The Error Analysis of Linearized Modification of Galerkin Finite Element Method for Numerical Reservoir Simulation

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Abstract

In this paper we discuss two phase immiscible flow in porous media governed by a system of nonlinear partial differential equations raised from reservoir simulation. Crank-Nicholson –Galerkin finite element method and lineari zed Crank-Nicolson –Galerkin finite element Predictor- corrector method are constructed respectively. We show that the discretization error of linearized Crank-Nicholson-Galerkin method **is** $o(k^2)$, where k refers to the time step for both schemes.

1-Introdaction

By reservoir simulation (Peaceman(1977)**)**, we mean the process of inferring the behavior of real reservoir from performance of a model of that reservoir. The model may be physical or mathematical such as a scaled laboratory model. For our purpose, a mathematical model of a physical system is a set of partial differential equation, together with an appropriate set of boundary conditions which we believe adequately describes the significant physical processes taking place in that system. The process occurring in petroleum reservoir are basically fluid and mass transfer. Up to three immiscible phases (water, oil, and gas) flow simultaneously, while mass transfer may take place between the phases (chiefly between process. The study of petroleum reservoir is characterized by strongly nonlinear equation. There are two simple models used to describe the movement of two fluids in a porous media. In the model the fluid are assumed to be completely miscible, one example is given by oil and a detergent solution. In other model the fluids, such as, water and oil, one considers to be completely immiscible.

In this paper we discuss two phase, immiscible flow in porous media governed by a system of nonlinear partial differential equation raised from reservoir simulation. There are many researcher solved these kinds of problems by Euler-Gale kin and Crank-Nicolson-Galerkin finite element method(Jie.cheng(1999),Kashkool(2005)). These methods have disadvantage that a nonlinear system of algebraic equation have to be solved at each time step, as a result of the presence $a(S^{n)}, f(S^n)$ and $\alpha(S^n)$. We shall

the gas and oil phases). Gravity, capillary, and viscous forces have a role in the fluid flow

therefore consider a linearzed modification of Crank-Nicolson-Galerkin finite element method in which this difficulty is avoided by replacing S^n by $\frac{3}{2}S^{n-1}$-$\frac{1}{2}S^{n-2}$ for this method. Since d the equations contain S^{n-2} it may be only used for $n \geq 2$, and have to be supplement it with another for determining S^1, for this purpose we shall use Predictor- corrector method. We show that the discretization error of linearized Crank-Nicholson-Galerkin method is $o(k^2)$, where k refers to the time step.

2-Mathimatical Preliminaries

Let $L^p(\Omega)$ denotes the linear space of measurable p^{th} power integrable function on Ω, endowed with norm $\|.\|_{0,p,\Omega}$ and W_p^m is the sobolev space of function which together with their derivatives up to m is in $L^p(\Omega)$. The norm in $W_p^m(\Omega)$ is defined by (Ciarlet(1978)).

$$\|u\|_{m,p,\Omega} = \Big(\sum_{|a|\leq m}\|D^\alpha u\|^p_{o,p,\Omega}\Big)^{\frac{1}{p}} \text{, for } 1 \leq p \leq \infty,$$

For $p = \infty$

$$\|u\|_{m,\infty.\Omega} = \max_{|a|\leq m}\|D^\alpha u\|_{0,\infty,\Omega}$$

and the semi norm

$$|u|_{m,p,\Omega} = (\sum_{|a|\leq m} \|D^{\alpha} u\|_{0,p,\Omega}^{p})^{\frac{1}{p}} .$$

For $p = \infty$

$$|u|_{m,\infty,\Omega} = \max_{|a|=m} \|D^{\alpha} u\|_{0,\infty,\Omega}$$

Let

$$H(div,\Omega) = \{u \in (L^2(\Omega))^2\}, \nabla \cdot u \in L^2(\Omega)\} ,$$

endowed with norm

$$\|u\|_{H(div\Omega)} = (\|u\|_{0,2,\Omega}^2 + \|\nabla \cdot \mathrm{u}\|_{0,2,\Omega}^2)^{\frac{1}{2}} .$$

We define the space of square integral functions denoted by $L^2(\Omega)$ as follows ,

$$L^2(\Omega) = \{u : u \text{ is a real function defined on } \Omega \text{ and } \int_{\Omega} u^2\, dx < \infty\} .$$

The space $L^2(\Omega)$ is Hilbert space with inner product $(u,v) = \int_{\Omega} uv\, dx,\ u,v \in L^2(\Omega)$ *and* the corresponding norm ,

$$\|u\|_{L^2(\Omega)} = (u,u)^{\frac{1}{2}} = (\int_{\Omega} u^2\, dx)^{\frac{1}{2}} .$$

By $H^r(\Omega)$, for any non-negative integer r, we denote the Hilbert space consist of those functions $u \in L^2(\Omega)$ for which all partial derivatives $\partial^{\alpha} u$ with $|\alpha| \leq r$ belong to the space $L^2(\Omega)$, such space is equipped with the norm

$$\|u\|_r = \|u\|_{H^r(\Omega)} = (\sum_{|\alpha|\leq r} \|\partial^{\alpha} u\|^2 dx)^{\frac{1}{2}}$$

And the semi norm

$$|u|_r = \|u\|_{H^r(\Omega)} = (\sum_{|\alpha|=r} \|\partial^{\alpha} u\|^2 dx)^{\frac{1}{2}}.$$

The space $H^r(\Omega)$ is a particular case of general class of Sobolev space. In this case, we have $H^r(\Omega) = W_2^r(\Omega)$.

By $L^q(0,T,W_p^k(\Omega))$ we denote the space (Solid pavel(2004))

$$L^q(0,T,W_p^k(\Omega)) = \{\mathrm{u}:(0,T) \to W_p^k(\Omega); u \text{ is measurable and } \int_0^T \|u(t)\|_{k,p,\Omega}^q dt < \infty\} .$$

3- The Finite Element Space

Let us consider a regular triangulation $T_h = \{e\}$ (see Kashkool(2002)) defined over $\bar{\Omega}$ where each of T_h is a closed triangle. We denote the internal vertexes by p_i with $i = 1, \dots . N$, the boundary vertexes on Γ_N by p_i with $i = N+1, \dots \dots . . M$ and on of Γ_D by p_i with $i = M+1, \dots \dots . . K$. We put h_s to be

the maximum side length of triangle and $\aleph$ to be the minimum perpendicular length of triangles for all $e \in T_h$.

Assumption for the triangulation (Ewing(1984)**) .**

(A1): Triangulation is regular and weakly a cute type, i.e., all interior angles γ of the triangles are bounded as follow: $\gamma \in [\gamma_{0,\frac{\pi}{2}}]$, where $\gamma_0 \in (0.\frac{\pi}{2})$ is independent of the mesh parameter.

Let $\varphi_i(p)$ $(1 \le i \le M)$ be the continuous function in $\bar{\Omega}$ such that $\varphi(p)$ is linear on $e \in T_h$and $\varphi_i(p_i) = \delta_{ij}$ $(1 \le i, j \le K)$ for any nodal point p_j, denote by $M_h$the linear span of φ_i , i.e., a finite dimensional of $H^1(\Omega)$.

$$M_{h=}\{x : x \in C(\bar{\Omega}\ ); x\ is\ linear\ function\ on\ e, \forall e \in T_h\}$$

And a subspace of $H_0^1(\Omega)$

$$M_{0h} = \left\{x : x \in M_{h,x}, x(p_h\ \ ) = 0,\ k = M+1, \dots \dots . . K\ \right\}.$$

Let $\{W_h\}$ be a family of finite dimensional subspace of $C^1(\bar{\Omega})$, which is piecewise polynomial of degree less than or equal to r with step length h_p and the following property: for $p \in [1, \infty], r \ge 2,$ there exists a constant M such that for $0 \le q \le 2$ and $\emptyset \in W_p^{r+1}(\Omega)$:

$$inf_{\sigma \in \{W_h\}} \|\emptyset - \sigma\|_{q,p} \le M h^{r+1+q}\, |\emptyset|_{r+1,p}.$$

Similarly, we define $\{V_h\}$ be a family of finite-dimensional subspace of $C(\bar{\Omega}) \times C(\bar{\Omega})$ which is piecewise W_h and $0 \le q \le 1$.

We also assume that the families $\{W_h\}$ and $\{V_h\}$ satisfy inverse equality:

$$\|\emptyset\|_{L^\infty} \le M h_p^{-1}\|\emptyset\|, \qquad \|\nabla\emptyset\|_{L_\infty} \le M h_p^{-1}\|\nabla\emptyset\| \qquad \forall \emptyset \in W_h$$

4-Model Problem:

Let Ω be a bounded domain (oil field) in the plane $R^2$with smooth boundary and $T > 0$. We consider the situation source and sink, the system of non-linear equation governing incompressible immiscible displacement in two-dimensional porous media with neglecting gravity are (Kashkool(2002))

$$\nabla \cdot v = 0, \qquad (x,t) \in \Omega \times (0,T) \tag{1}$$

$$v = -\alpha(s)\nabla p \ , \qquad (x,t) \in \Omega \times (0,T) \tag{2}$$

$$\emptyset \frac{\partial s}{\partial t} + \nabla \cdot (f(s)v) - \nabla \cdot (\alpha(s)\nabla s) = 0, \qquad (x,t) \in \Omega \times (0,T) \tag{3}$$

With the boundary and initial condition

$$\frac{\partial p}{\partial n} = 0, \qquad (x,t) \in \Gamma \times \ (0,T)$$

$$\frac{\partial s}{\partial n} = 0 \qquad (x,t) \in \Gamma \times \ (0,T)$$

$$s(x,0) = s^0(x) \ , \quad x \in \Omega$$

Where $s(x,t)$ reduced water saturation, $p(x,t)$ is global pressure, $v(x,t)$ is total Darcy velocity and $\emptyset$ is the porosity. Additional parameters are given by

$$\alpha(s,x) = (\lambda_w + \lambda_o)\begin{bmatrix} k_{x_1} & 0 \\ 0 & k_{x_2} \end{bmatrix}, \quad f(s) = \frac{\lambda_w}{\lambda_w + \lambda_o}, \qquad \text{a(s, x)}=\frac{\partial p_c}{\partial s} \cdot \frac{\lambda_w \lambda_o}{\lambda_w + \lambda_o}\begin{bmatrix} k_{x_1} & 0 \\ 0 & k_{x_2} \end{bmatrix},$$

Where λ_w and λ_o are the water and oil mobility respectively, k_{x_1} and $k_{x_2}$are the permeability's in the x_1 and x_2 direction and p_c capillarity pressure.

Some assumptions for the model (Kashkool (2002)):

(A2.1) $\boldsymbol{a}(s), \boldsymbol{\alpha}(s)$, and $f(s) \in C_0^1(R)$, $\qquad \emptyset(x), s_0(x) \in H^1(\Omega) \cap C(\bar{\Omega})$

(A2.2) There exists constant m_1 and m_2 such that ,

$$0 \le m_1 \le a(s), \alpha(s), f(s), \emptyset(x) \le m_2, \quad \forall s \in R, x \in \bar{\Omega}$$

(A3) The solution of equations (1), (2) and (3) are regular,

$$s(x,t) \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;W_\infty^1(\Omega)), \quad p(x,t) \in L^\infty(0,T;H^{k+3}(\Omega)), \ k > 0$$

$$s_t \in L^\infty(0,T;H^2(\Omega)), \quad s_{tt} \in L^\infty(0,T;H^1(\Omega)),$$

(A4) The inductive assumption (Solid pavel(2004)) that $\|V^L\|_{L^\infty} \le K^*, \quad 0 \le L \le n$,

The weak form of the model problem:

Let $V = H(div,\Omega) \cap \{v \cdot n = 0, in\ \Gamma\}, \quad W = L_2(\Omega)/\{z: z\ is\ constant\ in\ \Omega,$ M=$H^1(\Omega)\}$

then equations (1) and (2) are equivalent to solve the equation ,

$$\nabla \cdot (\alpha(s)\nabla p) = 0 \tag{4}$$

Multiply this equation by the test function $u \in L^2(\Omega)$ and integrate over Ω, and use green formula we have

$$(\alpha(s)\nabla p, \nabla u) = 0, \quad \forall u \in W. \tag{5}$$

The saturation equation (3) can be put in the weak form by finding a differentiable map,

$s: [0,T] \to M$, subject to $s(0) = s^0$ such that :

$$\left(\emptyset \frac{\partial s}{\partial t}, \varphi\right) + (\alpha(s)\nabla s, \nabla\varphi) + (\vec{v} \cdot \nabla f(s), \varphi) = 0, \ \forall \varphi \in M \tag{6}$$

5- Some important lemmas

Lemma (1)(Thomee(1984)). Let $\tilde{S}: (0,T] \to M_h$ be the H^1- projection of s in M_h such that $(\nabla(s - \tilde{S}), \nabla u) + (s - \tilde{S}, u) = 0, \quad \forall u \in M_h$ and $\rho = \tilde{S} - s$, then

$$\|\rho\|_{0,2,\Omega} \le Ch_s^2, \quad \|\rho\|_{1,2,\Omega} \le Ch_s, \quad \left\|\frac{\partial\rho}{\partial t}\right\|_{0,2,\Omega} \le Ch_s^2, \quad \left\|\frac{\partial\rho}{\partial t}\right\|_{0,2,\Omega} \le Ch_s$$

Lemma (2)(Thomee(1984)). we have independently of $h\ and\ t$.

$$\|\nabla\tilde{S}\|_{L^\infty} \le C(s)$$

Lemma (3)(Thomee(1984)). Assuming the appropriate regularity of s for the elliptic projection defined by

$$\left(\alpha(s)\left(\nabla\tilde{S}-s\right),\nabla x\right)=0 \quad \forall x\in M_h \text{ , we have } \quad \left\|\nabla\tilde{S}_{tt}\right\|\leq C(s)$$

Lemma (4)(Kashkool(2002)). If T_h is regular triangulation of weakly a cute type we have

$$|w_h|_{1,2,\Omega}\leq\frac{\sqrt{6}}{K}\|w_h\|_{0,2,\Omega} \qquad \forall w_h\in M_h \;.$$

Lemma (5)(Manna(2000)). Let $\tilde{p}\in W_h$ be elliptic projection of $p\in H^1(\Omega)$ into W_h defined by

$(\alpha(s)\nabla\tilde{p},\nabla x)=\left((\alpha(s)\nabla p,\nabla x)\right),\forall x\in W_h$. Then there exists C_1 such that

$$\|\tilde{p}-p\|+h_p\|\nabla\tilde{p}-\nabla p\|\leq C_1\|p\|_{r+1}h_p^{r+1}$$

6- Crank –Nicholson-Galerkin Method:

For our purpose of obtaining higher accuracy in time we shall consider the Crank-Nicholson -Galerkin method, let

$\bar{S}^n=\frac{1}{2}(S^n+S^{n-1})$ and $\bar{P}^n=\frac{1}{2}(P^n+P^{n-1})$, then equation (5) and (6) become

$$(\alpha(\bar{S}^n)\nabla\bar{P}^n,\nabla x)=0\,, \qquad \forall x\in W_h \tag{7}$$

$$(\emptyset\bar{\partial}_t S^n,\varphi)+(\,\alpha(\bar{S}^n)\nabla\bar{S}^n,\nabla\varphi)+(V^n\cdot\nabla f(\bar{S}^n),\varphi)=0\,, \qquad \forall\varphi\in M_h \tag{8}$$

Lemma (6): Let $\tilde{\bar{P}}^n$ be elliptic projection of $\bar{p}^n$ into W_h defined by,

$(\alpha(\bar{s}^n)\nabla\tilde{\bar{p}}^n,\nabla x)=(\alpha(\bar{s}^n)\nabla\bar{p}^n,\nabla x)$, then we have

$$\|\nabla\bar{P}^n-\nabla\tilde{\bar{p}}^n\|\leq\|\,\bar{S}^n-\bar{s}^n\|$$

Proof: we have $(\alpha(\bar{S}^n)\nabla\bar{P}^n,\nabla x)=0$ and $(\alpha(\bar{s}^n)\nabla\tilde{\bar{p}}^n,\nabla x)=0$

Subtract these equations we get $(\alpha(\bar{S}^n)\nabla\bar{P}^n-\;\alpha(\bar{s}^n)\nabla\tilde{\bar{p}}^n,\nabla x)=0$

Adding and subtracting $\alpha(\bar{S}^n)\nabla\tilde{\bar{p}}^n$

$$(\alpha(\bar{S}^n)(\nabla\bar{P}^n-\nabla\tilde{\bar{p}}^n),\nabla x)=((\alpha(\bar{s}^n)-\alpha(\bar{S}^n))\nabla\tilde{\bar{p}}^n,\nabla x)=0,$$

Let $\bar{P}^n - \tilde{\tilde{p}}^n \in W_h$, then

$$\|\nabla(\bar{p}^n - \tilde{\tilde{p}}^n)\|^2 \leq |(\alpha(\bar{S}^n)\nabla(\bar{P}^n - \tilde{\tilde{p}}^n), \nabla(\bar{P}^n - \tilde{\tilde{p}}^n))|$$

$$= \left|\left(\left(\alpha(\bar{s}^n) - \alpha(\bar{S}^n)\right)\nabla\tilde{\tilde{p}}^n, \nabla(\bar{P}^n - \tilde{\tilde{p}}^n)\right)\right|$$

$$\leq c\|\nabla\tilde{\tilde{p}}^n\| \, \|\bar{S}^n - \bar{s}^n\| \, \|\nabla(\bar{P}^n - \tilde{\tilde{p}}^n)\|$$

By lemma (5) and $(A3)$, if $h_p > 0$ is sufficiently small we have ,

$$\|\nabla\tilde{p}\| \leq \|\nabla p\| + c\|p\|_{r+1}h_p^r \leq c$$

Hence,

$$\|\nabla\bar{P}^n - \nabla\tilde{\tilde{p}}^n\| \leq \|\bar{S}^n - \bar{s}^n\|$$

Lemma (7): There exist a positive constant c such that

$$\|V^n - v^n\| \leq c\left(\|\bar{S} - \bar{s}\| + h_p^r\right)$$

Proof: We have $\|V^n - v^n\| = \|\alpha(\bar{S}^n)\nabla\bar{P}^n - \alpha(\bar{s}^n)\nabla\bar{p}^n\|$ adding and subtracting $\alpha(\bar{S}^n)\nabla\bar{p}^n$ we get,

$$\|V^n - v^n\| = \|\alpha(\bar{S}^n)(\nabla\bar{P}^n - \nabla\bar{p}^n)\| + \left\|\left(\alpha(\bar{S}^n) - \alpha(\bar{s}^n)\right)\nabla\bar{p}^n\right\|$$

$$\leq c \;\; \|(\nabla\bar{P}^n - \nabla\bar{p}^n)\| + \|\bar{S}^n - \bar{s}^n\|\|\nabla\bar{p}^n\|_{L^\infty}$$

Since

$\|\nabla\bar{P}^n - \nabla\bar{p}^n\| = \|\nabla\bar{P}^n - \nabla\tilde{\tilde{p}}^n\| + \|\nabla\tilde{\tilde{p}}^n - \nabla\bar{p}^n\|$ and from lemma (6) and (5), we get

$$\|\nabla\bar{P}^n - \nabla\bar{p}^n\| \leq c_2\|\bar{S}^n - \bar{s}^n\| + c_1\|\bar{p}^n\|_{r+1}h_p^r \quad \leq \left(\|\bar{S}^n - \bar{s}^n\| + h_p^r\right).$$

Hence

$$\|V^n - v^n\| \leq c\left(\|\bar{S}^n - \bar{s}^n\| + h_p^r\right)$$

Theorem (1): For small k and with $c = c(s)$, we have

$$\|S^n - s^n\| \leq c(\|S^0 - s^0\| + h_s + h_p^r + k^2)$$

Proof: Let $S - s = S - \tilde{S} + \tilde{S} - s = \theta + \rho$ and from lemma (1) we have, $\|\rho^n\| \leq ch^2$ and it remains to consider θ^n. In this case we have

$$\left(\emptyset\bar{\partial}_t\theta^n,\varphi\right) + (\alpha(\bar{S}^n)\nabla\bar{\theta}^n,\nabla\varphi) = \left(\emptyset\bar{\partial}_t S^n,\varphi\right) + (\alpha(\bar{S}^n)\nabla\bar{S}^n,\nabla\varphi) - \left(\emptyset\bar{\partial}_t\tilde{S}^n,\varphi\right) - \left(\alpha(\bar{S}^n)\nabla\bar{\tilde{S}}^n,\nabla\varphi\right), \qquad (9)$$

where $\tilde{S}^n$ is elliptic projection of s^n By using equation (8) we get ,

$$\left(\emptyset\bar{\partial}_t\theta^n,\varphi\right) + (\alpha(\bar{S}^n)\nabla\bar{\theta}^n,\nabla\varphi) = (V^n\cdot\nabla f(\bar{S}^n),\varphi) - \left(\emptyset\bar{\partial}_t\tilde{S}^n,\varphi\right) - \left(\alpha(\bar{S}^n)\nabla\bar{\tilde{S}}^n,\nabla\varphi\right)$$

Adding and subtracting $\emptyset s_t^{n-\frac{1}{2}}$ $and\ \alpha\left(s^{n-\frac{1}{2}}\right)\nabla\tilde{S}^{n-\frac{1}{2}}$

$$\left(\emptyset\bar{\partial}_t\theta^n,\varphi\right) + (\alpha(\bar{S}^n)\nabla\bar{\theta}^n,\nabla\varphi) = (V^n\cdot\nabla f(\bar{S}^n),\varphi) - \emptyset s_t^{n-\frac{1}{2}},\varphi) - \left(\emptyset\bar{\partial}_t\tilde{S}^n - \emptyset s_t^{n-\frac{1}{2}},\varphi\right) -$$

$$(\alpha(s^{n-\frac{1}{2}})\nabla\tilde{S}^{n-\frac{1}{2}},\nabla\varphi) - (\alpha(\bar{S}^n)\nabla\bar{\tilde{S}}^n - \alpha\left(s^{n-\frac{1}{2}}\right)\nabla\tilde{S}^{n-\frac{1}{2}},\nabla\varphi)$$

$$= (V^n\cdot\nabla f(\bar{S}^n),\varphi) - \left(v^n\cdot\nabla f\left(s^{n-\frac{1}{2}}\right),\varphi\right) - (\emptyset\bar{\partial}_t\tilde{S}^n - \emptyset s_t^{n-\frac{1}{2}},\varphi)$$

$$-((\alpha(\bar{S}^n) - \alpha(s^{n-\frac{1}{2}}))\nabla\bar{\tilde{S}}^n + \alpha(s^{n-\frac{1}{2}}))\nabla(\bar{\tilde{S}}^n - \tilde{S}^{n-\frac{1}{2}}),\nabla\varphi)$$

Using $\varphi = \bar{\theta}^n$ and noting that $\left(\emptyset\bar{\partial}_t\theta^n,\theta^n\right) = \frac{\emptyset}{2}\bar{\partial}_t\|\theta^n\|^2$, we get

$$\frac{\emptyset}{2}\bar{\partial}_t\|\theta^n\|^2 + \mu\left\|\nabla\bar{\theta}^n\right\|^2 \le \|I\| + c\left[\left\|\bar{\partial}_t\tilde{S}^n - s_t^{n-\frac{1}{2}}\right\|\|\bar{\theta}^n\| + \left(\left\|\bar{S}^n - s^{n-\frac{1}{2}}\right\| + \left\|\nabla(\bar{\tilde{S}}^n - \tilde{S}^{n-\frac{1}{2}})\right\|\right)\|\nabla\bar{\theta}^n\|\right]$$

Where $I = (V^n\cdot\nabla f(\bar{S}^n),\varphi) - \left(v^n\cdot\nabla f\left(s^{n-\frac{1}{2}}\right),\bar{\theta}^n\right)$ then, by Young-equality

$$\bar{\partial}_t\|\theta^n\|^2 \le C\ (\|I\| + \left\|\bar{\partial}_t\tilde{S}^n - s_t^{n-\frac{1}{2}}\right\|^2 + \left\|\bar{S}^n - s^{n-\frac{1}{2}}\right\|^2 + \left\|\nabla(\bar{\tilde{S}}^n - \tilde{S}^{n-\frac{1}{2}})\right\|^2 + \|\bar{\theta}^n\|^2)$$

To estimate I, adding and subtracting $V^n\cdot\nabla f(s^{n-\frac{1}{2}})$

$$I = (V^n \cdot \nabla f(\bar{S}^n) - V^n \cdot \nabla f\left(s^{n-\frac{1}{2}}\right) + V^n \cdot \nabla f\left(s^{n-\frac{1}{2}}\right) - v^n \cdot \nabla f\left(s^{n-\frac{1}{2}}\right), \bar{\theta}^n)$$

$$=(V^n \cdot \nabla\left(f(\bar{S}^n) - f\left(s^{n-\frac{1}{2}}\right)\right), \bar{\theta}^n) + ((V^n - v^n) \cdot \nabla f\left(s^{n-\frac{1}{2}}\right), \bar{\theta}^n)$$

$$\|I\| \leq \|I_1\| + \|I_2\|,$$

to estimate I_1 ,

$$\|I_1\| \leq \left\|(V^n \cdot \nabla\left(f(\bar{S}^n) - f\left(s^{n-\frac{1}{2}}\right)\right), \bar{\theta}^n)\right\| \leq \|V^n\| \left\|\bar{S}^n - s^{n-\frac{1}{2}}\right\| \|\bar{\theta}^n\|,$$

By $(A4$) we have, $\|V^n\| \leq K^*$ hence, by young- equality, we have

$$\|I_1\| \leq c\left(\left\|\bar{S}^n - s^{n-\frac{1}{2}}\right\|^2 + \|\bar{\theta}^n\|^2\right).$$

To estimate I_2

$$\|I_2\| \leq \left\|\left((V^n - v^n) \cdot \nabla f\left(s^{n-\frac{1}{2}}\right), \bar{\theta}^n\right)\right\| = \left\|(V^n - v^n) \cdot \acute{f}\left(s^{n-\frac{1}{2}}\right) \nabla s^{n-\frac{1}{2}}, \bar{\theta}^n\right\|$$

$$\leq ch_s \|V^n - v^n\| \|\bar{\theta}^n\|,$$

By lemma (7) we have,

$$\|V^n - v^n\| \leq c(\|\bar{S}^n - \bar{s}^n\| + h_p^r),$$

Then,

$$\|I_2\| \leq ch_s(\|\bar{S}^n - \bar{s}^n\| + h_p^r) \|\bar{\theta}^n\| \leq ch_s(\|\bar{\theta}^n\| - \|\bar{\rho}^n\| + ch_p^r)\|\bar{\theta}^n\|$$

$$\leq ch_s^2 + c(\|\bar{\theta}^n\|^2 + \|\bar{\rho}^n\|^2 + ch_p^{2r},$$

Hence

$$\|I\| \leq c(\left\|\bar{S}^n - s^{n-\frac{1}{2}}\right\|^2 + \|\bar{\theta}^n\|^2 + h_s^2 + h_p^{2r})$$

Then we get

$$\bar{\partial}_t\|\theta^n\|^2 \le c(\|\bar{\theta}^n\|^2 + \left\|\bar{\partial}_t\tilde{S}^n - s_t^{n-\frac{1}{2}}\right\|^2 + \left\|\bar{S}^n - s^{n-\frac{1}{2}}\right\|^2 + \left\|\nabla\left(\bar{\tilde{S}}^n - \tilde{S}^{n-\frac{1}{2}}\right)\right\|^2$$

$$+h_s^2 + h_p^{2r}) \qquad (10)$$

Since,

$$\left\|\bar{S}^n - s^{n-\frac{1}{2}}\right\| \le \|\bar{\theta}^n\| + \|\bar{\rho}^n\| + \left\|\bar{s}^n - s^{n-\frac{1}{2}}\right\|$$

By lemma (1), we have $\|\bar{\rho}^n\| \le \frac{1}{2}(\|\rho^n\| + \|\rho^{n-1}\|) \le c(s)h_s^2,$ and

$$\left\|\bar{s}^n - s^{n-\frac{1}{2}}\right\| \le c(s)k^2, \text{ then}$$

$$\left\|\bar{\partial}_t\tilde{S}^n - s_t^{n-\frac{1}{2}}\right\| \le \|\bar{\partial}_t\rho^n\| + \left\|\bar{\partial}_t s^n - s_t^{n-\frac{1}{2}}\right\| \le c(s)(h_s^2 + k^2)$$

And $\left\|\nabla\left(\bar{\tilde{S}}^n - \tilde{S}^{n-\frac{1}{2}}\right)\right\| \le \; c(s)\, k^2$

Then

$$\bar{\partial}_t\|\theta^n\|^2 \le \|\bar{\theta}^n\|^2 + (h_s + h_s^2 + k^2 + h_p^r)^2 \quad \le \|\bar{\theta}^n\|^2 + (h_s + k^2 + h_p^r)^2$$

Then we have (kashkool(2002))

$$\text{(1-ck)}\ \|\bar{\theta}^n\|^2 \le (1 + ck)\,)\ \|\bar{\theta}^{n-1}\|^2 + ck(h_s + k^2 + h_p^r)^2$$

$$\|\bar{\theta}^n\|^2 \le c\ \|\bar{\theta}^0\|^2 + c(h_s + k^2 + h_p^r)^2.$$

Hence,

$$\|\theta^n\| \le c(\|S^0 - s^0\| + h_s + k^2 + h_p^r \qquad (11)$$

Which completes the proof.

Theorem (2): with assumptions $(A2)$ and$(A3)$, and if $h_s, k = O\left(h_p\right)$, then

$$\|S^n - s^n\|_{L^\infty\{L_2\}} + \quad \|V^n - v^n\|_{L^\infty\{L_2\}} \le \; c\left(\|S^0 - s^0\| + h_s + k^2 + h_p^r\right)$$

Proof: from lemma (3.2)(Ewing(1984)) we have

$$\|V^n - v^n\| \le \; c(\|S^n - s^n\| + h_p^r) \qquad (12)$$

By theorem (1), we have $\|S^n - s^n\| \leq c(\|S^0 - s^0\| + h_s + k^2 + h_p^r)$

Hence

$$\|S^n - s^n\|_{L^\infty(L_2)} + \|V^n - v^n\|_{L^\infty(L_2)} \leq c\left(\|S^0 - s^0\| + h_s + k^2 + h_p^r\right) \qquad (13)$$

Which completes the proof.

6- Linearized Crank-Nicholson-Galerkin finite element method.

We shall consider a linearized modification in which the argument of α, a and f is obtained by extrapolation from S^{n-1} and S^{n-2} ,with

$$\hat{S}^n = \frac{3}{2}S^{n-1} - \frac{1}{2}S^{n-2}$$

$$\left(\emptyset\bar{\partial}_t S^n, \varphi\right) + \left(a\left(\hat{S}^n\right)\nabla\bar{S}^n, \nabla\varphi\right) + \left(V^n \cdot \nabla f\left(\hat{S}^n\right), \varphi\right) = 0, \forall\varphi \in M_h \qquad (14)$$

$$\left(\alpha\left(\hat{S}^n\right)\nabla\bar{P}^n, \nabla x\right) = 0, \forall x \in W_h \qquad (15)$$

The nonlinear equation (14) will be solvable for S^n when S^{n-1} and S^{n-2} are given.

We observe that since the equation now contains S^{n-2} it may be used for $n \geq 2$, and we have to be supplemented with another method for determiningS^1. We shall use here a predictor- corrector method for this purpose. Using as first approximating the value $S^{1,0}$ determined by the case $n = 1$ of equation (14) with $\hat{S}^1$ replaced by S^0 and then as the final approximation the result of the same equation with $\hat{S}^1$ replaced by$\frac{1}{2}(S^{1,0} + S^0)$, our procedure is defined by S^0 as follows

$$\emptyset k^{-1}(S^{1,0} + S^0, \varphi) + \left(a(S^0)\nabla\left(\frac{1}{2}(S^{1,0} + S^0)\right), \nabla\varphi\right) + (V^1 \cdot \nabla f(S^0), \varphi) = 0,\ \varphi \in M_h \qquad (16)$$

$$\left(\alpha(S^0)\nabla\left(\frac{1}{2}(P^{1,0} + P^0)\right), \nabla x\right) = 0.\ \ \forall x \in W_h \qquad (17)$$

Then

$$\emptyset\left(\bar{\partial}_t S^1, \varphi\right) + \left(a\left(\frac{1}{2}(S^{1,0} + S^0)\right)\nabla\bar{S}^1, \nabla\varphi\right) + (V^1 \cdot f\left(\frac{1}{2}(S^{1,0} + S^0)\right), \varphi = 0 \qquad (18)$$

$$(\alpha\left(\frac{1}{2}(S^{1,0}+S^{0})\right)\nabla\bar{P}^{1},\nabla x)=0 \qquad (19)$$

Lemma (8): If $\tilde{\tilde{p}}^{n}\in W_{h}$ be elliptic projection of $\bar{p}^{n}\in W_{h}$ define by

$$(\alpha(\hat{s}^{n})\ \nabla\tilde{\tilde{p}}^{n},\nabla x)=\ (\alpha(\hat{s}^{n})\ \nabla\bar{p}^{n},\nabla x)$$

$\forall x\in W_{h}$ Then $\|\nabla\bar{P}^{n}-\nabla\tilde{\tilde{p}}^{n}\|\leq c\left\|\hat{S}^{n}-\hat{s}^{n}\right\|$, where c is a positive constant.

Proof: since $(\alpha(\hat{S}^{n})\ \nabla\bar{P}^{n},\nabla x)=0$ and $(\alpha(\hat{s}^{n})\ \nabla\tilde{\tilde{p}}^{n},\nabla x)=0$, from these equation we have $(\alpha(\hat{S}^{n})\ \nabla\bar{P}^{n}-(\alpha(\hat{s}^{n})\ \nabla\tilde{\tilde{p}}^{n},\ \nabla x)=0$ by adding and subtracting $\alpha(\hat{S}^{n})\ \nabla\tilde{\tilde{p}}^{n}$, we get

$$(\alpha(\hat{S}^{n})(\ \nabla\bar{P}^{n}-\nabla\tilde{\tilde{p}}^{n}),\nabla x)=((\ \alpha(\hat{s}^{n}-\ \alpha(\hat{S}^{n}))\ \nabla\tilde{\tilde{p}}^{n},\nabla x)$$

Let $x=\bar{p}^{n}-\tilde{\tilde{p}}^{n}\in W_{h}$, then

$$\|\nabla(\bar{P}^{n}-\tilde{\tilde{p}}^{n})\|^{2}\leq\left|(\alpha\left(\hat{S}^{n}\right)(\bar{P}^{n}-\tilde{\tilde{p}}^{n}),\nabla(\bar{P}^{n}-\tilde{p}^{n}))\right|$$

$$=\left|(\alpha(\hat{s}^{n}-\ \alpha\left(\hat{S}^{n}\right)\nabla\tilde{\tilde{\bar{P}}}^{n},\nabla(\bar{P}^{n}-\tilde{\tilde{p}}^{n}))\right|$$

$$\leq c\|\nabla\tilde{\tilde{p}}^{n}\|\left\|\hat{S}^{n}-\hat{s}^{n}\right\|\|\nabla\bar{P}^{n}-\nabla\tilde{\tilde{p}}^{n}\|$$

With lemma (5) and $(A3)$ and if $h_{p}>0$ is sufficiently small, we get ,

$$\|\nabla\tilde{\tilde{p}}^{n}\|\leq\|\nabla\mathrm{p}^{n}\|+c\|p^{n}\|_{r+1}h_{p}^{r}\leq c$$

Then

$$\|\nabla(\bar{P}^{n}-\tilde{\tilde{p}}^{n})\|^{2}\leq c\left\|\hat{S}^{n}-\hat{s}^{n}\right\|\|\nabla\bar{P}^{n}-\nabla\tilde{\tilde{p}}^{n}\|$$

Hence

$$\|\nabla(\bar{P}^{n}-\tilde{\tilde{p}}^{n})\|\leq c\left\|\hat{S}^{n}-\hat{s}^{n}\right\|$$

Lemma (9): There exists a positive constant c such that ,

$$\|V^{n}-v^{n}\|\leq c\left(\left\|\hat{S}^{n}-\hat{s}^{n}\right\|+h_{p}^{r}\right)$$

Proof: Since $\|V^{n}-v^{n}\|=\left\|\alpha\left(\hat{S}^{n}\right)\nabla\bar{P}^{n}-\alpha(\hat{s}^{n})\nabla\bar{p}^{n}\right\|$

Adding and subtracting $\alpha(\hat{S}^n)\nabla\bar{p}^n$, we get

$$\|V^n - v^n\| \leq \|\alpha(\hat{S}^n)(\nabla\bar{P}^n - \nabla\bar{p}^{n)})\| + \|(\alpha(\hat{S}^n) - \alpha(\hat{s}^n))\nabla\bar{p}^{n)}\|$$

$$\leq C\|\nabla\bar{P}^n - \nabla\bar{p}^{n)}\| + \|\hat{S}^n - \hat{s}^n\|\|\nabla\bar{p}^n\|_{1,\infty}$$

We have ,

$$\|\nabla\bar{P}^n - \nabla\bar{p}^{n)}\| = \|\nabla\bar{P}^n - \nabla\tilde{\bar{p}}^n + \nabla\tilde{\bar{p}}^n - \nabla\bar{p}^n\| \leq \|\nabla\bar{P}^n - \nabla\tilde{\bar{p}}^n\| + \|\nabla\tilde{\bar{p}}^n - \nabla\bar{p}^n\|$$

Using lemma (5) and lemma (8) we get

$$\|\nabla\bar{P}^n - \nabla\bar{p}^{n)}\| \leq c\|\hat{S}^n - \hat{s}^n\| + ch_p^r\|p^n\|_{r+1}$$

Hence

$$\|V^n - v^n\| \leq c(\|\hat{S}^n - \hat{s}^n\| + h_p^r)$$

Theorem (3): Under the appropriate regularity assumption and for k is small enough with $c = c(s)$

$$\|S^n - s^n\| \leq c(\|S^0 - s^0\| + h_s + h_p^{r+}k^2)$$

Proof: From equation (9) for $n \geq 2$ we have,

$$(\emptyset\bar{\partial}_t\theta^n, \varphi) + (\alpha(\hat{S}^n)\nabla\bar{\theta}^n, \nabla\varphi) = (V^n \cdot \nabla f(\hat{S}^n), \varphi) - \left(v^n \cdot \nabla f\left(s^{n-\frac{1}{2}}\right), \varphi\right) -$$

$$(\emptyset\bar{\partial}_t\tilde{S}^n - \emptyset s_t^{n-\frac{1}{2}}, \varphi) \qquad - ((\alpha(\hat{S}^n) -$$

$$\alpha(s^{n-\frac{1}{2}}))\nabla\bar{\tilde{S}}^n + \alpha(s^{n-\frac{1}{2}}))\nabla(\bar{\tilde{S}}^n - \tilde{S}^{n-\frac{1}{2}}), \nabla\varphi)$$

By the similar way as we got equation (10) we have

$$\bar{\partial}_t\|\theta^n\|^2 \leq c(\|\bar{\theta}^n\|^2 + \left\|\bar{\partial}_t\tilde{S}^n - s_t^{n-\frac{1}{2}}\right\|^2 + \left\|\hat{S}^n - s^{n-\frac{1}{2}}\right\|^2 + \left\|\nabla\left(\tilde{S}^n - \tilde{S}^{n-\frac{1}{2}}\right)\right\|^2 + h_s^2 + h_p^{2r})$$

Since $\hat{S}^n = \frac{3}{2}S^{n-1} - \frac{1}{2}S^{n-2}$ and equation (18), we get

$$\left\|\hat{S}^n - s^{n-\frac{1}{2}}\right\| \le \|\hat{\theta}^n\| + \|\hat{\rho}^n\| + \left\|\hat{s}^n - s^{n-\frac{1}{2}}\right\| \le c(\|\theta^{n-1}\| + \|\theta^{n-2}\| + h_s^2 + k^2$$

We obtain (Kashkool(2002))

$$\|\theta^n\|^2 \le (1+ck)\|\theta^{n-1}\|^2 + ck\|\theta^{n-2}\|^2 + ck(h_s + h_p^r + k^2)^2$$

$$\le c(\|\theta^{n-1}\|^2 + \|\theta^{n-2}\|^2 + h_s^2 + h_p^{2r} + k^5)$$

By (Theorem3.1 Kashkool(2005)) we have

$$\|\theta^n\|^2 \le c(\|\theta^1\|^2 + \|\theta^0\|^2 + h_s^2 + h_p^{2r} + k^5)$$

We shall now estimate $\|\theta^1\|$ from the equation (16) and (8). In the same way as we got (14) from equation (10) with $\theta^{1,0} = S^{1,0} - \tilde{S}^0,\ \theta^{0,0} = \theta^0$

$$\bar{\partial}_t\|\theta^{1,0}\|^2 \le c(\left\|\bar{\theta}^{1,0}\right\|^2 + \left\|S^0 - s^{\frac{1}{2}}\right\|^2 + h_s^2 + h_p^{2r} + k^4)$$

Since,

$$\left\|S^0 - s^{\frac{1}{2}}\right\| \le \|\theta^0\| + \|\rho^0\| + \left\|s^0 - s^{\frac{1}{2}}\right\| \le \|\theta^0\| + c(h_s^2 + k)$$

So, $$\bar{\partial}_t\|\theta^{1,0}\|^2 \le c(\left\|\bar{\theta}^{1,0}\right\|^2 + \|\theta^0\|^2 + h_s^2 + h_p^{2r} + k^2)$$

And hence (Kashkool(2002))

$$(1\text{-ck})\left\|\bar{\theta}^{1,0}\right\|^2 \le (1+k)\|\theta^0\|^2 + ck(h_s^2 + h_p^{2r} + k^2)$$

Since,

$$\left\|\bar{\theta}^{1,0}\right\|^2 \le (1+k)\|\theta^0\|^2 + ck(h_s^2 + h_p^{2r} + k^2) \le c(\|\theta^0\|^2 + h_s^2 + h_p^{2r} + k^3)$$

Then,

$$\bar{\partial}_t\|\theta^{1,0}\|^2 \le c(\left\|\bar{\theta}^1\right\|^2 + \left\|\tfrac{1}{2}(S^{1,0} + S^0) - s^{\frac{1}{2}}\right\|^2 + h_s^2 + h_p^{2r} + k^4) \qquad (20)$$

Since,

$$\left\|\frac{1}{2}(S^{1,0}+S^{0})-s^{\frac{1}{2}}\right\| \le \left\|\frac{1}{2}(\theta^{1,0}+\theta^{0})-\theta^{0}\right\| + \left\|\bar{\bar{S}}^{1}-s^{\frac{1}{2}}\right\|$$

$$\le \frac{1}{2}(\|\theta^{1,0}\|+\|\theta^{0}\|)+c(h_s^2+k^2)$$

$$\le c\|\theta^{0}\|+c(h_s+h_p^r+k^{\frac{5}{2}})$$

Hence from equation (20), we get

$$(1\text{-ck})\left\|\bar{\theta}^{1}\right\|^{2} \le (1+k)\|\theta^{0}\|^{2}+ck(h_s^2+h_p^{2r}+k^3)$$

Then,

$$\left\|\bar{\theta}^{1}\right\|^{2} \le (1+k)\|\theta^{0}\|^{2}+ck(h_s^2+h_p^{2r}+k^3 \le c(\|\theta^{0}\|^{2}+h_s^2+h_p^{2r}+k^4)$$

Substitute this in equation (20) we get

$$\|\theta^{n}\| \le c\|\theta^{0}\|+c(h_s+h_p^r+k^2) \le c(\|S^{0}-s^{0}\|+h_s+h_p^r+k^2$$

The proof is complete.

Theorem (3): With assumption $(A2)$ and $(A3)$, and if $h_s, k = 0(h_p)$ then

$$\|S^{n}-s^{n}\|_{L^{\infty}(L_2)}+\|V^{n}-v^{n}\|_{L^{\infty}(L_2)} \le c(\|S^{0}-s^{0}\|+h_s+h_p^r+k^2)$$

Proof: from lemma (9) we have

$$\|V^{n}-v^{n}\| \le c(\left\|\hat{S}^{n}-\hat{s}^{n}\right\|+h_p^r) \le c(\|S^{0}-s^{0}\|+h_s+h_p^r+k^2)$$

From theorem (3), we have

$$\|S^{n}-s^{n}\| \le c(\|S^{0}-s^{0}\|+h_s+h_p^r+k^2)$$

Hence

$$\|S^n - s^n\|_{L^\infty(L_2)} + \|V^n - v^n\|_{L^\infty(L_2)} \le c(\|S^0 - s^0\| + h_s + h_p^r + k^2).$$

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