Error Estimate and Stability for the Finite Volume Method of the Convection – Diffusion Problem

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Abstract

In this paper, a finite volume method is studied for two-dimensional linear convection-diffusion problem. A linear convection term is approximated by upwind finite element scheme considered over a mesh to the triangular grid, whereas the diffusion term is approximated by using divergence theorem and approximate the direction derivative by difference quotient. The stability and error estimate of this method are proved under some assumption on the numerical fluxes.

Keywords. Finite volume method, convection-diffusion problem, error estimate, stability.

1. Introduction

The finite volume method (FVM) represent an efficient and robust method for the solution of conservation laws and inviscid compressible. This technique is based on expressing the balance of fluxes of conserved quantities through boundaries of control volumes, on the other hand, the finite element method (FEM) based on the concept of a weak solution defined with the aid of suitable test functions is quite natural for the solution of elliptic and parabolic problems. However, it is not mandatory to adhere to these paths of discretization in their respective regimes of common use. The finite (control) volume method (cell-centered or vertex-centered) may also be used for the discretization of elliptic problems (see Eymard(2000),Heinrich(1987)). Often the control volume approach is used in the framework of the finite element method in order to gain stability from upwinding (see Angermann(1995),Ohmori(1984),Schieweck(1989)). In the solution of convection-diffusion problems, it is quite natural to try to employ the advantages of both FV – FE methods in such a way that the FV method is used for the discretization of inviscid Euler fluxes. This idea leads (Feistauer(1995)) to the combined FV – FE method. The analysis and application of this method were investigated in (Angot(1998),Feistauer(1997),Feistauer(1999),Feistauer and Slavik(1999), Fort(2001),Kashkool(2002) and Manna(2000). In this paper, we consider the finite volume method for the elliptic form of the convection-diffusion problem. We present the theoretical of the error estimate and stability.

This paper consists of five sections. In section 2, the notation and the linear problem is given. In section 3, the finite volume space is defined, and the finite volume scheme. The stability for the solution of the discretized system is shown in section 4. In section 5, discrete error estimates is proven.

2. Formulation of the Problem and Some Notations

Throughout this paper, we will use C (with or without subscript or superscript) to denote generic constant independent of discrete parameter. $w_p^m(\Omega)$ denotes usual Sobolev spaces, where $\Omega \subset \mathbb{R}^2$ is a convex polygonal domain, m, p are nonnegative integer. The corresponding norm and semi-norm are $\|\cdot\|_{m,p,\Omega}$ and $|\cdot|_{m,p,\Omega}$ (Oden and Reddy(1976)). Particular, for $p = 2$, $H^m(\Omega) = w_2^m(\Omega)$, the corresponding norm and semi-norm are $\|\cdot\|_{m,2,\Omega}$ and $|\cdot|_{m,2,\Omega}$ respectively. Let $(\cdot,\cdot)$ denote the inner product of $L^2(\Omega)$, then

$$(u, v) = \int_\Omega uv\, dx.$$

As usual $H_0^1(\Omega) = \{v \in H^1(\Omega);\ v|_{\partial\Omega} = 0\}$ denote the subspaces of $H^1(\Omega)$.

We consider the following two-dimensional linear convection-diffusion initial boundary problem:

$$\frac{\partial u}{\partial t} - \varepsilon\, \Delta u + b \cdot \nabla u + cu = f \qquad (x,t) \in \Omega \times (0,T] = D; \tag{2.1}$$

$$u(x,t) = 0 \qquad (x,t) \in \Gamma \times (0,T]; \tag{2.2}$$

$$u(x,0) = u^o(x) \qquad x \in \overline{\Omega}, \tag{2.3}$$

where Γ is the boundary of Ω. The positive parameter ε is called diffusion coefficient, the vector $b: D \longrightarrow \mathbb{R}^2$ is called convection coefficient and $c, f: D \longrightarrow \mathbb{R}$ are given functions, and $u^o: \Omega \longrightarrow \mathbb{R}$ is given function.

We assume the coefficient of problem (2.1)-(2.3) satisfy the following conditions:

(A1) $b = (b_1, b_2) \in [W_\infty^1(\Omega)]^2$, $c \in W_\infty^1(\Omega)$, $f \in W_q^1(\Omega)$ with some $q > 2$,

(A2) $c-\frac{1}{2}\nabla\cdot b\geq a_o>0$ on Ω, where a_o does not depend on ε and $x\in\Omega$.

The weak form of problem (2.1)-(2.3) is, find $u:[0,T]\longrightarrow H_0^1(\Omega)$ such that,

$$(u_t^n,v)+a_\varepsilon(u^n,v)=(f^n,v),\ \text{for all } v\in H_0^1(\Omega), \tag{2.4}$$

$$u(0)=u^o, \tag{2.5}$$

where

$$(u_t^n,v)=\int_\Omega u_t^n\,v dx,\qquad a_\varepsilon(u^n,v)=\int_\Omega [\varepsilon\nabla u\cdot\nabla v+(b\cdot\nabla u+cu)v]dx,$$

$$(f^n,v)=\int_\Omega f^n v\,dx.$$

We assume that the weak solution u of problem (2.1)-(2.3) satisfies the following regularity:

(A3) $u\in L^\infty\big(0,T;H^2(\Omega)\big)\cap L^\infty\big(0,T;W_\infty^1(\Omega)\big),\ u_t,u_{tt},u_{ttt}\in L^\infty\big(0,T;L^\infty(\Omega)\big)$.

3. The Finite Volume Space and Finite Volume Scheme

Let us consider a family of regular triangulation $\{\mathcal{T}_h\}$ in $\overline{\Omega}$ (see Ciarlet(1978)). For a fixed triangulation $\mathcal{T}_h$, we defined the mesh parameter h by $h=\max_{\mathbb{T}\in\mathcal{T}_h} h_{\mathbb{T}}$, where $h_{\mathbb{T}}$ is the diameter of the triangle $\mathbb{T}$.

We assume that the triangulation family $\{\mathcal{T}_h\}$ is regular and weakly acute type, i.e.

(A4) There exists $\alpha_o\in(0,\frac{\pi}{2})$ independent of h, such that all interior angles α of the triangles are bounded as follows:

$$\alpha\in[\alpha_o,\frac{\pi}{2}],$$

For a given triangulation $\mathcal{T}_h$ with nodes $\{x_i\}\in\overline{\Omega}\,(1\leq i\leq K)$, where K is positive integer dependent on the triangulation, we construct a secondary partition. Namely, we introduce regions

$$\Omega_i^{\mathbb{T}}=\{x:x\in\mathbb{T},|x-x_i|\leq|x-x_j|\text{ for all } x_j\in\mathbb{T}\},$$

where $|x-x_i|$ is the distance of node x and node x_i. We consider the dual decomposition $\tilde{\mathcal{T}}_h=\{\Omega_i\}$, where Ω_i is circumcentric domain associated with nodal point x_i (Ikeda(1983))

$$\Omega_i=\cup\,\Omega_i^{\mathbb{T}}$$

We say that two nodes x_i, x_j are adjacent if and only if $\Gamma_{ij}=\partial\Omega_i\cap\partial\Omega_j\neq\emptyset$. The set of indices of all interior nodes $x_i\in\Omega$ is denoted by Λ whereas the set Λ_i contain the indices of all nodal points in $\overline{\Omega}$ adjacent to $x_i\in\overline{\Omega}$. Moreover we defined $d_{ij}=|x_i-x_j|$ and $x_{ij}=\frac{1}{2}(x_i+x_j)$. The area of Ω_i is denoted by $m_i=meas_1(\Omega_i)$ and for the length of the straight-line segment $\Gamma_{ij}\,(i\in\Lambda_i)$ we use the notation $m_{ij}=meas_2(\Gamma_{ij})$. If $m_{ij}>0$, then Γ_{ij} has a uniquely defined unit outward normal ν_{ij} with respect to Ω_i. The elements of this partition are defined as follows. Obviously, the straight-line segment Γ_{ij} and the node x_i can be regarded as an edge and the corresponding opposite vertex, respectively, of some triangle $\mathbb{T}_{ij}$. Now, an element Q_{ij} of the partition is defined by $Q_{ij} = \mathbb{T}_{ij} \cup \mathbb{T}_{ji}$.

Furthermore, the triangle $\mathbb{T}_{ij}$ can be represented as the union of two triangles $\mathbb{T}_{ij}^{(k)}(k = 1,2,$ the common boundary of which is part of the edge connecting the node xi with the node xj (see Figure(1)). We set $\Gamma_{ij}^{(k)} = \Gamma_{ij} \cap \mathbb{T}_{ij}^{(k)}$ and $m_{ij}^{(k)} = meas_2(\Gamma_{ij}^{(k)})$.

It is not difficult to see that Q_{ij} can also be decomposed as $Q_{ij} = Q_{ij}^{(1)} \cup Q_{ij}^{(2)}$,

where $Q_{ij}^{(k)} = \mathbb{T}_{ij}^{(k)} \cup \mathbb{T}_{ji}^{(3-k)}(k = 1,2)$.

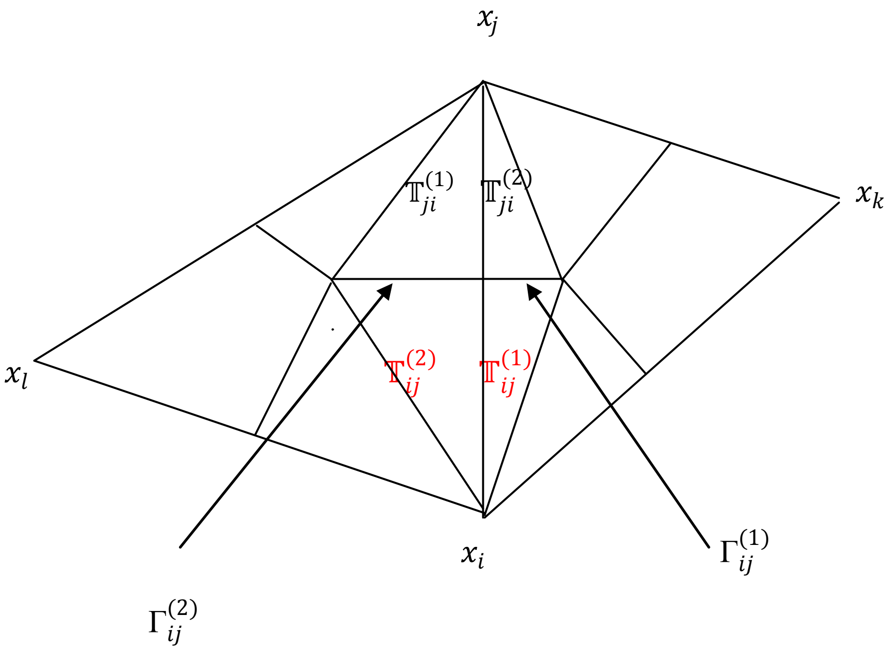

Figure (1)The auxiliary triangles $\mathbb{T}_{ij}^{(k)}$and $\mathbb{T}_{ji}^{(k)}$

We mention the following relation:

$$meas_1(Q_{ij}^{(k)}) = 2\, meas_1(\mathbb{T}_{ij}^{(k)}) = \tfrac{1}{2} m_{ij}^{(k)} d_{ij}. \tag{3.1}$$

For $\ell = \{0,1,2, \dots\}$, $\mathbb{T} \in \mathcal{T}_h$ we denote by $P_\ell(\mathbb{T})$ the space of all polynomials on $\mathbb{T}$ of degree $\leq \ell$. In what follows the following finite element spaces

$X_h = \{v_h | v_h \in C(\overline{\Omega}); v_h|_{\mathbb{T}} \in P_1(\mathbb{T}), \forall \mathbb{T} \in \mathcal{T}_h\} \subset H^1(\Omega)$,

$V_h = \{v_h | v_h \in X_h ; v_h = 0 \text{ on } \partial\Omega\} \subset H_0^1(\Omega)$,

and the finite volume space

$Y_h = \{v_h | v_h \in L^2(\Omega) ; v_h|_{\Omega_i} \in P_0(\Omega_i), \forall \Omega_i \in \tilde{\mathcal{T}}_h \}$.

By making use of the characteristic function $\hat{\mu}_i$ of circumcentric domain Ω_i, the mass lumping operator $\wedge$ is now defined by $\wedge : w_h \in C(\overline{\Omega}) \longrightarrow \widehat{w}_h \in Y_h$, such that

$$\hat{w}_h(x) = \sum_{i=1}^{K} w_h(x_i)\hat{\mu}_i(x),$$

where $\hat{w}_h(x) = w_h(x_i) = w_{hi}$.

Then we have some important lemmas:

Lemma (3.1)(Ikeda(1983)). If $\mathcal{T}_h$ is regular triangulation of weakly acute type we have

$$|w_h|^2_{1,2,\Omega} \le \frac{6}{\hat{k}^2} \| \hat{w}_h \|^2_{0,2,\Omega} \ , \forall\, w_h \in X_h,$$

where $\hat{k} = \min\{\hat{k}_{\mathbb{T}};\ \mathbb{T} \in \mathcal{T}_h\}$, $\hat{k}_{\mathbb{T}}$ =minimum perpendicular length of $\mathbb{T}$.

Lemma (3.2)(Ikeda(1983). For all $w \in X_h$ with $p \ge 1$and all $w \in W^1_p(\Omega)$ with $p > 2$

$$\| \hat{w} - w \|_{0,p,\Omega} \le Ch|w|_{1,p,\Omega}.$$

Lemma (3.3)(Ikeda(1983). For all $w \in X_h$

$$(\nabla w, \nabla \varphi_i) = -\sum_{j\in\Lambda_i} (w_j - w_i)\frac{m_{ij}}{d_{ij}}, \quad 1 \le i \le K$$

where φ_i is base function of finite element space X_h.

Lemma (3.4)(Kashkool and Chechan(2011)). Let conditions (A3.1), (A3.2) and (A2.1) be fulfilled. Then, for sufficiently small $h_o > 0$ there exists a constant $\lambda > 0$ such that for all $h \in (0, h_o]$ and $v_h \in V_h$ the relation.

$$a_h(v_h, v_h) \ge \lambda \| v_h \|^2_{\varepsilon},$$

hold, where h_o can be chosen independently of ε and λ does not depend on ε and h.

The finite volume scheme is found by integration equation (2.1) on a given control volume of discretization mesh and finding an approximation of the fluxes on the control volume boundary in terms of the discrete unknowns.

We turn to the derivation of the discrete scheme. We start by integrating the equation (2.1) over Ω_i and using the relation

$$\nabla\cdot(b\,u^n) = b\cdot\nabla u^n + (\nabla\cdot b)u^n$$

$$\int_{\Omega_i} \frac{\partial u}{\partial t}\,dx - \int_{\Omega_i} \nabla\cdot(\varepsilon\,\nabla u^n)dx + \int_{\Omega_i} \nabla\cdot(b\,u^n)dx - \int_{\Omega_i} (\nabla\cdot b)u^n\,dx + \int_{\Omega_i} c\,u^n\,dx$$

$$= \int_{\Omega_i} f^n\,dx. \tag{3.2}$$

We approximate $\frac{\partial u}{\partial t}$ by the forward difference

$$\int_{\Omega_i} \frac{\partial u}{\partial t}\, dx \approx \tfrac{1}{\tau}\left(u_i^{n+1} - u_i^n\right) m_i\,, \tag{3.3}$$

where $\tau > 0$ is time step, for $n = 0,1,\ldots,N_\tau - 1$ with $N_\tau = {}^{T}/_{\tau}$.

Applying Gauss's theorem in second term of equation (3.2) we obtain

$$\int_{\Omega_i} \nabla\cdot(\varepsilon\,\nabla u^n)\,dx = \int_{\partial\Omega_i} \nu\cdot(\varepsilon\,\nabla u^n)\,ds\,,$$

where ν is the unit outer normal on $\partial\Omega_i$. Then, using the concrete structure of the boundary of Ω_i, we can write

$$\int_{\Omega_i} \nabla\cdot(\varepsilon\,\nabla u^n)\,dx = \sum_{j\in\Lambda_i} \int_{\Gamma_{ij}} \nu_{ij}\cdot(\varepsilon\,\nabla u^n)\,ds\,.$$

we approximate the direction derivatives by difference quotients

$$\nu_{ij}\cdot\nabla u^n \approx \frac{u_j^n - u_i^n}{d_{ij}}, \tag{3.4}$$

then

$$\int_{\Omega_i} \nabla\cdot(\varepsilon\,\nabla u^n)\,dx \approx \sum_{j\in\Lambda_i} \frac{\varepsilon}{d_{ij}}\left(u_j^n - u_i^n\right) m_{ij}\,. \tag{3.5}$$

To approximate the third term

$$\int_{\Omega_i} \nabla\cdot(b\,u^n)\,dx = \sum_{j\in\Lambda_i} \int_{\Gamma_{ij}} \nu_{ij}\cdot b\,u^n\,ds$$

$$\approx \sum_{j\in\Lambda_i} \gamma_{ij}\left(r_{ij}\,u_i^n + (1 - r_{ij})\,u_j^n\right) m_{ij}\,, \tag{3.6}$$

where $\nu_{ij}\cdot b|_{\Gamma_{ij}} \approx \gamma_{ij}$ is constant, and $u^n|_{\Gamma_{ij}} \approx r_{ij}\,u^n(x_i) + (1 - r_{ij})\,u^n(x_j)$

$r_{ij} \in [0,1]$ is a parameter and depends on ε, γ_{ij} and d_{ij}.

It remains in the left-hand side of equation (3.2), we approximate as follows:

$$\int_{\Omega_i} (\nabla\cdot b)\,u^n\,dx = \sum_{j\in\Lambda_i} u_i^n \int_{\Omega_i} \nabla\cdot b\,dx$$

$$= \sum_{j \in \Lambda_i} u_i^n \int_{\Gamma_{ij}} \nu_{ij} \cdot b \, ds$$

$$\approx \sum_{j \in \Lambda_i} u_i^n \gamma_{ij} m_{ij}, \qquad (3.7)$$

and

$$\int_{\Omega_i} c\, u^n \, dx \approx c_i \, u_i^n m_i \, . \qquad (3.8)$$

The approximation of the right-hand side in equation (3.2) is as follows

$$\int_{\Omega_i} f^n dx \approx f_i^n m_i \, . \qquad (3.9)$$

Thus we obtain the following discrete version of equation (3.2) :

$$\frac{1}{\tau}\left(u_{hi}^{n+1} - u_{hi}^n\right)m_i + \sum_{j \in \Lambda_i} \frac{\varepsilon}{d_{ij}}\left(1-(1-r_{ij})\frac{\gamma_{ij}d_{ij}}{\varepsilon}\right)(u_{hi}^n - u_{hj}^n)m_{ij} + c_i u_{hi}^n m_i$$

$$= f_i^n m_i \, . \quad (3.10)$$

Taking an arbitrary function $v_h \in V_h \subset H_0^1(\Omega)$ multiplying equation (3.10) by v_{hi} and summing all these expression over $i \in \Lambda$, the resulting discrete problem can be written in the form

$$(D_\tau u_h^n, v_h) + a_h(u_h^n, v_h) = (\mathcal{F}^n, v_h) \quad \text{for all } v_h \in V_h,$$

(3.11)

where

$$(D_\tau u_h^n, v_h) = \sum_{i \in \Lambda} v_{hi} \left(\frac{u_{hi}^{n+1} - u_{hi}^n}{\tau}\right) m_i \, ,$$

$$a_h(u_h^n, v_h) = \sum_{i \in \Lambda} v_{hi} \left\{ \sum_{j \in \Lambda_i} \frac{\varepsilon}{d_{ij}} \left(1-(1-r_{ij})\frac{\gamma_{ij}d_{ij}}{\varepsilon}\right)(u_{hi}^n - u_{hj}^n)m_{ij} + c_i u_{hi}^n m_i \right\},$$

$$\left(\mathcal{F}^n, v_h\right) = \sum_{i \in \Lambda} v_{hi} \; f_i^n \; m_i \, .$$

Moreover, we introduce the following norm (Angermann(1995)).

$$\| v_h \|_h = \sqrt{(v_h, v_h)_h} = \| \hat{v}_h \|_{0,2,\Omega} \, ,$$

(3.12)

$$\| v_h \|_\varepsilon = \sqrt{\varepsilon \, |v_h|_{1,2,\Omega}^2 + \| v_h \|_{0,2,\Omega}^2} .$$

(3.13)

The scheme (3.11) can also be defined for control functions $r : \mathbb{R} \longrightarrow [0,1]$, where the control function r is defined as (Angermann(1991))

$$r(\gamma_{ij} d_{ij}/\varepsilon) = r(\mathrm{z}) = 1 - \frac{1}{\mathrm{z}} + \frac{1}{e^{\mathrm{z}} - 1}.$$

However, we have that these functions satisfy the following properties:

(P1) $\lim_{\mathrm{z}\to-\infty} r(\mathrm{z}) = 0$, $\lim_{\mathrm{z}\to\infty} r(\mathrm{z}) = 1$,

(P2) $1+\mathrm{z}r(\mathrm{z}) \geq 0$ for all z,

(P3) $[1 - r(\mathrm{z}) - r(-\mathrm{z})]\mathrm{z} = 0$ for all z,

(P4) $[\frac{1}{2} - r(\mathrm{z})]\mathrm{z} \leq 0$ for all z.

4. The Stability of the Scheme

Theorem (4.1)

Let $u_h \in V_h$ be the approximation solution of equation (3.11) at $t = t^{n+\frac{1}{2}}$, $n = 0,1,\dots,m-1$, then

$$\| u_h^m \|_{0,2,\Omega}^2 \leq \| u_h^0 \|_{0,2,\Omega}^2 + \frac{2T}{c} \sup_{s\in(0,T)} \| f(s) \|_{o,q,\Omega}^2,$$

where c is a positive constant .

Proof

In equation (3.11), we take the particular test function $v_h = u_h^{n+\frac{1}{2}}$, we have

$$\left(\frac{u_h^{n+1} - u_h^n}{\tau}, u_h^{n+\frac{1}{2}}\right) + a_h\left(u_h^{n+\frac{1}{2}}, u_h^{n+\frac{1}{2}}\right) = \left(\mathcal{F}^{n+\frac{1}{2}}, u_h^{n+\frac{1}{2}}\right),$$

Since

$$\frac{1}{\tau}\left(u_h^{n+1} - u_h^n, \frac{u_h^{n+1} + u_h^n}{2}\right) = \frac{1}{2\tau}\left(\| u_h^{n+1} \|_{0,2,\Omega}^2 - \| u_h^n \|_{0,2,\Omega}^2\right),$$

from lemma (3.4) and Schwarz inequality, we have

$$\frac{1}{2\tau}\left(\| u_h^{n+1} \|_{0,2,\Omega}^2 - \| u_h^n \|_{0,2,\Omega}^2\right) + c\,|u_h^{n+\frac{1}{2}}|_{\varepsilon}^2 \leq \| \mathcal{F}^{n+\frac{1}{2}} \|_{0,q,\Omega} \| u_h^{n+\frac{1}{2}} \|_{0,2,\Omega} \qquad (4.1)$$

Now, applying the Young's inequality with $\varepsilon = {}^{c}/_{2}$ to the right-hand side of equation (4.1) and using (3.13), we have

$$\frac{1}{2\tau}\left(\| u_h^{n+1} \|_{0,2,\Omega}^2 - \| u_h^n \|_{0,2,\Omega}^2\right) \leq \frac{1}{c} \| \mathcal{F}^{n+\frac{1}{2}} \|_{0,q,\Omega}^2,$$

by taking the summation from 0 to $m-1$ for both side

$$\sum_{n=0}^{m-1}\left(\| u_h^{n+1} \|_{0,2,\Omega}^2 - \| u_h^n \|_{0,2,\Omega}^2\right) \leq \frac{2}{c}\sum_{n=0}^{m-1} \tau \| \mathcal{F}^{n+\frac{1}{2}} \|_{0,q,\Omega}^2,$$

thus

$$\| u_h^m \|_{0,2,\Omega}^2 - \| u_h^0 \|_{0,2,\Omega}^2 \leq \frac{2}{c} \sum_{n=0}^{m-1} \tau \ \| \hat{f}^{n+\frac{1}{2}} \|_{0,q,\Omega}^2,$$

$$\| u_h^m \|_{0,2,\Omega}^2 \leq \| u_h^0 \|_{0,2,\Omega}^2 + \frac{1}{c} \sum_{n=0}^{m-1} \tau \ (\| \hat{f}^{n+1} \|_{0,q,\Omega}^2 + \| \hat{f}^{n} \|_{0,q,\Omega}^2)$$

$$\leq \| u_h^0 \|_{0,2,\Omega}^2 + \frac{2T}{C} \sup_{s\in(0,T)} \| f(s) \|_{o,q,\Omega}^2.$$

□

5. The Error Estimate

Theorem (5.1)

Let u be the weak solution of equation (2.4) which satisfies the conditions (A1), (A2) and (A3), u_h be the solution of equation (3.11), and τ be the time step, then for $\tau > 0$ small enough and $\tau = O(h)$, then we have

$$\| u_h - u \|_{0,2,\Omega} \leq C(h + \tau^2),$$

where C is constant independent of h and τ.

proof

Let $w_h = \mathrm{I}_h u$, where I_h is an interpolation operator from $C(\overline{\Omega})$ to V_h, we set $u_h - u = (u_h - w_h) + (w_h - u) = \theta_h + \rho$. It is clear that (Thom$\acute{e}$e(1984))

$$\| \mathrm{I}_h u - u \| = \| \rho \|_{0,2,\Omega} \leq Ch^2 |u|_{0,2,\Omega}, \qquad \| \rho \|_{1,2,\Omega} \leq Ch,$$

where, C is constant independent of u, h and τ.

We prove this theorem by mathematics induction. It is clear that at n=0,$\| u_h^0 - u^0 \|_{0,2,\Omega} \leq Ch$, we assume $\| u_h^m - u^m \|_{0,2,\Omega} \leq C(h + \tau^2)$ at $0 \leq m \leq n$. Then, we must prove

$$\| u_h^{n+1} - u^{n+1} \|_{0,2,\Omega} \leq C(h + \tau^2).$$

From equation (3.11) and equation (2.4), with $D_\tau \hat{\theta}_h^n = \frac{\hat{\theta}_h^{n+1} - \hat{\theta}_h^n}{\tau}$, at $t = t^{n+\frac{1}{2}}$ we have

$$\left(D_\tau \hat{\theta}_h^n, \hat{v}_h\right) + a_h(\theta_h^{n+\frac{1}{2}}, v_h) = (\hat{f}^{n+\frac{1}{2}}, \hat{v}_h) - (D_\tau \hat{w}_h^n, \hat{v}_h) - a_h(w_h^{n+\frac{1}{2}}, v_h)$$

$$= [(\hat{f}^{n+\frac{1}{2}}, \hat{v}_h) - (f^{n+\frac{1}{2}}, v)] - [(D_\tau \hat{w}_h^n, \hat{v}_h) - (u_t^{n+\frac{1}{2}}, v)] - [a_h(w_h^{n+\frac{1}{2}}, v_h) - a_\varepsilon(u^{n+\frac{1}{2}}, v)].$$

Put $v = \theta^{n+\frac{1}{2}}$

$$(D_\tau \hat{\theta}_h^n, \hat{\theta}_h^{n+\frac{1}{2}}) = \left(\frac{\hat{\theta}_h^{n+1} - \hat{\theta}_h^n}{\tau}, \frac{\hat{\theta}_h^{n+1} + \hat{\theta}_h^n}{2}\right) = \frac{1}{2} D_\tau \| \hat{\theta}_h^n \|_{0,2,\Omega}^2.$$

From Lemma (3. 4) , we have

$$a_h(\theta_h^{n+\frac{1}{2}}, \theta_h^{n+\frac{1}{2}}) \geq m \, |\theta^{n+\frac{1}{2}}|_{1,2,\Omega}^2.$$

Then,

$$\frac{1}{2}D_\tau \parallel \hat{\theta}_h^n \parallel_{0,2,\Omega}^2 + m\,|\theta^{n+\frac{1}{2}}|_{1,2,\Omega}^2 \le [(\hat{f}^{n+\frac{1}{2}}, \hat{\theta}_h^{n+\frac{1}{2}}) - (f^{n+\frac{1}{2}}, \theta^{n+\frac{1}{2}})]$$

$$-[(D_\tau \widehat{w}_h^n, \hat{\theta}_h^{n+\frac{1}{2}}) - (u_t^{n+\frac{1}{2}}, \theta^{n+\frac{1}{2}})] - [a_h(w_h^{n+\frac{1}{2}}, \theta_h^{n+\frac{1}{2}}) - a_\varepsilon(u^{n+\frac{1}{2}}, \theta^{n+\frac{1}{2}})].$$

Thus, we can write

$$\frac{1}{2}D_\tau \parallel \hat{\theta}_h^n \parallel_{0,2,\Omega}^2 + m|\theta^{n+\frac{1}{2}}|_{1,2,\Omega}^2 \le \sum_{k=1}^{3} A^{(k)}. \quad (5.1)$$

To estimate $A^{(1)}$,

$$A^{(1)} = (\hat{f}^{n+\frac{1}{2}}, \hat{\theta}_h^{n+\frac{1}{2}}) + (f^{n+\frac{1}{2}}, \hat{\theta}_h^{n+\frac{1}{2}}) - (f^{n+\frac{1}{2}}, \hat{\theta}_h^{n+\frac{1}{2}}) - (f^{n+\frac{1}{2}}, \theta^{n+\frac{1}{2}})$$

$$= (\hat{f}^{n+\frac{1}{2}} - f^{n+\frac{1}{2}}, \hat{\theta}_h^{n+\frac{1}{2}}) + (f^{n+\frac{1}{2}}, \hat{\theta}_h^{n+\frac{1}{2}} - \theta^{n+\frac{1}{2}}) = \sum_{k=1}^{2} A^{(1k)}.$$

To estimate $A^{(11)}$, we use Lemma (3.2) and Schwarz inequality

$$|A^{(11)}| = |(\hat{f}^{n+\frac{1}{2}} - f^{n+\frac{1}{2}}, \hat{\theta}_h^{n+\frac{1}{2}}) \le \parallel \hat{f}^{n+\frac{1}{2}} - f^{n+\frac{1}{2}} \parallel_{0,q,\Omega} \parallel \hat{\theta}_h^{n+\frac{1}{2}} \parallel_{0,2,\Omega}$$

$$\le Ch|f^{n+\frac{1}{2}}|_{1,q,\Omega} \parallel \hat{\theta}_h^{n+\frac{1}{2}} \parallel_{0,2,\Omega} \le Ch \parallel \hat{\theta}_h^{n+\frac{1}{2}} \parallel_{0,2,\Omega}.$$

By using Young's inequality, we get

$$\left|A^{(11)}\right| \le Ch^2 + \beta \parallel \hat{\theta}_h^{n+\frac{1}{2}} \parallel_{0,2,\Omega}^2.$$

To estimate $A^{(12)}$

$$|A^{(12)}| \le \parallel f^{n+\frac{1}{2}} \parallel_{0,q,\Omega} \parallel \hat{\theta}_h^{n+\frac{1}{2}} - \theta^{n+\frac{1}{2}} \parallel_{0,2,\Omega} \le Ch \parallel f^{n+\frac{1}{2}} \parallel_{0,q,\Omega} |\theta^{n+\frac{1}{2}}|_{1,2,\Omega}$$

$$\le Ch|\theta^{n+\frac{1}{2}}|_{1,2,\Omega} \le Ch^2 + \beta|\theta^{n+\frac{1}{2}}|_{1,2,\Omega}^2.$$

Then,

$$|A^{(1)}| \le Ch^2 + \beta \parallel \hat{\theta}_h^{n+\frac{1}{2}} \parallel_{0,2,\Omega}^2 + \beta|\,\theta^{n+\frac{1}{2}}|_{1,2,\Omega}^2\,. \quad (5.2)$$

To estimate $A^{(2)}$

$$A^{(2)} = (D_\tau \widehat{w}_h^n, \hat{\theta}_h^{n+\frac{1}{2}}) + (\hat{u}_t^{n+\frac{1}{2}}, \hat{\theta}_h^{n+\frac{1}{2}}) - (\hat{u}_t^{n+\frac{1}{2}}, \hat{\theta}_h^{n+\frac{1}{2}}) + (u_t^{n+\frac{1}{2}}, \hat{\theta}_h^{n+\frac{1}{2}}) - (u_t^{n+\frac{1}{2}}, \hat{\theta}_h^{n+\frac{1}{2}})$$

$$- (u_t^{n+\frac{1}{2}}, \theta_h^{n+\frac{1}{2}})$$

$$= (D_\tau \widehat{w}_h^n - \hat{u}_t^{n+\frac{1}{2}}, \hat{\theta}_h^{n+\frac{1}{2}}) + (\hat{u}_t^{n+\frac{1}{2}} - u_t^{n+\frac{1}{2}}, \hat{\theta}_h^{n+\frac{1}{2}}) + (u_t^{n+\frac{1}{2}}, \hat{\theta}_h^{n+\frac{1}{2}} - \theta_h^{n+\frac{1}{2}}) = \sum_{k=1}^{3} A^{(2k)}$$

To estimate $A^{(21)}$, by Taylor expansion, we have

$\hat{u}_t^{n+\frac{1}{2}} = \frac{\hat{u}^{n+1} - \hat{u}^n}{\tau} + \hat{\alpha}^{n+\frac{1}{2}}$, where $\parallel \hat{\alpha}^{n+\frac{1}{2}} \parallel_{0,2,\Omega} \le C\tau^2 \parallel u_{ttt} \parallel_{0,2,\Omega}$

Now

$$|A^{(21)}| = |(\frac{\widehat{w}_h^{n+1} - \widehat{w}_h^n}{\tau} - \frac{\hat{u}^{n+1} - \hat{u}^n}{\tau}, \hat{\theta}_h^{n+\frac{1}{2}}) + (\hat{\alpha}^{n+\frac{1}{2}}, \hat{\theta}_h^{n+\frac{1}{2}})|$$

$$= |(\frac{\widehat{w}_h^{n+1} - \hat{u}_h^{n+1}}{\tau} - \frac{\widehat{w}^n - \hat{u}^n}{\tau}, \hat{\theta}_h^{n+\frac{1}{2}}) + (\hat{\alpha}^{n+\frac{1}{2}}, \hat{\theta}_h^{n+\frac{1}{2}})|$$

$$\leq |\tfrac{1}{\tau}(\hat{\rho}^{n+1}, \hat{\theta}_h^{n+\frac{1}{2}})| + |\tfrac{1}{\tau}(\hat{\rho}^n, \hat{\theta}_h^{n+\frac{1}{2}})| + |(\hat{\alpha}^{n+\frac{1}{2}}, \hat{\theta}_h^{n+\frac{1}{2}})|$$

$$\leq (\tfrac{1}{\tau} \| \hat{\rho}^{n+1} \|_{0,2,\Omega} + \tfrac{1}{\tau} \| \hat{\rho}^n \|_{0,2,\Omega} + \| \hat{\alpha}^{n+\frac{1}{2}} \|_{0,2,\Omega}) \| \hat{\theta}_h^{n+\frac{1}{2}} \|_{0,2,\Omega}$$

$$\leq \frac{Ch^2}{\tau} \| \hat{\theta}_h^{n+\frac{1}{2}} \|_{0,2,\Omega} + C\tau^2 \| \hat{\theta}_h^{n+\frac{1}{2}} \|_{0,2,\Omega}.$$

Put $\tau = O(h)$ and using Young's inequality, we get

$$|A^{(21)}| \leq C(h^2 + \tau^4) + 2\beta \| \hat{\theta}_h^{n+\frac{1}{2}} \|_{0,2,\Omega}^2.$$

To estimate $A^{(22)}$

$$|A^{(22)}| = |(\hat{u}_t^{n+\frac{1}{2}} - u_t^{n+\frac{1}{2}}, \hat{\theta}_h^{n+\frac{1}{2}})|$$

$$\leq \| \hat{u}_t^{n+\frac{1}{2}} - u_t^{n+\frac{1}{2}} \|_{0,2,\Omega} \| \hat{\theta}_h^{n+\frac{1}{2}} \|_{0,2,\Omega} \leq Ch|u_t^{n+\frac{1}{2}}|_{1,2,\Omega} \| \hat{\theta}_h^{n+\frac{1}{2}} \|_{0,2,\Omega}$$

$$\leq Ch \| \hat{\theta}_h^{n+\frac{1}{2}} \|_{0,2,\Omega} \leq Ch^2 + \beta \| \hat{\theta}_h^{n+\frac{1}{2}} \|_{0,2,\Omega}^2.$$

To estimate $A^{(23)}$

$$|A^{(23)}| = |(u_t^{n+\frac{1}{2}}, \hat{\theta}_h^{n+\frac{1}{2}} - \theta_h^{n+\frac{1}{2}})|$$

$$\leq \| u_t^{n+\frac{1}{2}} \|_{0,2,\Omega} \| \hat{\theta}_h^{n+\frac{1}{2}} - \theta^{n+\frac{1}{2}} \|_{0,2,\Omega} \leq Ch \| u_t^{n+\frac{1}{2}} \|_{0,2,\Omega} |\theta^{n+\frac{1}{2}}|_{1,2,\Omega}$$

$$\leq Ch|\theta^{n+\frac{1}{2}}|_{1,2\Omega} \leq Ch^2 + \beta|\theta^{n+\frac{1}{2}}|_{1,2,\Omega}^2.$$

Then,

$$|A^{(2)}| \leq C(h^2 + \tau^4) + 3\beta \| \theta_h^{n+\frac{1}{2}} \|_{0,2,\Omega}^2 + \beta|\theta^{n+\frac{1}{2}}|_{1,2,\Omega}^2. \qquad (5.3)$$

To estimate $A^{(3)}$

$$A^{(3)} = a_h(w_h^{n+\frac{1}{2}}, \theta_h^{n+\frac{1}{2}}) - a_\varepsilon(u^{n+\frac{1}{2}}, \theta^{n+\frac{1}{2}})$$

Since

$$\frac{\varepsilon}{d_{ij}} (1 - (1 - r_{ij}) \frac{\gamma_{ij} d_{ij}}{\varepsilon}) (w_{hi}^{n+\frac{1}{2}} - w_{hj}^{n+\frac{1}{2}})$$

$$= \frac{\varepsilon}{d_{ij}} (w_{hi}^{n+\frac{1}{2}} - w_{hj}^{n+\frac{1}{2}}) + (\tfrac{1}{2} - r_{ij})\gamma_{ij} (w_{hj}^{n+\frac{1}{2}} - w_{hi}^{n+\frac{1}{2}}) + \tfrac{1}{2}\gamma_{ij} (w_{hj}^{n+\frac{1}{2}} - w_{hi}^{n+\frac{1}{2}}).$$

Then,

$$A^{(3)} = [\varepsilon \sum_{i \epsilon \Lambda} \theta_{hi}^{n+\frac{1}{2}} \sum_{j \epsilon \Lambda i} (w_{hi}^{n+\frac{1}{2}} - w_{hj}^{n+\frac{1}{2}}) \frac{m_{ij}}{d_{ij}} - (\varepsilon \nabla u^{n+\frac{1}{2}}, \nabla \theta^{n+\frac{1}{2}})]$$

$$+\sum_{i\epsilon\Lambda}\theta_{hi}^{n+\frac{1}{2}}\sum_{j\varepsilon\Lambda i}(\tfrac{1}{2}-\ r_{ij})\gamma_{ij}\,(w_{hj}^{n+\frac{1}{2}}-w_{hi}^{n+\frac{1}{2}})\,m_{ij}$$

$$+[\tfrac{1}{2}\sum_{i\epsilon\Lambda}\theta_{hi}^{n+\frac{1}{2}}\sum_{j\epsilon\Lambda i}(w_{hj}^{n+\frac{1}{2}}-w_{hi}^{n+\frac{1}{2}})\,\gamma_{ij}\,m_{ij}\ -\ (b.\nabla u^{n+\frac{1}{2}},\theta^{n+\frac{1}{2}})]$$

$$+[\sum_{i\epsilon\Lambda}\theta_{hi}^{n+\frac{1}{2}}c_i\,w_{hi}^{n+\frac{1}{2}}m_i-(cu^{n+\frac{1}{2}},\theta^{n+\frac{1}{2}})].$$

Thus, we can write

$$A^{(3)}=\sum_{k=1}^{4}A^{(3k)}\,.$$

To estimate $A^{(31)}$, by applying Lemma (3.3) and adding and subtracting $(\varepsilon\nabla\hat{u}^{n+\frac{1}{2}},\nabla\theta^{n+\frac{1}{2}})$, we obtain

$$A^{(31)}=(\varepsilon\nabla\widehat{w}_h^{n+\frac{1}{2}}-\varepsilon\nabla\hat{u}^{n+\frac{1}{2}},\nabla\theta^{n+\frac{1}{2}})+(\varepsilon\nabla\hat{u}^{n+\frac{1}{2}}-\varepsilon\nabla u^{n+\frac{1}{2}},\nabla\theta^{n+\frac{1}{2}})=\sum_{k=1}^{2}A^{(31k)}$$

To estimate $A^{(311)}$

Let $\hat{u}^{n+\frac{1}{2}}=\frac{\hat{u}^{n+1}+\hat{u}^{n}}{2}+\hat{\sigma}^{n+\frac{1}{2}},$

then $\|\hat{\sigma}^{n+\frac{1}{2}}\|_{0,2,\Omega}\leq C\tau^2\|u_{tt}\|_{0,2,\Omega}$ and $\|\nabla\hat{\sigma}^{n+\frac{1}{2}}\|_{0,2,\Omega}\leq C\tau^2\|\nabla u_{tt}\|_{0,2,\Omega}$

$$\begin{aligned}|A^{(311)}|&=|(\varepsilon(\frac{\nabla\widehat{w}_h^{n+1}+\nabla\widehat{w}_h^{n}}{2}-\frac{\nabla\hat{u}^{n+1}+\nabla\hat{u}^{n}}{2}),\nabla\theta^{n+\frac{1}{2}})+(\nabla\hat{\sigma}^{n+\frac{1}{2}},\nabla\theta^{n+\frac{1}{2}})|\\
&\leq(\frac{\varepsilon}{2}\|\nabla\hat{\rho}^{n+1}\|_{0,2,\Omega}+\frac{\varepsilon}{2}\|\nabla\hat{\rho}^{n}\|_{0,2,\Omega}+\|\nabla\sigma^{n+\frac{1}{2}}\|_{0,2,\Omega})\|\nabla\theta^{n+\frac{1}{2}}\|_{0,2,\Omega}\\
&\leq(\frac{\varepsilon}{2}\|\hat{\rho}^{n+1}\|_{1,2,\Omega}+\frac{\varepsilon}{2}\|\hat{\rho}^{n}\|_{1,2,\Omega}+\|\nabla\sigma^{n+\frac{1}{2}}\|_{0,2,\Omega})\|\nabla\theta^{n+\frac{1}{2}}\|_{0,2,\Omega}\\
&\leq Ch\|\nabla\theta^{n+\frac{1}{2}}\|_{0,2,\Omega}+C\tau^2\|\nabla\theta^{n+\frac{1}{2}}\|_{0,2,\Omega}\\
&\leq C(h^2+\tau^4)+2\beta\|\nabla\theta^{n+\frac{1}{2}}\|_{0,2,\Omega}^2\leq C(h^2+\tau^4)+2\beta|\theta^{n+\frac{1}{2}}|_{1,2,\Omega}^2.\end{aligned}$$

To estimate $A^{(312)}$

$$\begin{aligned}|A^{(312)}|&=|(\varepsilon(\nabla\hat{u}^{n+\frac{1}{2}}-\nabla u^{n+\frac{1}{2}}),\nabla\theta^{n+\frac{1}{2}})|\\
&\leq\varepsilon\|\nabla\hat{u}^{n+\frac{1}{2}}-\nabla u^{n+\frac{1}{2}}\|_{0,2,\Omega}\|\nabla\theta^{n+\frac{1}{2}}\|_{0,2,\Omega}\leq Ch|\nabla u^{n+\frac{1}{2}}|_{1,2,\Omega}\|\nabla\theta^{n+\frac{1}{2}}\|_{0,2,\Omega}\\
&\leq Ch|u^{n+\frac{1}{2}}|_{2,2,\Omega}\|\nabla\theta^{n+\frac{1}{2}}\|_{0,2,\Omega}\leq Ch\|\nabla\theta^{n+\frac{1}{2}}\|_{0,2,\Omega}\\
&\leq Ch^2+\beta\|\nabla\theta^{n+\frac{1}{2}}\|_{0,2,\Omega}^2\leq Ch^2+\beta|\theta^{n+\frac{1}{2}}|_{1,2,\Omega}^2.\end{aligned}$$

Then,

$$|A^{(31)}|\leq C(h^2+\tau^4)+3\beta|\theta^{n+\frac{1}{2}}|_{1,2,\Omega}^2. \qquad (5.4)$$

To estimate $A^{(32)}$, applying asymmetry argument, and we use the relations

$(\frac{1}{2}-r_{ji})\gamma_{ji} = (\frac{1}{2}-r_{ij})\gamma_{ij}$ and $m_{ij} = m_{ji}$

$$A^{(32)} = \frac{1}{2}\sum_{i\in\Lambda}\sum_{j\in\Lambda_i}\{(\tfrac{1}{2}-r_{ij})\gamma_{ij}\,(w_{hj}^{n+\frac{1}{2}}-w_{hi}^{n+\frac{1}{2}})\theta_{hi}^{n+\frac{1}{2}}+(\tfrac{1}{2}-r_{ji})\gamma_{ji}\,(w_{hi}^{n+\frac{1}{2}}-w_{hj}^{n+\frac{1}{2}})\theta_{hj}^{n+\frac{1}{2}}\}m_{ij}.$$

$$= \frac{1}{2}\sum_{i\epsilon\Lambda}\sum_{j\epsilon\Lambda i}(\tfrac{1}{2}-r_{ij})\gamma_{ij}\,(w_{hj}^{n+\frac{1}{2}}-w_{hi}^{n+\frac{1}{2}})\ (\theta_{hi}^{n+\frac{1}{2}}-\theta_{hj}^{n+\frac{1}{2}})m_{ij}.$$

Now, in view of $\left|r_{ij}-\frac{1}{2}\right|\le\frac{1}{2}$ and Sobolev's imbedding theorem, which implies $|\gamma_{ij}|\le C\parallel b\parallel_{1,\infty,\Omega^2}$, we have

$$|A^{(32)}|\le C\parallel b\parallel_{1,\infty,\Omega^2}\sum_{i\epsilon\Lambda}\sum_{j\epsilon\Lambda i}|w_{hj}^{n+\frac{1}{2}}-w_{hi}^{n+\frac{1}{2}}||\theta_{hi}^{n+\frac{1}{2}}-\theta_{hj}^{n+\frac{1}{2}}|m_{ij}.$$

Now, we may write

$$|\theta_{hj}^{n+\frac{1}{2}}-\theta_{hi}^{n+\frac{1}{2}}| = |\frac{\theta_{hj}^{n+1}+\theta_{hj}^{n}}{2}-\frac{\theta_{hi}^{n+1}+\theta_{hi}^{n}}{2}| = |\frac{\theta_{hj}^{n+1}-\theta_{hi}^{n+1}}{2}+\frac{\theta_{hj}^{n}-\theta_{hi}^{n}}{2}|$$

$$= \frac{1}{2}|\frac{\theta_{hj}^{n+1}-\theta_{hi}^{n+1}}{d_{ij}}+\frac{\theta_{hj}^{n}-\theta_{hi}^{n}}{d_{ij}}|d_{ij}$$

$$= \frac{1}{2}|\nu_{ij}.\nabla\theta_h^{n+1}+\nu_{ij}.\nabla\theta_h^{n}|d_{ij} \quad = |\nu_{ij}.\nabla\theta_h^{n+\frac{1}{2}}|d_{ij}.$$

Using equation (3.1) we get

$$|\theta_{hj}^{n+\frac{1}{2}}-\theta_{hi}^{n+\frac{1}{2}}| = \frac{4}{m_{ij}}|\nu_{ij}.\nabla\theta_h^{n+\frac{1}{2}}|meas_1(\mathbb{T}_{ij})$$

$$= \frac{4}{m_{ij}}\int_{\mathbb{T}_{ij}}|\nu_{ij}\,.\nabla\theta_h^{n+\frac{1}{2}}|\,dx$$

$$= \frac{4}{m_{ij}}\sum_{k=1}^{2}\int_{\mathbb{T}_{ij}^{(k)}}|\nu_{ij}\,.\nabla\theta_h^{n+\frac{1}{2}}|\,dx \quad \le\frac{4}{m_{ij}}\ |\theta_h^{n+\frac{1}{2}}|_{1,2,\mathbb{T}_{ij}}\sqrt{meas_1(\mathbb{T}_{ij})}. \tag{5.5}$$

It follows that

$$|w_{hj}^{n+\frac{1}{2}}-w_{hi}^{n+\frac{1}{2}}||\theta_{hi}^{n+\frac{1}{2}}-\theta_{hj}^{n+\frac{1}{2}}|\le\frac{16}{m_{ij}^2}|w_h^{n+\frac{1}{2}}|_{1,2,\mathbb{T}_{ij}}|\theta_h^{n+\frac{1}{2}}|_{1,2,\mathbb{T}_{ij}}meas_1(\mathbb{T}_{ij})$$

$$= \frac{4d_{ij}}{m_{ij}}\ |w_h^{n+\frac{1}{2}}|_{1,2,\mathbb{T}_{ij}}|\theta_h^{n+\frac{1}{2}}|_{1,2,\mathbb{T}_{ij}},$$

and consequently

$$|A^{(32)}|\le Ch\parallel b\parallel_{1,\infty,\Omega^2}\sum_{i\epsilon\Lambda}\sum_{j\epsilon\Lambda i}|w_h^{n+\frac{1}{2}}|_{1,2,\mathbb{T}_{ij}}|\theta_h^{n+\frac{1}{2}}|_{1,2,\mathbb{T}_{ij}}$$

$$\le Ch\parallel b\parallel_{1,\infty,\Omega^2}|w_h^{n+\frac{1}{2}}|_{1,2,\Omega}|\theta_h^{n+\frac{1}{2}}|_{1,2,\Omega}\le Ch|\theta_h^{n+\frac{1}{2}}|_{1,2,\Omega}.$$

By using Young's inequality, we get

$|A^{(32)}| \leq Ch^2 + \beta|\theta_h^{n+\frac{1}{2}}|_{1,2,\Omega}^2.$

From Lemma (3.1), we have

$$\left|A^{(32)}\right| \leq Ch^2 + \beta \parallel \hat{\theta}_h^{n+\frac{1}{2}} \parallel_{0,2,\Omega}^2. \tag{5.6}$$

To estimate $A^{(33)}$

$$A^{(33)} = \frac{1}{2}\sum_{i\epsilon\Lambda} \theta_{hi}^{n+\frac{1}{2}} \sum_{j\epsilon\Lambda i} \int_{\Gamma_{ij}} \nu_{ij} \cdot b(w_{hj}^{n+\frac{1}{2}} - w_{hi}^{n+\frac{1}{2}})ds - (\nabla \cdot bu^{n+\frac{1}{2}}, \theta^{n+\frac{1}{2}})$$

$$+(u^{n+\frac{1}{2}}\nabla \cdot b, \theta^{n+\frac{1}{2}}) + (\nabla \cdot bu^{n+\frac{1}{2}}, \hat{\theta}_h^{n+\frac{1}{2}}) - (\nabla \cdot bu^{n+\frac{1}{2}}, \hat{\theta}_h^{n+\frac{1}{2}}) + (u^{n+\frac{1}{2}}\nabla \cdot b, \hat{\theta}_h^{n+\frac{1}{2}})$$

$$-(u^{n+\frac{1}{2}}\nabla \cdot b, \hat{\theta}_h^{n+\frac{1}{2}}) + (\hat{u}^{n+\frac{1}{2}}\nabla \cdot b, \hat{\theta}_h^{n+\frac{1}{2}}) - (\hat{u}^{n+\frac{1}{2}}\nabla \cdot b, \hat{\theta}_h^{n+\frac{1}{2}})$$

$$= \frac{1}{2}\sum_{i\epsilon\Lambda} \theta_{hi}^{n+\frac{1}{2}} \sum_{j\epsilon\Lambda i} \int_{\Gamma_{ij}} \nu_{ij} \cdot b(w_{hj}^{n+\frac{1}{2}} - w_{hi}^{n+\frac{1}{2}})ds - (\nabla \cdot bu^{n+\frac{1}{2}} - \hat{u}^{n+\frac{1}{2}}\nabla \cdot b, \hat{\theta}_h^{n+\frac{1}{2}})$$

$$+((u^{n+\frac{1}{2}} - \hat{u}^{n+\frac{1}{2}})\nabla \cdot b, \hat{\theta}_h^{n+\frac{1}{2}}) + (u^{n+\frac{1}{2}}\nabla \cdot b, \theta^{n+\frac{1}{2}} - \hat{\theta}_h^{n+\frac{1}{2}}) + (\nabla \cdot bu^{n+\frac{1}{2}}, \hat{\theta}_h^{n+\frac{1}{2}} - \theta^{n+\frac{1}{2}})$$

Thus, we can write

$$A^{(33)} = \sum_{k=1}^{4} A^{(33k)}.$$

To estimate $A^{(332)}$

$$|A^{(332)}| \leq \parallel u^{n+\frac{1}{2}} - \hat{u}^{n+\frac{1}{2}} \parallel_{0,2,\Omega} \parallel \nabla \cdot b \parallel_{0,\infty,\Omega^2} \parallel \hat{\theta}_h^{n+\frac{1}{2}} \parallel_{o,2,\Omega}$$

$$\leq Ch\, |u^{n+\frac{1}{2}}|_{1,2,\Omega} \parallel \hat{\theta}_h^{n+\frac{1}{2}} \parallel_{0,2,\Omega}$$

$$\leq Ch \parallel \hat{\theta}_h^{n+\frac{1}{2}} \parallel_{0,2,\Omega} \leq Ch^2 + \beta \parallel \hat{\theta}_h^{n+\frac{1}{2}} \parallel_{0,2,\Omega}^2. \tag{5.7}$$

To estimate $A^{(333)}$

$$|A^{(333)}| \leq \parallel u^{n+\frac{1}{2}} \parallel_{0,2,\Omega} \parallel \nabla \cdot b \parallel_{0,\infty,\Omega^2} \parallel \theta^{n+\frac{1}{2}} - \hat{\theta}_h^{n+\frac{1}{2}} \parallel_{0,2,\Omega}$$

$$\leq Ch\, |\theta^{n+\frac{1}{2}}|_{1,2,\Omega} \leq Ch^2 + \beta|\theta^{n+\frac{1}{2}}|_{1,2,\Omega}^2. \tag{5.8}$$

To estimate $A^{(334)}$

$$|A^{(334)}| \leq \parallel \nabla \cdot b \parallel_{0,\infty,\Omega^2} \parallel u^{n+\frac{1}{2}} \parallel_{0,2,\Omega} \parallel \hat{\theta}_h^{n+\frac{1}{2}} - \theta^{n+\frac{1}{2}} \parallel_{0,2,\Omega}$$

$$\leq Ch\, |\theta^{n+\frac{1}{2}}|_{1,2,\Omega} \leq Ch^2 + \beta|\theta^{n+\frac{1}{2}}|_{1,2,\Omega}^2. \tag{5.9}$$

To estimate $A^{(331)}$

$$A^{(331)} = \frac{1}{2}\sum_{i\epsilon\Lambda}\theta_{hi}^{n+\frac{1}{2}}\sum_{j\epsilon\Lambda i}\int_{\Gamma_{ij}}\nu_{ij}\cdot b(w_{hj}^{n+\frac{1}{2}} - w_{hi}^{n+\frac{1}{2}})ds \quad -\sum_{i\epsilon\Lambda}\theta_{hi}^{n+\frac{1}{2}}\sum_{j\epsilon\Lambda i}\int_{\Gamma_{ij}}\nu_{ij}\cdot b(u^{n+\frac{1}{2}} - u_i^{n+\frac{1}{2}})ds$$

$$= \sum_{i\epsilon\Lambda}\theta_{hi}^{n+\frac{1}{2}}\sum_{j\epsilon\Lambda i}\int_{\Gamma_{ij}}\nu_{ij}\cdot b\,(\tfrac{1}{2}w_{hj}^{n+\frac{1}{2}} + \tfrac{1}{2}w_{hi}^{n+\frac{1}{2}} - w_{hi}^{n+\frac{1}{2}})ds$$

$$-\sum_{i\epsilon\Lambda}\theta_{hi}^{n+\frac{1}{2}}\sum_{j\epsilon\Lambda i}\int_{\Gamma_{ij}}\nu_{ij}\cdot b(u^{n+\frac{1}{2}} - u_i^{n+\frac{1}{2}})ds + \sum_{i\epsilon\Lambda}\theta_{hi}^{n+\frac{1}{2}}\sum_{j\epsilon\Lambda i}\int_{\Gamma_{ij}}\nu_{ij}\cdot b\,u_i^{n+\frac{1}{2}}ds$$

$$-\sum_{i\epsilon\Lambda}\theta_{hi}^{n+\frac{1}{2}}\sum_{j\epsilon\Lambda i}\int_{\Gamma_{ij}}\nu_{ij}\cdot b\,u_i^{n+\frac{1}{2}}ds.$$

$$= [\sum_{i\epsilon\Lambda}\theta_{hi}^{n+\frac{1}{2}}\sum_{j\epsilon\Lambda}\int_{\Gamma_{ij}}\nu_{ij}\cdot b(u_i^{n+\frac{1}{2}} - u^{n+\frac{1}{2}})ds] + [\sum_{i\epsilon\Lambda}\theta_{hi}^{n+\frac{1}{2}}\sum_{j\epsilon\Lambda i}\int_{\Gamma_{ij}}\nu_{ij}\cdot b(u_i^{n+\frac{1}{2}} - w_{hi}^{n+\frac{1}{2}})ds]$$

$$+[\sum_{i\epsilon\Lambda}\theta_{hi}^{n+\frac{1}{2}}\sum_{j\epsilon\Lambda i}\int_{\Gamma_{ij}}\nu_{ij}\cdot b(\tfrac{1}{2}w_{hj}^{n+\frac{1}{2}} + \tfrac{1}{2}w_{hi}^{n+\frac{1}{2}} - u_i^{n+\frac{1}{2}})ds]$$

Thus, we can write

$$A^{(331)} = \sum_{k=1}^{3}A^{(331k)}$$

To estimate $A^{(3311)}$

$$\left|A^{(3311)}\right| \le \|\nabla\cdot b\|_{0,\infty,\Omega^2}\,\|\hat{u}^{n+\frac{1}{2}} - u^{n+\frac{1}{2}}\|_{0,2,\Omega}\|\hat{\theta}_h^{n+\frac{1}{2}}\|_{0,2,\Omega}$$

$$\le Ch\,|u^{n+\frac{1}{2}}|_{1,2,\Omega}\ \|\hat{\theta}_h^{n+\frac{1}{2}}\|_{0,2,\Omega} \le Ch\ \|\hat{\theta}_h^{n+\frac{1}{2}}\|_{0,2,\Omega}$$

$$\le Ch^2 + \beta\,\|\hat{\theta}_h^{n+\frac{1}{2}}\|_{0,2,\Omega}^2.$$

(5.10)

To estimate $A^{(3312)}$

$$\left|A^{(3312)}\right| = |(\nabla\cdot b(\hat{u}^{n+\frac{1}{2}} - \hat{w}_h^{n+\frac{1}{2}}), \hat{\theta}_h^{n+\frac{1}{2}})|$$

$$= |(\nabla\cdot b(\frac{\hat{u}^{n+1}+\hat{u}^n}{2} - \frac{\hat{w}_h^{n+1}+\hat{w}_h^n}{2}),\ \hat{\theta}_h^{n+\frac{1}{2}}) + (\nabla\cdot b\,\hat{\sigma}^{n+\frac{1}{2}}, \hat{\theta}_h^{n+\frac{1}{2}})|$$

$$\le (\tfrac{1}{2}\ \|\nabla\cdot b\|_{0,\infty,\Omega^2}\,\|\hat{\rho}^{n+1}\|_{0,2,\Omega} + \tfrac{1}{2}\,\|\nabla\cdot b\|_{0,\infty,\Omega^2}\|\hat{\rho}^n\|_{0,2,\Omega}$$

$$+\|\nabla\cdot b\|_{0,\infty,\Omega^2}\|\hat{\sigma}^{n+\frac{1}{2}}\|_{0,2,\Omega})\ \|\hat{\theta}_h^{n+\frac{1}{2}}\|_{0,2,\Omega}$$

$$\le Ch^2\,\|\hat{\theta}_h^{n+\frac{1}{2}}\|_{0,2,\Omega} + C\tau^2\,\|\hat{\theta}_h^{n+\frac{1}{2}}\|_{0,2,\Omega} \le C(h^4+\tau^4) + 2\beta\,\|\hat{\theta}_h^{n+\frac{1}{2}}\|_{0,2,\Omega}^2. \qquad (5.11)$$

To estimate $A^{(3313)}$

$$A^{(3313)}=\sum_{i\epsilon\Lambda}\theta_{hi}^{n+\frac{1}{2}}\sum_{j\epsilon\Lambda i}\int_{\Gamma_{ij}}\nu_{ij}\cdot b(\tfrac{1}{2}w_{hj}^{n+\frac{1}{2}}+\tfrac{1}{2}w_{hi}^{n+\frac{1}{2}}-u_i^{n+\frac{1}{2}})ds.$$

Let $w_{hij}^{n+\frac{1}{2}}=\frac{1}{2}(w_{hi}^{n+\frac{1}{2}}+w_{hj}^{n+\frac{1}{2}})$, $u_{ij}^{n+\frac{1}{2}}=u^{n+\frac{1}{2}}(x_{ij})$

$$A^{(3313)}=\sum_{i\epsilon\Lambda}\theta_{hi}^{n+\frac{1}{2}}\sum_{j\epsilon\Lambda i}\int_{\Gamma_{ij}}\nu_{ij}\cdot b(w_{hij}^{n+\frac{1}{2}}+u_{ij}^{n+\frac{1}{2}}-u_{ij}^{n+\frac{1}{2}}-u_i^{n+\frac{1}{2}})ds$$

$$=\sum_{i\epsilon\Lambda}\theta_{hi}^{n+\frac{1}{2}}\sum_{j\epsilon\Lambda i}\int_{\Gamma_{ij}}\nu_{ij}\cdot b(w_{hij}^{n+\frac{1}{2}}-u_{ij}^{n+\frac{1}{2}})ds+\sum_{i\epsilon\Lambda}\theta_{hi}^{n+\frac{1}{2}}\sum_{j\Lambda i}\int_{\Gamma_{ij}}\nu_{ij}\cdot b(u_{ij}^{n+\frac{1}{2}}-u_i^{n+\frac{1}{2}})\,ds$$

$$=\mathrm{I}^{(1)}+\mathrm{I}^{(2)}$$

To estimate $\mathrm{I}^{(1)}$

$$\mathrm{I}_{ij_k}^{(1)}=\int_{\Gamma_{ij}^{(k)}}\nu_{ij}\cdot b(w_{hij}^{n+\frac{1}{2}}-u_{ij}^{n+\frac{1}{2}})ds.$$

By an affine mapping $FQ_{ij}^{(k)}:Q_{ij}^{(k)}\longrightarrow\tilde{\mathbb{T}}$ such that the image Γ of $\Gamma_{ij}^{(k)}$ has unit length. Then taking into consideration the relation $w_{hij}^{n+\frac{1}{2}}=\frac{1}{2}(u_i^{n+\frac{1}{2}}+u_j^{n+\frac{1}{2}})$, we have

$$I_{ij_k}^{(1)}=m_{ij}^{(k)}\mathrm{J}_{ij_k}^{(1)}(\tilde{u}),$$

with

$$\mathrm{J}_{ij_k}^{(1)}(\tilde{u})=\int_{\Gamma}(\nu_{ij}\cdot b)\,(\tfrac{1}{2}(\tilde{u}_i^{n+\frac{1}{2}}+\tilde{u}_j^{n+\frac{1}{2}})-\tilde{u}_{ij}^{n+\frac{1}{2}})\,d\tilde{s}.$$

From Sobolev's imbedding theorem implies

$$|\tilde{u}_i|,|\tilde{u}_j|,|\tilde{u}_{ij}|\le C\parallel\tilde{u}\parallel_{2,2,\tilde{\mathbb{T}}},$$

i.e.

$$|\mathrm{J}_{ij_k}^{(1)}(\tilde{u})|\le C\parallel b\parallel_{0,\infty,\tilde{\Gamma}}\parallel u\parallel_{2,2,\tilde{\mathbb{T}}}\le C\parallel u\parallel_{2,2,\tilde{\mathbb{T}}}.$$

Returning to the original triangle $Q_{ij}^{(k)}$, we obtain

$$|\mathrm{J}_{ij_k}^{(1)}(\tilde{u})|\le C\parallel(DFQ_{ij}^{(k)})^{-1}\parallel^2\{meas_1(Q_{ij}^{(k)})\}^{-\frac{1}{2}}|u|_{2,2,Q_{ij}^{(k)}}$$

$$\le ch_{Q_{ij}^{(k)}}^2 meas_1(Q_{ij}^{(k)})\}^{-\frac{1}{2}}|u|_{2,2,Q_{ij}^{(k)}},$$

where $DFQ_{ij}^{(k)}$ is the Jacobian of $FQ_{ij}^{(k)}$. From lemma(2)(Angermann(1995)) and using equation (3.1)

$$h_{Q_{ij}^{(k)}}\{meas_1(Q_{ij}^{(k)})\}^{-1/2} = 2\sqrt{2}\frac{h_{Q_{ij}^{(k)}}}{m_{ij}^{(k)}d_{ij}}\sqrt{meas_1\left(\mathbb{T}_{ij}^{(k)}\right)} \leq \frac{C}{m_{ij}^{(k)}}\sqrt{meas_1\left(\mathbb{T}_{ij}^{(k)}\right)}.$$

It follows that

$$|\mathrm{I}_{ij_k}^{(1)}| \leq Ch|u|_{2,2,Q_{ij}^{(k)}}\sqrt{meas_1(\mathbb{T}_{ij}^{(k)})} \ .$$

In view of $|\theta_{hi}^{n+\frac{1}{2}}|\sqrt{meas_1(\mathbb{T}_{ij}^{(k)})} = \| \hat{\theta}_h^{n+\frac{1}{2}} \|_{0,2,\mathbb{T}_{ij}^{(k)}}$ and used the relation

$$\sum_{i\epsilon\Lambda}\sum_{j\epsilon\Lambda i}|u|_{2,2,Q_{ij}}^2 \leq 2\,|u|_{2,2,\Omega}^2$$

Then

$$\left|\mathrm{I}^{(1)}\right| \leq Ch\sum_{i\epsilon\Lambda}\sum_{j\epsilon\Lambda i}|u|_{2,2,Q_{ij}} \ \| \hat{\theta}_h^{n+\frac{1}{2}} \|_{0,2,\mathbb{T}_{ij}}$$

$$\leq Ch|u|_{2,2,\Omega} \ \| \hat{\theta}_h^{n+\frac{1}{2}} \|_{0,2,\Omega} \ \leq Ch \ \| \hat{\theta}_h^{n+\frac{1}{2}} \|_{0,2,\Omega}$$

$$\leq Ch^2 + \beta \ \| \hat{\theta}_h^{n+\frac{1}{2}} \|_{0,2,\Omega}^2.$$

To estimate $\mathrm{I}^{(2)}$

$$\mathrm{I}^{(2)} = \sum_{i\epsilon\Lambda}\theta_{hi}^{n+\frac{1}{2}}\sum_{j\epsilon\Lambda i}\int_{\Gamma_{ij}}\nu_{ij}\cdot b\,(u_{ij}^{n+\frac{1}{2}} - u^{n+\frac{1}{2}} + u^{n+\frac{1}{2}} - u_i^{n+\frac{1}{2}})ds$$

$$= \sum_{i\epsilon\Lambda}\theta_{hi}^{n+\frac{1}{2}}\sum_{j\epsilon\Lambda i}\int_{\Gamma_{ij}}\nu_{ij}\cdot b(u_{ij}^{n+\frac{1}{2}} - u^{n+\frac{1}{2}})ds + \sum_{i\epsilon\Lambda}\theta_{hi}^{n+\frac{1}{2}}\sum_{j\epsilon\Lambda i}\int_{\Gamma_{ij}}\nu_{ij}\cdot b\,(u^{n+\frac{1}{2}} - u_i^{n+\frac{1}{2}})ds$$

$$= \mathrm{I}^{(21)} + \mathrm{I}^{(22)}.$$

To estimate $\mathrm{I}^{(21)}$, applying the symmetry argument we have

$$\mathrm{I}^{(21)} = \frac{1}{2}\sum_{i\epsilon\Lambda}\sum_{j\epsilon\Lambda i}(\theta_{hi}^{n+\frac{1}{2}} - \theta_{hj}^{n+\frac{1}{2}})\mathrm{I}_{ij}^{(21)},$$

where

$$\mathrm{I}_{ij}^{(21)} = \int_{\Gamma_{ij}}\left(\nu_{ij}\cdot b\right)(u_{ij}^{n+\frac{1}{2}} - u^{n+\frac{1}{2}})ds$$

The transformation $F_{ij}:\mathbb{T}_{ij} \longrightarrow \widetilde{\mathbb{T}}$ and the image Γ of Γ_{ij} we have

$\mathrm{I}_{ij}^{(21)} = m_{ij}\ \mathrm{J}_{ij}^{(2)}(\tilde{u})$, with

$$\mathrm{J}_{ij}^{(2)}(\tilde{u}) = \int_{\tilde{\Gamma}}\nu_{ij}\cdot b\left(u_{ij}^{n+\frac{1}{2}} - u^{n+\frac{1}{2}}\right)ds$$

In contrast to estimation of $\mathrm{J}_{ij}^{(2)}(\tilde{u})$, here we use the relation .

$|\tilde{u}_{ij}^{n+\frac{1}{2}}|, |\tilde{u}^{n+\frac{1}{2}}| \le C \parallel u^{n+\frac{1}{2}} \parallel_{0,\acute{q},\widetilde{\mathbb{T}}}$, with some $\acute{q} > 2$. Hence we get

$|\mathrm{J}_{ij}^{(2)}(\tilde{u})| \le C \parallel b \parallel_{1,\infty,T^2} |\tilde{u}^{n+\frac{1}{2}}|_{1,\acute{q},\widetilde{\mathbb{T}}} \le C|\tilde{u}^{n+\frac{1}{2}}|_{1,\acute{q},\widetilde{\mathbb{T}}}$.

The back-transformation gives

$$|\mathrm{J}_{ij}^{(2)}(\tilde{u})| \le C \parallel (DF_{ij})^{-1} \parallel \{meas_1(\mathbb{T}_{ij})\}^{-1/\acute{q}} |u^{n+\frac{1}{2}}|_{1,\acute{q},\mathbb{T}_{ij}}$$

$$\le Ch_{\mathbb{T}_{ij}} \parallel \{meas_1(\mathbb{T}_{ij})\}^{-1/\acute{q}} |u^{n+\frac{1}{2}}|_{1,\acute{q},\mathbb{T}_{ij}}.$$

Moreover, from equation (5.5) we obtain

$$|(\theta_{hi}^{n+\frac{1}{2}} - \theta_{hj}^{n+\frac{1}{2}})\mathrm{I}_{ij}^{(21)}| \le Ch_{T_{ij}}\{meas_1(\mathbb{T}_{ij})\}^{1/2-1/\acute{q}} |u^{n+\frac{1}{2}}|_{1,\acute{q},\mathbb{T}_{ij}} |\theta_h^{n+\frac{1}{2}}|_{1,2,\mathbb{T}_{ij}}.$$

From lemma (1)(Angermann(1995)) and using equation (3.1), we get

$$\left|\mathrm{I}^{(21)}\right| \le Ch \sum_{i\epsilon\Lambda} \{m_i\}^{1/2-1/\acute{q}} \parallel u^{n+\frac{1}{2}} \parallel_{1,\acute{q},\Omega_i} |\theta_h^{n+\frac{1}{2}}|_{1,2,\Omega_i}$$

$$\le Ch\{meas_1(\Omega)\}^{1/2-1/\acute{q}} \parallel u^{n+\frac{1}{2}} \parallel_{1,\acute{q},\Omega} |\theta_h^{n+\frac{1}{2}}|_{1,2,\Omega}.$$

The continuous imbedding $w_2^2(\Omega) \subset w_{\acute{q}}^2(\Omega)$ implies

$$\left|\mathrm{I}^{(21)}\right| \le Ch \parallel u^{n+\frac{1}{2}} \parallel_{2,2,\Omega} |\theta_h^{n+\frac{1}{2}}|_{1,2,\Omega}$$

$$\le Ch\, |\theta_h^{n+\frac{1}{2}}|_{1,2,\Omega} \le Ch \parallel \hat{\theta}_h^{n+\frac{1}{2}} \parallel_{0,2,\Omega} \le Ch^2 + \beta \parallel \hat{\theta}_h^{n+\frac{1}{2}} \parallel_{0,2,\Omega}^2.$$

To estimate $\mathrm{I}^{(22)}$

$$\mathrm{I}^{(22)} = (\nabla \cdot b(u^{n+\frac{1}{2}} - \hat{u}^{n+\frac{1}{2}}), \hat{\theta}_h^{n+\frac{1}{2}})$$

$$\left|\mathrm{I}^{(22)}\right| \le \parallel \nabla \cdot b \parallel_{0,2,\Omega} \parallel u^{n+\frac{1}{2}} - \hat{u}^{n+\frac{1}{2}} \parallel_{0,2,\Omega} \parallel \hat{\theta}_h^{n+\frac{1}{2}} \parallel_{0,2,\Omega}$$

$$\le Ch|u^{n+\frac{1}{2}}|_{1,2,\Omega} \parallel \hat{\theta}_h^{n+\frac{1}{2}} \parallel_{0,2,\Omega} \le Ch \parallel \hat{\theta}_h^{n+\frac{1}{2}} \parallel_{0,2,\Omega}$$

$$\le Ch^2 + \beta \parallel \hat{\theta}_h^{n+\frac{1}{2}} \parallel_{0,2,\Omega}^2$$

Then,

$$\left|\mathrm{I}^{(2)}\right| \le Ch^2 + 2\beta \parallel \hat{\theta}_h^{n+\frac{1}{2}} \parallel_{0,2,\Omega}^2,$$

$$\left|A^{(3313)}\right| \le Ch^2 + 3\beta \parallel \hat{\theta}_h^{n+\frac{1}{2}} \parallel_{0,2,\Omega}^2. \qquad (5.12)$$

From equations (5.10)-(5.12), we get

$$\left|A^{(331)}\right| \le C(h^2 + \tau^4) + 6\beta \parallel \hat{\theta}_h^{n+\frac{1}{2}} \parallel_{0,2,\Omega}^2. \qquad (5.13)$$

From equations (5.7), (5.8), (5.9) and (5.13), we get

$$\left|A^{(33)}\right| \le C(h^2 + \tau^4) + 7\beta \parallel \hat{\theta}_h^{n+\frac{1}{2}} \parallel_{0,2,\Omega}^2 + 2\beta|\theta^{n+\frac{1}{2}}|_{1,2,\Omega}^2. \qquad (5.14)$$

To estimate $A^{(34)}$

$$A^{(34)} = \sum_{i\epsilon\Lambda} \theta_{hi}^{n+\frac{1}{2}} c_i w_{hi}^{n+\frac{1}{2}} m_i - (cu^{n+\frac{1}{2}}, \theta^{n+\frac{1}{2}})$$

$$= (\hat{c}\widehat{w}_h^{n+\frac{1}{2}}, \hat{\theta}_h^{n+\frac{1}{2}}) - (cu^{n+\frac{1}{2}}, \theta^{n+\frac{1}{2}})$$

$$= (\hat{c}\widehat{w}_h^{n+\frac{1}{2}}, \hat{\theta}_h^{n+\frac{1}{2}}) + (\hat{c}\hat{u}^{n+\frac{1}{2}}, \hat{\theta}_h^{n+\frac{1}{2}}) - (\hat{c}\hat{u}^{n+\frac{1}{2}}, \hat{\theta}_h^{n+\frac{1}{2}}) + (c\hat{u}^{n+\frac{1}{2}}, \hat{\theta}_h^{n+\frac{1}{2}})$$

$$-(c\hat{u}^{n+\frac{1}{2}}, \hat{\theta}_h^{n+\frac{1}{2}}) + (cu^{n+\frac{1}{2}}, \hat{\theta}_h^{n+\frac{1}{2}}) - (cu^{n+\frac{1}{2}}, \hat{\theta}_h^{n+\frac{1}{2}}) - (cu^{n+\frac{1}{2}}, \theta^{n+\frac{1}{2}})$$

$$= (\hat{c}(\widehat{w}_h^{n+\frac{1}{2}} - \hat{u}^{n+\frac{1}{2}}), \hat{\theta}_h^{n+\frac{1}{2}}) + ((\hat{c}-c)\hat{u}^{n+\frac{1}{2}}, \hat{\theta}_h^{n+\frac{1}{2}}) + (c(\hat{u}^{n+\frac{1}{2}} - u^{n+\frac{1}{2}}), \hat{\theta}_h^{n+\frac{1}{2}})$$

$$+(cu^{n+\frac{1}{2}}, \hat{\theta}_h^{n+\frac{1}{2}} - \theta^{n+\frac{1}{2}}) = \sum_{k=1}^{4} A^{(34k)}$$

To estimate $A^{(341)}$

$$|A^{(341)}| = |(\hat{c}(\frac{\widehat{w}_h^{n+1} + \widehat{w}_h^{n}}{2} - \frac{\hat{u}^{n+1} + \hat{u}^{n}}{2}), \hat{\theta}_h^{n+\frac{1}{2}}) + (\hat{c}\hat{\sigma}^{n+\frac{1}{2}}, \hat{\theta}_h^{n+\frac{1}{2}})|$$

$$= |(\hat{c}(\frac{\widehat{w}_h^{n+1} - \hat{u}_h^{n+1}}{2} + \frac{\widehat{w}^{n} - \hat{u}^{n}}{2}), \hat{\theta}_h^{n+\frac{1}{2}}) + (\hat{c}\hat{\sigma}^{n+\frac{1}{2}}, \hat{\theta}_h^{n+\frac{1}{2}})|$$

$$\leq (\frac{1}{2} \parallel \hat{c} \parallel_{0,\infty\Omega} \parallel \hat{\rho}^{n+1} \parallel_{0,2,\Omega} + \frac{1}{2} \parallel \hat{c} \parallel_{0,\infty,\Omega} \parallel \hat{\rho}^{n} \parallel_{0,2,\Omega}$$

$$+ \parallel \hat{c} \parallel_{0,\infty,\Omega} \parallel \hat{\sigma}^{n+\frac{1}{2}} \parallel_{0,2,\Omega}) \parallel \hat{\theta}_h^{n+\frac{1}{2}} \parallel_{0,2,\Omega}$$

$$\leq C(h^2 + \tau^4) + 2\beta \parallel \hat{\theta}_h^{n+\frac{1}{2}} \parallel_{0,2,\Omega}^2.$$

To estimate $A^{(342)}$

$$|A^{(342)}| \leq \parallel \hat{c} - c \parallel_{0,\infty,\Omega} \parallel \hat{u}^{n+\frac{1}{2}} \parallel_{0,2,\Omega} \parallel \hat{\theta}_h^{n+\frac{1}{2}} \parallel_{0,2,\Omega}$$

$$\leq Ch\, |c|_{1,\infty,\Omega} \parallel \hat{\theta}_h^{n+\frac{1}{2}} \parallel_{0,2,\Omega} \leq Ch \parallel \hat{\theta}_h^{n+\frac{1}{2}} \parallel_{0,2,\Omega}$$

$$\leq Ch^2 + \beta \parallel \hat{\theta}_h^{n+\frac{1}{2}} \parallel_{0,2,\Omega}^2.$$

To estimate $A^{(343)}$

$$|A^{(343)}| \leq \parallel c \parallel_{0,\infty,\Omega} \parallel \hat{u}^{n+\frac{1}{2}} - u^{n+\frac{1}{2}} \parallel_{0,2,\Omega} \parallel \hat{\theta}_h^{n+\frac{1}{2}} \parallel_{0,2,\Omega}$$

$$\leq Ch|u^{n+\frac{1}{2}}|_{1,2,\Omega} \parallel \hat{\theta}_h^{n+\frac{1}{2}} \parallel_{0,2,\Omega} \leq Ch \parallel \hat{\theta}_h^{n+\frac{1}{2}} \parallel_{0,2,\Omega}$$

$$\leq Ch^2 + \beta \parallel \hat{\theta}_h^{n+\frac{1}{2}} \parallel_{0,2,\Omega}^2.$$

To estimate $A^{(344)}$

$$|A^{(344)}| \leq \parallel c \parallel_{0,\infty,\Omega} \parallel u^{n+\frac{1}{2}} \parallel_{0,2,\Omega} \parallel \hat{\theta}_h^{n+\frac{1}{2}} - \theta^{n+\frac{1}{2}} \parallel_{0,2,\Omega}$$

$$\leq Ch\, |\theta^{n+\frac{1}{2}}|_{1,2,\Omega} \leq Ch^2 + \beta|\theta^{n+\frac{1}{2}}|_{1,2,\Omega}^2.$$

Then,

$|A^{(34)}| \leq C(h^2+\tau^4)+4\beta \parallel \hat{\theta}_h^{n+\frac{1}{2}} \parallel_{0,2,\Omega}^2 + \beta|\theta^{n+\frac{1}{2}}|_{1,2,\Omega}^2$. (5.15)

From equations (5.4), (5.6), (5.14) and (5.15), we get

$|A^{(3)}| \leq C(h^2+\tau^4)+12\beta \parallel \hat{\theta}_h^{n+\frac{1}{2}} \parallel_{0,2,\Omega}^2 + 6\beta|\theta^{n+\frac{1}{2}}|_{1,2,\Omega}^2$. (5.16)

By substituting of equations (5.2), (5.3) and (5.16) in equation (5.1) and we note that $\|\cdot\|$ is equivalent to $\|\hat{\cdot}\|$, then we take $\beta > 0$ small enough such that $m - C_\beta > 0$ and move $C_\beta|\theta^{n+\frac{1}{2}}|_{1,2,\Omega}^2$ into the left hand side we get

$$\frac{1}{2}D_\tau \parallel \hat{\theta}_h^n \parallel_{0,2,\Omega}^2 \leq C\left(h^2+\tau^4+\parallel \hat{\theta}_h^n \parallel_{0,2,\Omega}^2 + \parallel \hat{\theta}_h^{n+1} \parallel_{0,2,\Omega}^2\right),$$

then

$$\parallel \hat{\theta}_h^{n+1} \parallel_{0,2,\Omega}^2 \leq \frac{(1+2C\tau)}{(1-2C\tau)} \parallel \hat{\theta}_h^n \parallel_{0,2,\Omega}^2 + \frac{2C\tau}{(1-2C\tau)}(h^2+\tau^4).$$

This implies that if $\tau > 0$ is small enough such that $(1-2C\tau) > 0$, then

$$\parallel \hat{\theta}_h^{n+1} \parallel_{0,2,\Omega}^2 \leq (1+C\tau) \parallel \hat{\theta}_h^n \parallel_{0,2,\Omega}^2 + C\tau(h^2+\tau^4).$$

By induction over n=0,1,2,…,$N_\tau - 1$, this easily deduce that

$$\parallel \hat{\theta}_h^{n+1} \parallel_{0,2,\Omega}^2 \leq (1+C\tau)^{n+1} \parallel \hat{\theta}_h^0 \parallel_{0,2,\Omega}^2 + [(1+C\tau)^{n+1}-1](h^2+\tau^4).$$

Since $(1+C\tau)^{n+1} \leq (1+C\tau)^{N_\tau} \leq \exp(C\tau)$, it follows that

$$\parallel \hat{\theta}_h^{n+1} \parallel_{0,2,\Omega}^2 \leq \exp(C\tau) \parallel \hat{\theta}_h^0 \parallel_{0,2,\Omega}^2 + [\exp(C\tau)-1](h^2+\tau^4),$$

that is

$$\parallel \hat{\theta}_h^{n+1} \parallel_{0,2,\Omega} \leq C(h+\tau^2),$$

and also

$$\parallel \hat{\rho}^{n+1} \parallel_{0,2,\Omega} \leq Ch^2$$

Hence the theorem is complete

□

6. Conclusion

In this paper, we proved that the finite volume method is converge with error of order $(h+\tau^2)$ which better than partial upwind finite element scheme which converges with the error of order $(h+\tau)$ (see Kashkool(2002), Manna(2000) and Zhi-yong(2004)), and we proved the stability of the discretized system.

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المستخلص

في هذا البحث، درسنا طريقة الحجم المحددة لمسألة الحمل والانتشار الخطية ذات البعدين ، حد الحمل الخطي تم تقريبه بواسطة طريقة أبوند (upwind) للعنصر المحدد على شبكة التثليث بينما حد الانتشار الخطي تم تقريبه باستخدام نظرية التباعد (divergence theorem) وتقريب المشقة الاتجاهية باستخدام الفروقات المحددة. ثم برهنا الاستقرارية و تخمين الخطأ تحت بعض الشروط المفروضة على الفيض العددي.