 Q_n'(\cos\theta)]}{c_2 P_n(\cos\theta) + c_3 Q_n(\cos\theta)}$$

Therefore $U(r,\theta) = e^{\int(\frac{b\tanh(b\ln(c_1 r))}{r} - \frac{1}{2r})\,dr + \int\frac{-\sin\theta[c_2 P_n'(\cos\theta) + c_3 Q_n'(\cos\theta)]\,d\theta}{c_2 P_n(\cos\theta) + c_3 Q_n(\cos\theta)}}$

$$= e^{\ln[\cosh(b\ln(c_1 r))] - \frac{1}{2}\ln r + \ln[c_2 P_n(\cos\theta) + c_3 Q_n(\cos\theta)] + g} \qquad ;$$

$[c_2 P_n(\cos\theta) + c_3 Q_n(\cos\theta)] \rangle 0$

$U(r,\theta) = A\, r^{-\frac{1}{2}} \cosh(b\ln(c_1 r))[c_2 P_n(\cos\theta) + c_3 Q_n(\cos\theta)]$ where $A = e^g$

$$\Rightarrow\ U(r,\theta) = (a_1 r^{b-\frac{1}{2}} + a_2 r^{-(b+\frac{1}{2})})\ [c_2\, P_n(\cos\theta) + c_3\, Q_n(\cos\theta)]$$

where $a_1 = A\,\frac{c_1^b}{2}$ and $a_2 = A\,\frac{c_1^{-b}}{2}$

$$\Rightarrow \quad U(r,\theta) = (\,a_1 r^{\sqrt{\lambda^2+\frac{1}{4}}\ -\frac{1}{2}} + a_2 r^{-(\sqrt{\lambda^2+\frac{1}{4}}\ +\frac{1}{2})})[c_2\, P_n(\cos\theta) + c_3\, Q_n(\cos\theta)]$$

But $\lambda^2 = n\,(n+1)$

$$\Rightarrow \qquad n = \sqrt{\lambda^2 + \frac{1}{4}} - \frac{1}{2} \quad \text{and} \quad n+1 = \sqrt{\lambda^2+\frac{1}{4}}\ + \frac{1}{2}$$

So, the complete solution of (17) will have the shape

$$U(r,\theta) = (\,a_1 r^n + a_2 r^{-(n+1)})\ (c_2 P_n(\cos\theta) + c_3 Q_n(\cos\theta))$$

where a_i and c_j ; $(i = 1, 2$ and $j = 2, 3)$ are arbitrary constants.

This solution becomes infinite when $r = 0$, this keep us to choose $a_2 = 0$, and if $\theta = 0$ or π then Legendre functions become infinite, we can get the boundedess when we take $n = 0, 1, 2, 3, \ldots$ So, $P_n(\cos\theta)$ is ***Legendre polynomials***, since the functions $Q_n(\cos\theta)$ are infinite if $\theta = 0\,,\,\pi$, accordingly, we choose $c_3 = 0$. Thus the solution becomes

$$U(r,\theta) = A\, r^n P_n(\cos\theta) \qquad ; \qquad A = a_1\, c_2$$

To obtain a solution that also satisfies the initial condition (16),we consider the series

$$U(r,\theta) = \sum_{n=0}^{\infty} a_n r^n P_n(\cos\theta)$$

Now, the values of the coefficients a_n can be determined from the condition (16), and hence

$$U(1,\theta) = f(\theta) = \sum_{n=0}^{\infty} a_n P_n(\cos\theta) \qquad \ldots (23)$$

To find the expansion of $f(\theta)$ using ***Legendre series*[9]**, and multiplying (23) by $\sin\theta\, P_m(\cos\theta)$ and integrating with respect to θ from 0 to π, we get

$$\int_0^{\pi} f(\theta)\, P_m(\cos\theta)\sin\theta\, d\theta = \sum_{n=0}^{\infty} a_n \int_0^{\pi} P_n(\cos\theta)\, P_m(\cos\theta)\sin\theta\, d\theta$$

$$= \sum_{n=0}^{\infty} a_n \int_{-1}^{1} P_n(x)\, P_m(x)\; dx = \begin{cases} 0 & \text{if } n \neq m \\ \dfrac{2\,a_n}{2\,n+1} & \text{if } n = m \end{cases}$$

We can realize that ***Legendre polynomials*** are orthogonal on the interval (-1, 1), so

$a_n = \frac{2\,n+1}{2} U_o \int_0^{\frac{\pi}{2}} P_n(\cos\theta)\sin\theta\, d\theta$ Putting n = 0,1,2,… and using the first few ***Legendre polynomials terms*** we get $P_o(\cos\theta) = 1,\, P_1(\cos\theta) = \cos\theta$,

$P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$, $P_3(\cos\theta) = \frac{1}{2}(5\cos^3\theta - 3\cos\theta)$, ...

So , $a_o = \frac{U_o}{2}$, $a_1 = \frac{3U_o}{4}$, $a_2 = 0$, $a_3 = \frac{-7\,U_o}{16}$,...

$$\Rightarrow \quad U(r,\theta) = \frac{U_o}{2}\left[1 + \frac{3}{2}\, r\, P_1(\cos\theta) - \frac{7}{8}\, r^3 P_3(\cos\theta) + \ldots\right]$$

Which represents the solution of the above partial differential equation .

Conclusion

We found a substitution to solve special kind of partial differential equations as follows :

$Z(x,y) = e^{\int \frac{U(x)}{x}\, dx + \int V(y)\, dy}$,

this a substitution help us to find solution of this kind of partial differential equations quickly with steps less than known methods.

This method used in search is method right to solve the heat equation in spherical coordinates and can applied its on the same equation in other coordinates, also can be applied on other equations as Laplace equation and wave equation in different coordinates.

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