Solve Inverse problem ofFractal Rendering 3D Shapes.

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Abstract

The inverse problem of fractal shapes can be considered as the main difficulty in fractal geometry, because the problem of determining and identifying the parameters of the affine mappings that constitute the Iterated Function System (IFS).

The purpose of this paper is to formulate and solve the inverse problem of Fractal Rendering 3D Shapes using a new approach in optimization theory. This can be made by finding and generating parameters of a set of affine maps, contractive mapping; which is IFS forpatches; that repeatedly iterative numerically by using the random iteration algorithm

المستخلص

المسألة العكسية للأشكال الكسورية يمكن أن توصف بالصعوبة الرئيسية في الهندسة الكسورية، بسببوجود مشكلةَ في حساب وتَمييز معاملاتمخططات التقريب الذي يشكلِ نظام الوظيفة المتكّرر (IFS).

إنّ غرضَ هذا البحثِ هو صياغة وحل المسألة العكسية للأشكال الكسورية المعادة الثلاثية الابعاد باستخدام طريقة جديدة في نظرية الأمثلية. ويمكن هذا بإيجاد وتوليد الثوابت أو العوامل لمجموعة الدوال، الذي يشكلِ نظام الوظيفة المتكَرر (IFS) والذي يتكرر عددياً باستخدام خوارزمية التكرار العشوائية.

1- Introduction

Fractals discovered by Benoit Mandelbrot [4] in 1970shave changed the way we see everything. Natural objectssuch as clouds, plants, landscapes and many other objects,complex in shape, can be efficiently modeled using this newtool. The main property of fractals – self-similarity is thecrucial feature used in finding fractal parameters needed inmodeling process.

Fractals possess many different aspects through which they can be characterized. Among these are:

- Self-similarity.
- Non-integer dimensionality.
- Being attractors of some peculiar dynamical system.

Self-similarity means that the wholeobject is similar to its parts. For commonly known fractalssuch as Koch's curve, Cantor's set and Sierpinski's gasketself-similarity property is evidently seen. But smooth curvesor patches usually we do not treat as fractals although inreality they are, as it has been shown in [3],[11]. Eg. Self – similarityof Bézier curves or patches is not obvious. Basingon self-similarity property one can define IFS (IteratedFunction System) that can be further used to generateiteratively the so-called attractor (fixed point of IFS) that is afractal. So fractals may be considered as attractors of IFSs. Itis worth mentioning that if one knows IFS then one cangenerate its attractor easily using one of the methods –deterministic or probabilistic one. But the solution of theinverse problem to find IFS of any attractor, in general, isnot solved yet.

Inspirited by [3], [10] in this paper, using self-similarityproperty we derive formulas that give fractal description ofpatches based on 3 control points.

Having IFSs for fractal rendering of patches one may use itfor fractal generation of any 3D shape that may be presentedas a finite collection of single patches. Precisely speaking, weuse the so-called PIFS (Partitioned IFS), not IFS, becauseevery patch is modeled separately by its own IFS.We presentexamples of fractal rendering both for single patches and forsome more complex 3D shapes consisting of several patches.

2- The Inverse Problem

The so – called *"Inverse Problem"* of fractal shapes can be considered as the main difficulty in fractal geometry, because the problem of determining and identifying the parameters of the affine mappings that constitute the Iterated Function System (IFS). Originally,[2] solved it by laborious human manipulation (requiring about 100 hours of effort per image). Earlyautomated attempts (e.g. [4]) relied upon matching power moments of candidate IFS with those ofthe target image. They require huge (and slow) searches through parameter spaces, and tend to workpoorly for target images of dimension 2 and higher. Later attempts (e.g. [1], [8]) simplified the problemdrastically by making assumptions about the character of the transformations.

3- Fractal Rendering of 3D Shapes

Fractal rendering is progressive in nature, in contrary togeometrical modeling based on control points patches.Progressiveness is a very desirable property duringtransmitting of graphical information through the net. So

then, fractal rendering may be of the interests for practice.The results presented in the paper may be treated as anattempt of extension of fractal rendering method from 2D to 3D. Unfortunately, the results presented here are not socomplete as for 2D case for which a fractal renderingpipeline [13] has been proposed. Namely, any 2D contourcould be there automatically splatted into a finite number ofsegments with known IFS for each one. Exemplary twoshapes (a rabbit and a vase) presented in this paper have beenmodeled manually. Additionally, one more complex shape,the famous teapot from Utah, also had been renderedsuccessfully in the fractal way. Some examples of fractalmodeling of 2D shapes are presented in previous authors'papers [13], while 3D fractal shapes are displayed on [7].

4- IFS For Patches

Below in this section we give formulas that define IFSs forfractal rendering of 3-points-based patches.They seem to form a collection of patches sufficient to buildup a wide class of real and artificial 3D shapes. Also example of patches of different kinds rendered usingdeterministic or probabilistic methods are presented [13].

Any triangle in R^3 with vertices P$_1$=[x_1,y_1,z_1], P$_2$=[x_2,y_2,z_2],P$_3$=[x_3,y_3,z_3] can be divided into four smaller ones. Applyingthe mid-point subdivision strategy we can define thefollowing four subdivision matrices:

$$B_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_3 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Non-singular quadratic matrix *P* has the form:

$$P = \begin{bmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{x y z}$$

It should be pointed out that vertices of small triangles areenumerated in the same way (in clockwise direction) as inthe input triangle, as shown in Fig.1. Similarity between the input triangle and the 4th small triangle is defined by B_4P.The same is with the left three small triangles.

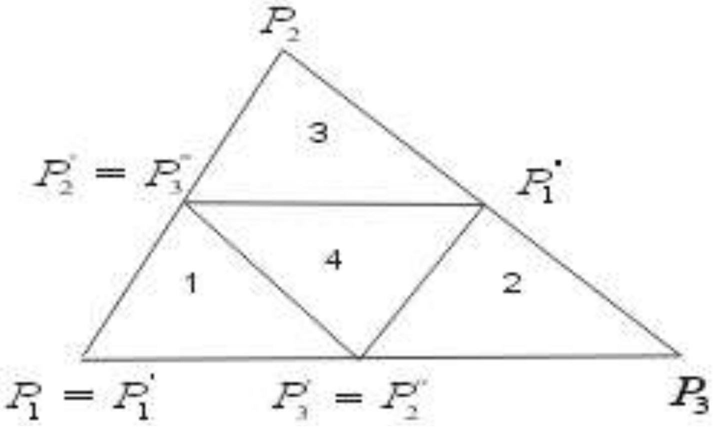

Fig.1. Subdivision of a triangle into four smaller ones.

Taking into account [13] we get the form ofIFS=*{F1,F2,F3,F4}*, where Fi=P-1·Bi·P, i=1,..,4. As theparticular case of the latest IFS we can obtain another one,more convenient, form of IFS. Namely, IFS=*{F1,F2,F3,F4}*$F_i = P^{-1} \cdot B_i \cdot P, i = 1, \cdots, 4.$As the particular case of the latest IFS we can obtain another one, more convenient, form of IFS. Namely, IFS=*{F1,F2,F3,F4}*, where

$$F_1 = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ \frac{x_1}{2} & \frac{y_1}{2} & \frac{z_1}{2} & 1 \end{bmatrix}, \quad F_2 = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ \frac{x_2}{2} & y_{\frac{2}{2}} & \frac{z_2}{2} & 1 \end{bmatrix}$$

$$F_3 = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ \frac{x_3}{2} & y_{\frac{3}{2}} & \frac{z_3}{2} & 1 \end{bmatrix}$$

$$F_4 = \begin{bmatrix} -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ \frac{x_1+x_2+x_3}{2} & \frac{y_1+y_2+y_3}{2} & \frac{z_1+z_2+z_3}{2} & 1 \end{bmatrix}$$

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The above given form of IFS can be obtained immediatelyusing subdivision matrices, if the fourth rows of zeros in $B_i, i=1,..4$, will be replaced by [*1/2,0,0,1/2*], [*0,0,1/2,1/2*],[*0,1/2,0,1/2*], [*1/2,1/2,1/2,1/2*], respectively. It means that IFSis not determined uniquely. For different IFSs it is easily toobserve different convergence to the attractor. IFSs givenabove generate attractor in R^4. After its projection on R^3 weobtain the fractal triangle. In *Fig.2* the fractal renderingprocess of a triangle is presented. The triangle appears in theprogressive way.

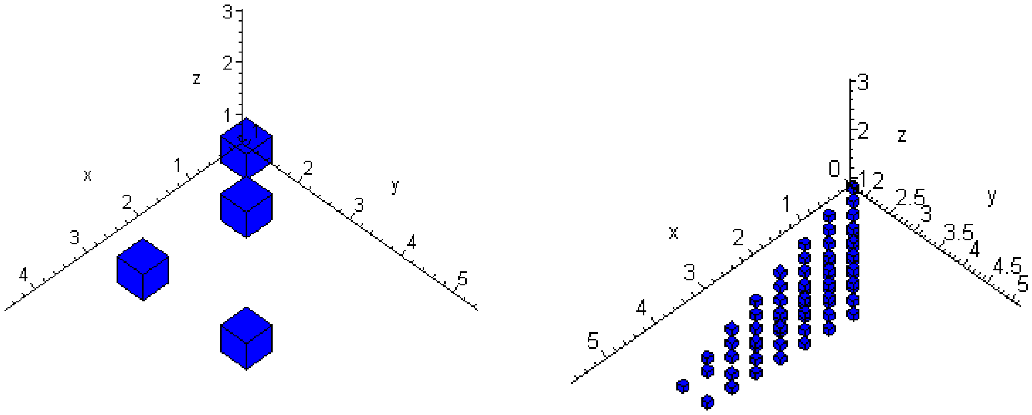

Fig. 2. A triangle rendered fractally using deterministicmethod starting from a box (iterations: 1,3,6).

5- An Minimization Problem Using Least Square Method

The minimization is due to the application of the discrete least square method which is the difference between the calculated and exact set of points constituting the attractor of the IFS, this will be done on minimizing the Hausdorff distancebetween these two sets [3].

Considering that, the fractal shape is given in advance, and the problem is to find the affine maps that constructing the IFS, the procedure is as follows:

i. Select n number of data points from the fractalrederning shape after scaling the figure using a prespecified coordinate axis and arrange the obtained data points into ascending order.

ii. Introducing an initial guess to the coefficients corresponding to thefractal renderingpatches.

iii. Applying the direct problem. The results of this program will produce a different shape, but has the property that their results depend on the initial parameters obtained from the inverse program.

iv. Select, also n-number of data points resulted by applying the direct problem on the produced results of the inverse problem and rearranging these points ascending order.

v. Evaluating the objective function using the discrete least square method. This function is given by:

$$d(A_e, A_a) = \sum_{i=1}^{n} (x_{e_i} - x_{a_i})^2$$

Where x_{e_i} is the x-coordinate of the original fractal shape, and x_{a_i} is the x-coordinate corresponding to the approximate fractal shape.

vi. Minimize the objective function proposed in step (v) usingHooke and Jeeves [5] or any optimization method.

6- Conclusion Remark

Due to the Simplest of direct optimization problem especial Hook & Jeeves method, the approximate method of “Hook & Jeeves” [5] will be used here in this work to find thenumerical solution of the inverse problem..

8- Future Work

1) Fractal dimension has been used to characterize data texture in a large number of physical and biological sciences. In medical imaging, Alan and Murray are presented in [2] a new method of estimating fractal dimension which is applicable to data-limited medical images and which shows promise of improving the reparability classes of images. On of the open future work to find relation ship between dimension of image in this method and the dimension of fractal shape in our work.
2) Using this subject in coding theory, so as to use a fractal shape as a password to the systems.

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