Modified treatment of initial boundary value problems for one dimensional heat-like and wave-like equations using Adomian decomposition method

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Abstract

In this paper, a new technique is applied to modified treatment of initial boundary value problems for one dimensional heat-like and wave-like partial differential equations (ordinary or fractional) by mixed initial and boundary conditions together to obtain a new initial solution at every iteration using Adomian decomposition method (ADM). The structure of a new successive initial solutions can give a more accurate solution in a first step.

Keywords: initial boundary value problems, one dimensional, heat-like and wave-like partial differential equations, Adomian decomposition method .

1. Introduction

Many researchers discussed the initial and boundary value problems. The Adomian decomposition method discussed for solving higher dimensional initial boundary value problems by Wazwaz [2000]. Analytic treatment for variable coefficient fourth-order parabolic partial differential equations discussed by Wazwaz [2001]. The solution of fractional heat-like and wave-like equations with variable coefficients using the decomposition method was found by Momani [2005] and so as by using variational iteration method was found by Yulita Molliq et.al [2009]. Solving higher dimensional initial boundary value problems by variational iteration decomposition method by Noor and Mohyud-Din [2008]. Exact and numerical solutions for non-linear Burger's equation by variational iteration method was applied by Biazar and Aminikhah [2009]. Weighted algorithm based on the homotopy analysis method is applied to inverse heat conduction problems and discussed by Shidfar and Molabahrami [2010]. The boundary value problems was applied by Niu and Wang [2010] to calculate a one step optimal homotopy analysis method for linear and nonlinear differential equations with boundary conditions only, and homotopy perturbation technique for solving two-point boundary value problems– compared it with other methods was discussed by Chun and Sakthivel [2010]. Fractional differential equations with initial boundary conditions by modified Riemann–Liouville derivative was solved by Wu and Lee [2010].

It is interesting to point out that all these researchers obtained the solutions of initial and boundary value problems by using either initial or boundary conditions only. So we present a reliable framework by applying a new technique for treatment initial and boundary value problems by mixed initial conditions with boundary conditions together to obtain a new initial solution at every iteration using variational iteration method. Such as technique was applied by Ali [2011] for treatment of initial boundary value problems. In this paper, a new technique is applied to modified treatment of initial boundary value problems for one dimensional heat-like and wave-like partial differential equations (ordinary or fractional) to construct a new successive initial solutions which can give a more accurate solution by Adomian decomposition method, some examples are given in this paper to illustrate the effectiveness and convenience of this technique.

We give some basic definitions and properties of the fractional calculus theory which are used further in this paper.

Definition 2.1. Jumarie is defined the fractional derivative [Jumarie, 2009] as the following limit form

$$f^{(\alpha)} = \lim_{h\to 0} \frac{\Delta^{\alpha}[f(x) - f(0)]}{h^{\alpha}}. \tag{1.1}$$

This definition is close to the standard definition of derivatives, and as a direct result, the αth derivative of a constant, $0 < \alpha < 1$ is zero.

Definition 2.2. Fractional integral operator of order $\alpha \geq 0$ is defined as

$$I_a^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} f(\tau) d\tau, \quad a < x < b, \ \alpha > 0. \tag{1.2}$$

Where Γ is a gamma function.

Definition 2.3. Fractional derivative of $f(x)$ in the Caputo sense [Caputo,1967] is defined as

$$D_x^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-\tau)^{m-\alpha-1} \frac{d^m f(\tau)}{d\tau^m} d\tau, \quad m-1 < \alpha \leq m, m \in \mathbb{N}, x > 0. \tag{1.3}$$

Definition 2.4. Fractional derivative of compounded functions [Jumarie, 2009] is defined as

$$d^{\alpha} f \cong \Gamma(1+\alpha) df, \ \ 0 < \alpha < 1. \tag{1.4}$$

Definition 2.5. The integral with respect to $(dx)^{\alpha}$ [Jumarie, 2009] is defined as the solution of the fractional differential equation

$$dy \cong f(x)(dx)^{\alpha}, \quad x \geq 0, \quad y(0) = 0, \ \ 0 < \alpha < 1. \tag{1.5}$$

Lemma 2.1. Let $f(x)$ denote a continuous function [Jumarie, 2009] then the solution of the Eq. (1.5) is defined as

$$y = \int_0^x f(\tau)(d\tau)^{\alpha} = \alpha \int_0^x (x-\tau)^{\alpha-1} f(\tau) d\tau, \ \ 0 < \alpha < 1. \tag{1.6}$$

For example $f(x) = x^{\gamma}$ in Eq. (1.6) one obtains

$$\int_0^x \tau^{\gamma} (d\tau)^{\alpha} = \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}, \ \ 0 < \alpha \leq 1. \tag{1.7}$$

3. Adomian decomposition method

Adomian [1994] has presented and developed a so-called decomposition method for solving linear or nonlinear problems such as ordinary differential equations. It consists of splitting the given equation into linear and nonlinear parts, inverting the highest-order derivative operator contained in the linear operator on both sides, identifying the initial and/or boundary conditions and the terms involving the independent variable alone as initial approximation, decomposing the unknown function into a series whose components are to be determined, decomposing the nonlinear function in terms of special polynomials called Adomian's polynomials, and finding the successive terms of the series solution by recurrent relation using Adomian's polynomials. Consider the equation

$$F\big(u(x)\big) = g(x), \tag{2.1}$$

where F represents a general nonlinear ordinary differential operator and g is a given function. The linear terms in $F\big(u(x)\big)$ are decomposed into $Lu + Ru$, where L is an easily invertible operator, which is taken as the highest order derivative and R is the remainder of the linear operator. Thus, Eq. (2.1) can be written as

$$Lu + Ru + Nu = g, \tag{2.2}$$

where Nu represents the nonlinear terms in $F\big(u(x)\big)$. Applying the inverse operator L^{-1} on both sides yields

$$u = \varphi + L^{-1}(g) - L^{-1}(Ru) - L^{-1}(Nu), \tag{2.3}$$

where φ is the constant of integration satisfies the condition $L\varphi = 0$. Now assuming that the solution u can be represented as infinite series of the form

$$u = \sum_{n=0}^{\infty} u_n \,, \tag{2.4}$$

Furthermore, suppose that the nonlinear term Nu can be written as infinite series in terms of the Adomian polynomials A_n of the form

$$Nu = \sum_{n=0}^{\infty} A_n \ , \tag{2.5}$$

where the Adomian polynomials A_n of Nu are evaluated using the formula

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n} N(\sum_{i=0}^{\infty}(\lambda^i u_i))\bigg|_{\lambda=0} \ , \quad n = 0,1,2,\ldots \tag{2.6}$$

Then substituting (2.4) and (2.5) in (2.3) gives

$$\sum_{n=0}^{\infty} u_n = \varphi + L^{-1}(g) - L^{-1}(R\sum_{n=0}^{\infty} u_n) - L^{-1}(\sum_{n=0}^{\infty} A_n) \tag{2.7}$$

Each term of series (2.4) is given by the recurrent relation

$$u_0 = \varphi + +L^{-1}(g), \qquad n = 0 \tag{2.8.a}$$

$$u_{n+1} = L^{-1}(Ru_n) - L^{-1}(A_n), \qquad n \geq 0 \tag{2.8.b}$$

To describe the solution procedure of the fractional Adomian decomposition method, we consider the following one dimensional heat-like or wave-like partial differential equations

$$\frac{\partial^\alpha}{\partial t^\alpha} u(x,t) = \mu(x)\, u_{xx}(x,t), \ \ 0 < x < 1, \ \ t > 0, \ \ \alpha > 0, \tag{2.9}$$

where $\mu(x)$ is the differential operator in x such as $\mu(x) = \frac{1}{2}x^2$. In an operator form, Eq. (2.9) becomes

$$D_t^\alpha u(x,t) = \mu(x)\, u_{xx}(x,t), \tag{2.10}$$

where the fractional differential operator D_t^α is $D_t^\alpha = \frac{\partial^\alpha}{\partial t^\alpha}$, so that D_t^α is the operator defined (1.3). Where the Caputo time-fractional derivative operator of order $\alpha > 0$ is defined as

$$D_t^{\alpha} u(x,\tau) = \frac{\partial^{\alpha}}{\partial \tau^{\alpha}} u(x,\tau)$$
$$= \begin{cases} \frac{\partial^m u(x,\tau)}{\partial \tau^m}, & \text{for } \alpha = m \in \mathbb{N}. \\ \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m u(x,\tau)}{\partial \tau^m} d\tau, & \text{for } m-1 < \alpha < m. \end{cases} \tag{2.11}$$

Operating with $I^{\alpha} = I_0^{\alpha}$ on both sides of Eq. (2.9) and using the initial conditions yields

$$u(x,t) = \sum_{k=0}^{m-1} \frac{\partial^k}{\partial t^k} u(x,0^+) \frac{t^k}{k!} + I^{\alpha}[\mu(x)\, u_{xx}(x,t)], \tag{2.12}$$

where the function $u(x,t)$ is a assumed to be a causal function of time, i.e., vanishing for $t < 0$. According to the Adomian decomposition method [Adomian (1994), Momani (2005)], assuming that the solution u can be represented as infinite series as the form (2.4). Substituting Eq. (2.4) into both of Eq. (2.12) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = \sum_{k=0}^{m-1} \frac{\partial^k}{\partial t^k} u(x,0^+) \frac{t^k}{k!} + I^{\alpha}\left[\mu(x) \left(\sum_{n=0}^{\infty} (u_n(x,t))_{xx}\right)\right], \tag{2.13}$$

Each term of series (2.4) is given by the recurrent relation

$$u_0(x,t) = \sum_{k=0}^{m-1} \frac{\partial^k}{\partial t^k} u(x,0^+) \frac{t^k}{k!}, \tag{2.14.a}$$

$$u_{n+1}(x,t) = I^{\alpha}\left[\mu(x)\big(u_n(x,t)\big)_{xx}\right], \quad n \geq 0. \tag{2.14.b}$$

The convergence of the decomposition series have investigated by several authors [Cherrualt (1989), Cherrualt and Adomian (1993)].

3. New technique for solving one dimensional heat-like and wave-like equations (ordinary or fractional) using VIM

To convey the basic idea for modified treatment of initial boundary value problems by Adomian decomposition method to solve one dimensional heat-like and wave-like equation of the form

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t)=\frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2},\quad t>0,\ \ 0<x<1,\ \ \alpha>0, \tag{3.1}$$

the initial conditions associated with Eq. (3.1) are of the form

$$u(x,0)=f_0(x),\qquad \frac{\partial u(x,0)}{\partial t}=f_1(x),\qquad ,\quad 0<x<1, \tag{3.2}$$

and the boundary conditions are given by

$$u(0,t)=g_0(t),\qquad u(1,t)=g_1(t),\quad t>0, \tag{3.3}$$

where $f_0(x), f_1(x), g_0(t)$ and $g_1(t)$ are given functions. The initial solution can be written as $u_0(x,t)=f_0(x)+tf_1(x)$.

The initial values are usually used for selecting the zeroth approximation u_0 but in this paper, we accredit to modified a new technique to calculate the zeroth approximation u_0^* by construct a new initial solutions u_n^* by mixed initial conditions in Eq. (3.2) with boundary conditions in Eq. (3.3) at every iteration as follows [Ali (2011)]

$$u_n^*(x,t)=u_n(x,t)+(1-x)[g_0(t)-u_n(0,t)]+x[g_1(t)-u_n(1,t)],\ \ n\geq 0. \tag{3.4}$$

It is obvious that the new successive initial solutions u_n^* in Eq. (3.4) satisfying the initial and boundary conditions together as follows

$$if\ \ x=0\ \ then\ \ u_n^*(0,t)=g_0(t),$$

$$if\ \ x=1\ \ then\ \ u_n^*(1,t)=g_1(t),$$

$$if\ \ t=0\ \ then\ \ u_n^*(x,0)=u_n(x,0). \tag{3.5}$$

The second and third terms in right side of Eq. (3.4) well be vanish when we applying the second derivative by x which is used in the right side of Eq. (2.8.b) or Eq. (2.14.b), so to establish these terms we can be modified Eq. (3.4) and rewritten in a new formulation as

$$u_n^*(x,t) = u_n(x,t) + (1-x^2)[g_0(t) - u_n(0,t)] + x^2[g_1(t) - u_n(1,t)], \ \ n \geq 0. \quad (3.6)$$

Eq. (2.14.b) associated with Eqs. (3.1) and (3.6) can be rewritten in a new formulation as

$$u_{n+1}(x,t) = \frac{1}{2}x^2 I^{\alpha}[(u_n^*(x,t))_{xx}], \quad n \geq 0. \quad (3.7)$$

Such as treatment is a very effective as shown in this paper.

4. Applications and results

Example 1: Consider the following one-dimensional heat-like problem

$$\frac{\partial u}{\partial t} - \frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2} = 0, \ \ 0 < x < 1, t > 0, \quad (4.1)$$

subject to the initial conditions

$$u(x,0) = x^2, \qquad 0 < x < 1, \quad (4.2)$$

and the boundary conditions

$$u(0,t) = 0\,, \quad u(1,t) = \mathrm{e}^{\mathrm{t}}\,, \quad t > 0. \quad (4.3)$$

The initial approximation is

$$u_0(x,t) = x^2. \quad (4.4)$$

By applying a new approximations u_n^* in Eq. (3.6) we have

$$u_n^*(x,t) = u_n(x,t) + (1-x^2)[0 - u_n(0,t)] + x^2[\mathrm{e}^{\mathrm{t}} - u_n(1,t)], \quad n = 0,1,2, \dots . \quad (4.5)$$

To begin with a new initial approximation u_0^* we applying Eq. (4.5) at $n = 0$ such as

$$u_0^*(x,t) = x^2\ \mathrm{e}^{\mathrm{t}}. \quad (4.6)$$

According to the Adomian decomposition method, we have an operater form for Eq.(4.1) as

$$Lu = \frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2}\ , \qquad 0 < x < 1, t > 0, \tag{4.7}$$

where the differential operator is $L = \frac{\partial}{\partial t}$, so that L^{-1} is integral operator

$$L^{-1}(.) = \int_0^t (.)dt. \tag{4.8}$$

By operating with L^{-1}on both sides of Eq.(4.7) and using a new technique of initial solutions u_n^* we have

$$u_{n+1}(x,t) = \int_0^t \left(\frac{1}{2}x^2\frac{\partial^2 u_n^*(x,t)}{\partial x^2}\right)dt. \tag{4.9}$$

By Eq. (4.6), so as soon we have

$$u_1(x,t) = x^2(e^t - 1). \tag{4.10}$$

We can readily check

$$u(x,y,t) = u_0(x,y,t) + u_1(x,y,t) = x^2\ \mathrm{e}^{\mathrm{t}}. \tag{4.11}$$

Which yields an exact solution of Eq. (4.1).

***Example 2*:** We next consider the one-dimensional wave-like equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2} = 0, \qquad 0 < x < 1, t > 0, \tag{4.12}$$

subject to the initial conditions

$$u(x,0) = x, \quad \frac{\partial u(x,0)}{\partial t} = x^2\ , \quad 0 < x < 1,, \tag{4.13}$$

and the boundary conditions

$$u(0,t) = 0 \quad , \quad u(1,t) = 1 + \sinh t\ , \ \ t > 0., \tag{4.14}$$

The initial approximation is

$$u_0(x,t) = x + x^2 t. \tag{4.15}$$

By applying a new approximations u_n^* in Eq. (3.6) we have

$$u_n^*(x,t) = u_n(x,t) + (1-x^2)[0 - u_n(0,t)] + x^2[1 + \sinh t - u_n(1,t)], \; n = 0,1,2,\ldots \quad (4.16)$$

To begin with a new initial approximation u_0^* we applying Eq. (4.16) at $n = 0$ such as

$$u_0^*(x,t) = x + x^2 \sinh t. \quad (4.17)$$

By Adomian decomposition method , we have an operator form for Eq.(4.12) as

$$Lu = \frac{1}{2}x^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, t > 0, \quad (4.18)$$

where the differential operator is $L = \frac{\partial^2}{\partial t^2}$, so that L^{-1} is a two-fold integral operator

$$L^{-1}(.) = \int_0^t \int_0^t (.) dtdt. \quad (4.19)$$

By operating with L^{-1} on both sides of (4.18) and using a new technique of initial solutions u_n^* we have

$$u_{n+1}(x,t) = \int_0^t \int_0^t \left(\frac{1}{2}x^2 \frac{\partial^2 u_n^*(x,t)}{\partial x^2}\right) dtdt, \qquad n \geq 0. \quad (4.20)$$

By Eq. (4.17), so as soon we have

$$u_1(x,t) = x^2 (\sinh t - t). \quad (4.21)$$

We can readily check

$$u(x,y,t) = u_0(x,y,t) + u_1(x,y,t) = x + x^2 \sinh t. \quad (4.22)$$

Which yields an exact solution of Eq. (4.12).

***Example 3*:** Consider the following one-dimensional fractional heat-like problem

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{1}{2}x^2 \frac{\partial^2 u}{\partial x^2} = 0, \;\; 0 < x < 1, \;\; 0 < \alpha \leq 1, t > 0, \quad (4.23)$$

subject to the initial conditions

$$u(x,0) = x^2, \qquad 0 < x < 1, \tag{4.24}$$

and the boundary conditions

$$u(0,t) = 0\,, \quad u(1,t) = \mathrm{e}^{\mathrm{t}}\,, \quad t > 0. \tag{4.25}$$

The exact solution is [Momani (2005), Yulita Molliq et.al (2009)]

$$u(x,t) = x^2 \mathrm{E}_{\alpha}(\mathrm{t}^{\alpha}), \tag{4.26}$$

where

$$\mathrm{E}_{\alpha}(\mathrm{t}^{\alpha}) = \lim_{\mathrm{m}\to\infty} \sum_{\mathrm{k}=0}^{\mathrm{m}} \frac{\mathrm{t}^{\mathrm{k}\alpha}}{\Gamma(1+\mathrm{k}\alpha)}\,. \tag{4.27}$$

The initial approximation is

$$u_0(x,t) = x^2. \tag{4.28}$$

By applying a new approximations u_n^* in Eq. (3.6) we have

$$u_n^*(x,t) = u_n(x,t) + (1-x^2)[0 - u_n(0,t)] + x^2[\mathrm{e}^{\mathrm{t}} - u_n(1,t)], \qquad n = 0,1,2,\ldots\,. \tag{4.29}$$

At $n = 0$, we begin with a new initial approximation u_0^* as

$$u_0^*(x,t) = x^2\ \mathrm{e}^{\mathrm{t}}$$

$$= x^2\left(1 + \frac{t}{\Gamma(2)} + \frac{t^2}{\Gamma(3)} + \cdots\right). \tag{4.30}$$

We choose $m = 1$ in Eq. (2.14.a)

$$u_0(x,t) = \sum_{k=0}^{0} \frac{\partial^k}{\partial t^k} u(x,0) \frac{t^k}{k!} = x^2, \tag{4.31}$$

by Eq. (3.7) and Eq. (1.2) we have

$$u_1(x,t) = \frac{1}{2}x^2 I^\alpha[(u_0^*(x,t))_{xx}]$$

$$= \frac{1}{2}x^2 I^\alpha\left[\frac{\partial^2(x^2\ \mathrm{e}^{\mathrm{t}}\ )}{\partial x^2}\right]$$

$$= \frac{x^2}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(1+\frac{\tau}{\Gamma(2)}+\frac{\tau^2}{\Gamma(3)}+\cdots\right)d\tau,$$

$$= \frac{x^2}{\Gamma(\alpha)}\left(\mathcal{B}(\alpha,1)t^\alpha+\frac{\mathcal{B}(\alpha,2)t^{\alpha+1}}{\Gamma(2)}+\frac{\mathcal{B}(\alpha,3)t^{\alpha+2}}{\Gamma(3)}+\cdots\right)$$

$$= x^2\left(\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{t^{\alpha+2}}{\Gamma(\alpha+3)}+\cdots\right), \quad (4.32)$$

We can readily check

$$u(x,t) = u_0(x,t)+u_1(x,t) = x^2\left(1+\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{t^{\alpha+2}}{\Gamma(\alpha+3)}+\cdots\right). \quad (4.33)$$

Let $\alpha = 1$, Eq.(4.33) becomes

$$u(x,t) = x^2\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots\right) = x^2 e^t. \quad (4.34)$$

Which yields an exact solution of Eq. (4.23) when $\alpha = 1$. But, generally the solution (4.33) is not exactly for Eq. (4.23), because it doesn't satisfying of vinery equation. In the other hand the boundary conditions (4.25) is not corresponding to the exact solution (4.26) which were given in [Momani (2005), Yulita Molliq et.al (2009)], that's mean the boundary conditions (4.25) are be not true for this problem. So, in the following example we present a reliable framework by applying another boundary conditions which can be satisfying by exact solution (4.26).

Example 4**:** Consider the following one-dimensional fractional heat-like problem

$$\frac{\partial^\alpha u}{\partial t^\alpha}-\frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2} = 0,\ \ 0<x<1,\ \ 0<\alpha<1, t>0, \quad (4.35)$$

subject to the initial conditions

$$u(x,0) = x^2, \qquad 0 < x < 1, \tag{4.36}$$

and the boundary conditions

$$u(0,t) = 0\,, \quad u(1,t) = \mathrm{E}_\alpha(\mathrm{t}^\alpha)\,, \quad t > 0, \tag{4.37}$$

where

$$\mathrm{E}_\alpha(\mathrm{t}^\alpha) = \lim_{\mathrm{m}\to\infty} \sum_{\mathrm{k}=0}^{\mathrm{m}} \frac{\mathrm{t}^{\mathrm{k}\alpha}}{\Gamma(1+\mathrm{k}\alpha)}\,. \tag{4.38}$$

The initial approximation is

$$u_0(x,t) = x^2. \tag{4.39}$$

By applying a new approximations u_n^* in Eq. (3.6) we have

$$u_n^*(x,t) = u_n(x,t) + (1-x^2)[0 - u_n(0,t)] + x^2[\mathrm{E}_\alpha(\mathrm{t}^\alpha) - u_n(1,t)], \qquad n = 0,1,2,\ldots. \tag{4.40}$$

At $n = 0$, we begin with a new initial approximation u_0^* as

$$\begin{aligned} u_0^*(x,t) &= x^2 \mathrm{E}_\alpha(\mathrm{t}^\alpha) \\ &= x^2\left(1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots\right). \end{aligned} \tag{4.41}$$

We choose $m = 1$ in Eq. (2.15.a)

$$u_0(x,t) = \sum_{k=0}^{0} \frac{\partial^k}{\partial t^k} u(x,0) \frac{t^k}{k!} = x^2, \tag{4.42}$$

by Eq. (3.7) and Eq. (1.2) we have

$$\begin{aligned} u_1(x,t) &= \frac{1}{2} x^2 I^\alpha[(u_0^*(x,t))_{xx}] \\ &= \frac{1}{2} x^2 I^\alpha\left[\frac{\partial^2(x^2\mathrm{E}_\alpha(\mathrm{t}^\alpha))}{\partial x^2}\right] \end{aligned}$$

$$= \frac{x^2}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left(1 + \frac{\tau^\alpha}{\Gamma(1+\alpha)} + \frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots \right) d\tau,$$

$$= \frac{x^2}{\Gamma(\alpha)} \left(\mathcal{B}(\alpha,1) t^\alpha + \frac{\mathcal{B}(\alpha,\alpha+1) t^{2\alpha}}{\Gamma(\alpha+1)} + \frac{\mathcal{B}(\alpha,2\alpha+1) t^{3\alpha}}{\Gamma(2\alpha+1)} + \cdots \right)$$

$$= x^2 \left(\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots \right), \tag{4.43}$$

where $\mathcal{B}$ is beta function. We can readily check

$$u(x,y,t) = u_0(x,y,t) + u_1(x,y,t) = x^2 \left(1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots \right). \tag{4.44}$$

Which yields an exact solution of Eq. (4.35), it is the same result which is writing by [Momani (2005), Yulita Molliq et.al (2009)] where are using initial conditions only.

Example 5**:** Consider the one-dimensional fractional wave-like equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{1}{2} x^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < 1, 1 < \alpha < 2, t > 0, \tag{4.45}$$

subject to the initial conditions

$$u(x,0) = x, \quad \frac{\partial u(x,0)}{\partial t} = x^2, \quad 0 < x < 1, \tag{4.46}$$

and the boundary conditions

$$u(0,t) = 0, \quad u(1,t) = 1 + \sinh t, \quad t > 0, \tag{4.47}$$

The exact solution is [Momani (2005), Yulita Molliq et.al (2009)]

$$u(x,t) = x + x^2 t \, \mathrm{E}_{\alpha,2}(\mathrm{t}^\alpha), \tag{4.48}$$

where

$$\mathrm{E}_{\alpha,2}(\mathrm{t}^\alpha) = \lim_{\mathrm{m}\to\infty} \sum_{\mathrm{k}=0}^{\mathrm{m}} \frac{\mathrm{t}^{\mathrm{k}\alpha}}{\Gamma(2+\mathrm{k}\alpha)} . \tag{4.49}$$

The initial approximation is

$$u_0(x,t) = x + x^2 t. \tag{4.50}$$

By applying a new approximations u_n^* in Eq. (3.6) we obtain

$$u_n^*(x,t) = u_n(x,t) + (1-x^2)[0 - u_n(0,t)] + x^2[1 + \sinh t - u_n(1,t)], \quad n = 0,1,2,\dots \tag{4.51}$$

To begin with a new initial approximation u_0^* we applying Eq. (4.51) at $n = 0$ such as

$$u_0^*(x,t) = x + x^2 \sinh t. \tag{4.52}$$

We choose $m = 1$ in Eq. (2.14.a)

$$u_0(x,t) = \sum_{k=0}^{0} \frac{\partial^k}{\partial t^k} u(x,0) \frac{t^k}{k!} = x + x^2 t, \tag{4.53}$$

by Eq. (3.7) and Eq. (1.2) we have

$$\begin{aligned} u_1(x,t) &= \frac{1}{2} x^2 I^\alpha [(u_0^*(x,t))_{xx}] \\ &= \frac{1}{2} x^2 I^\alpha \left[\frac{\partial^2 (x + x^2 \sinh t\ )}{\partial x^2}\right] \\ &= \frac{x^2}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left(\tau + \frac{\tau^3}{\Gamma(4)} + \frac{\tau^5}{\Gamma(6)} + \cdots\right) d\tau, \\ &= \frac{x^2}{\Gamma(\alpha)} \left(\mathcal{B}(\alpha,2) t^{\alpha+1} + \frac{\mathcal{B}(\alpha,4) t^{\alpha+3}}{\Gamma(4)} + \frac{\mathcal{B}(\alpha,6) t^{\alpha+5}}{\Gamma(6)} + \cdots\right) \\ &= x^2 \left(\frac{1}{\Gamma(\alpha+2)} t^{\alpha+1} + \frac{1}{\Gamma(\alpha+4)} t^{\alpha+3} + \frac{1}{\Gamma(\alpha+6)} t^{\alpha+5} + \cdots\right) \end{aligned} \tag{4.54}$$

We can readily check

$$\begin{aligned} u(x,t) &= u_0(x,t) + u_1(x,t) \\ &= x + x^2 \left(t + \frac{1}{\Gamma(\alpha+2)} t^{\alpha+1} + \frac{1}{\Gamma(\alpha+4)} t^{\alpha+3} + \frac{1}{\Gamma(\alpha+6)} t^{\alpha+5} + \cdots\right). \end{aligned} \tag{4.55}$$

Let $\alpha = 2$, Eq. (4.55) becomes

$$u(x,t) \ = x + x^2\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^7}{6!} + \cdots\right) = x + x^2 \sinh t. \tag{4.56}$$

Which yields an exact solution of Eq. (4.45) when $\alpha = 1$. But, generally the solution (4.55) is not exactly for Eq. (4.45), because it doesn't satisfying of vinery equation. In the other hand the boundary conditions (4.47) is not corresponding to the exact solution (4.48) which were given in [Momani (2005), Yulita Molliq et.al (2009)], that's mean the boundary conditions (4.47) are be not true. So, in the following example we present a reliable framework by applying another boundary conditions which can be satisfying by exact solution (4.48).

***Example 6*:** Consider the one-dimensional fractional wave-like equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2} = 0, \ \ 0 < x < 1, 1 < \alpha < 2, t > 0, \tag{4.57}$$

subject to the initial conditions

$$u(x,0) = x, \quad \frac{\partial u(x,0)}{\partial t} \ = x^2\,, \quad 0 < x < 1, \tag{4.58}$$

and the boundary conditions

$$u(0,t) = 0\,, \quad u(1,t) = 1 + t\,\mathrm{E}_{\alpha,2}(\mathrm{t}^\alpha)\,, \quad t > 0, \tag{4.59}$$

where

$$\mathrm{E}_{\alpha,2}(\mathrm{t}^\alpha) = \lim_{\mathrm{m}\to\infty}\sum_{\mathrm{k}=0}^{\mathrm{m}}\frac{\mathrm{t}^{\mathrm{k}\alpha}}{\Gamma(2+\mathrm{k}\alpha)}\ . \tag{4.60}$$

The initial approximation is

$$u_0(x,t) = x + x^2 t. \tag{4.61}$$

By applying a new approximations u_n^* in Eq. (3.6) we obtain

$$u_n^*(x,t) = u_n(x,t) + (1-x^2)[0 - u_n(0,t)] + x^2\left[1 + t\mathrm{E}_{\alpha,2}(\mathrm{t}^\alpha) - u_n(1,t)\right], n = 0,1,2,.. \tag{4.62}$$

At $n = 0$, the new initial approximation u_0^* is

$$u_0^*(x,t) = x + x^2 t\ \mathrm{E}_{\alpha,2}(\mathrm{t}^{\alpha})$$

$$= x + x^2 t\left(1 + \frac{t^{\alpha}}{\Gamma(2+\alpha)} + \frac{t^{2\alpha}}{\Gamma(2+2\alpha)} + \cdots\right). \tag{4.63}$$

We choose $m = 1$ in Eq. (2.14.a)

$$u_0(x,t) = \sum_{k=0}^{0} \frac{\partial^k}{\partial t^k} u(x,0)\frac{t^k}{k!} = x + x^2 t, \tag{4.64}$$

by Eq. (3.7) and Eq. (1.2) we have

$$u_1(x,t) = \frac{1}{2}x^2 I^{\alpha}[(u_0^*(x,t))_{xx}]$$

$$= \frac{1}{2}x^2 I^{\alpha}\left[\frac{\partial^2(x + x^2 t\ \mathrm{E}_{\alpha,2}(\mathrm{t}^{\alpha})\ )}{\partial x^2}\right]$$

$$= \frac{x^2}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(\tau + \frac{\tau^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{\tau^{2\alpha+1}}{\Gamma(2\alpha+2)} + \cdots\right)d\tau,$$

$$= \frac{x^2}{\Gamma(\alpha)}\left(\mathcal{B}(\alpha,2)t^{\alpha+1} + \frac{\mathcal{B}(\alpha,\alpha+2)t^{2\alpha+1}}{\Gamma(\alpha+2)} + \frac{\mathcal{B}(\alpha,2\alpha+2)t^{3\alpha+1}}{\Gamma(2\alpha+2)} + \cdots\right)$$

$$= x^2\left(\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} + \cdots\right), \tag{4.65}$$

We can readily check

$$u(x,t) = u_0(x,t) + u_1(x,t)$$

$$= x + x^2 t\left(1 + \frac{t^{\alpha}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+2)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+2)} + \cdots\right)$$

$$= x + x^2 t\ \mathrm{E}_{\alpha,2}(\mathrm{t}^{\alpha}). \tag{4.66}$$

Which yields an exact solution of Eq. (4.57), it is the same result which is writing by [Momani (2005), Yulita Molliq et.al (2009)] whrer are using initial conditions only.

5. Conclusions

Very effective to construct a new initial successive solutions u_n^* by mixed initial and boundary conditions together which explained in formula (3.6) and used it to find successive approximations u_n of the solution u by applying Adomian decomposition method to solve initial boundary value problems for one dimensional heat-like and wave-like partial differential equations (ordinary or fractional). Some examples are given in this paper to illustrate the effectiveness and convenience of a new technique. It is important and obvious to show that the exact solutions have found directly from a first iteration of these examples by applying a new technique which is determined in this paper, but if used initial conditions only [Momani (2005), Yulita Molliq et.al (2009)] or applied formula of Eq. (3.4) [Ali (2011)] we will have exact solution by calculating infinite successive solutions u_n which closed form by Eq. (2.4). In the other hand we note that the first a new initial approximation u_0^* are appearing in the same exact solution.

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معالجة معدلة للمسائل ذات القيم الأبتدائية والحدودية الخاصة بالمعادلات المثيلة للحرارية والموجية ذات البعد الواحد باستخدام طريقة تحليل أدومين

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المستخلص:

في هذا البحث، طبقنا معالجة معدلة لمسائل ذات قيم ابتدائية وحدودية الخاصة بالمعادلات المثيلة للحرارية والموجية ذات البعد الواحد (تفاضلية اعتيادية أو كسرية) وذلك بخلط الشروط الأبتدائية والحدودية معا بأسلوب معين لغرض الحصول على حل ابتدائي جديد عند كل خطوة تكرارية باستخدام طريقة تحليل أدومين (ADM) ، ان تكوين متتابعة الحلول الابتدائية بالاسلوب الجديد يعطي حلا دقيق في أول خطوة تكرارية.