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On Separable S^*-Non-Atomic Boolean Algebra

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Abstract

The main purpose of this paper is to study the characterization of separable S*-non-atomic Boolean algebra and to give a necessary facts about this concept, where $S^* = S^*[0,1]$ is the ring of all real measurable functions on [0,1].

المستخلص

الغرض الرئيسي من هذا البحث هو دراسة خصائص الجبر البولياني اللاذري المنفصل-S* وإعطاء الحقائق الضرورية حول هذه المبادئ ، عندما S* تمثل حلقة جميع الدوال الحقيقية القابلة للقياس على الفترة المغلقة [0,1].

1. Introduction

Throughout this paper, ∇ , $\widehat{\nabla}$, ∇_1 , $\widehat{m}$, P and ∇_τ , denote the Boolean algebra of all Lebesgue measurable subsets of $T = [0,1]$, complete separable Boolean algebra, regular Boolean subalgebra, strictly positive S^*-valued measure, Lebesgue measure on T and measurable field of Boolean algebra, respectively.

The algebraic structures implicit in Boole's analysis were first explicitly presented by Huntington in 1904 and termed " Boolean algebra " by Sheffer in 1913. As Huntington recognized, there are various equivalent ways of characterizing Boolean algebra.

In this paper a series of known notions, notations and facts of the theory of Boolean algebras and of the theory of measurable fields of metric spaces and of Boolean algebras with a measure is cited [6,7,9,10,11].

2. Basic Concepts

***Definition 2.1* : [10]**

A mapping $\rho: X \times X \to S^*$ is called a metric on a set X with values in S^* if

1. $\rho(x,y) \geq 0$ for any $x, y \in X$ and $\rho(x,y) = 0$ if and only if $x = y$.
2. $\rho(x,y) = \rho(y,x)$ for any $x, y \in X$.
3. $\rho(x,y) \leq \rho(x,z) + \rho(z,y)$ for any $x, y, z \in X$.

***Definition 2.2* : [10]**

The space (X, ρ) is called separable, if there exists a countable subset $M \subset X$ such that for any $x \in X$, there is $\{z_n\}_{n=1}^{\infty} \subset M$ for which $\rho(x, z_n) \xrightarrow{(t)} 0$.

Suppose that (X_τ, ρ_τ) be a complete separable metric space defined for P-almost every $\tau \in T$.

***Definition 2.3* :[1]**

An element from X_+ is called a Freudenthal unit and denoted by $\hat{1}$, if it follows from $x \in X$, $x \wedge \hat{1} = 0$, that $x = 0$.
where X_+ is the set of all non-negative elements from vector lattice X .

***Definition 2.4* : [10]**

A measurable field of metric space is a pair $\{X, \{X_\tau\}_{\tau \in T}\}$ where X is a totality of functions $x: \tau \longrightarrow x(\tau) \in X_\tau$ for P-almost every $\tau \in T$ such that:

1. There exists a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ such that $\{x_n(\tau)\}_{n=1}^{\infty}$ is dense in (X_τ, ρ_τ) for P- almost every $\tau \in T$.
2. The function $\tau \to \rho_\tau(x(\tau), y(\tau))$ is measurable on T for all $x, y \in X$.
3. If $\{y_n\}_{n=1}^{\infty} \subset X, y: \tau \longrightarrow y(\tau) \in X_\tau$ for P-almost every $\tau \in T$ and $\rho_\tau(y_n(\tau), y(\tau)) \longrightarrow 0$ as $n \longrightarrow \infty$ for P-almost every $\tau \in T$, then $y \in X$.

***Remark 2.5* : [11]**

For each complete separable space (Y, d) with a S^*- valued metric d of a measurable field of metric space is determined such that (X, ρ) and (Y, ρ) are S^*- isometric.

Let A_τ be a closed subsets of X_τ for P- almost every $\tau \in T$.

***Definition 2.6* : [7]**

A measurable field of closed sets is a pair $\{\mathrm{A}, \{\mathrm{A}_\tau\}_{\tau \in T}\}$ where $\mathrm{A} \in X$ and

1. If $x \in \mathrm{A}$,then $x(\tau) \in \mathrm{A}_\tau$ for P- almost every $\tau \in T$.
2. There exists $\{x_n\}_{n=1}^{\infty} \subset A$ such that $\{x_n(\tau)\}_{n=1}^{\infty}$ is dense in A_τ for P-almost every $\tau \in T$.
3. If $\{y_n\}_{n=1}^{\infty} \subset \mathrm{A}$, $y \in \mathrm{X}$ and $\rho_\tau(y_n(\tau), y(\tau) \longrightarrow 0$ as $n \longrightarrow \infty$ for P-almost every $\tau \in T$, then $y \in \mathrm{A}$.

***Definition 2.7* : [9]**

A measurable field of Boolean algebra is a pair $\{\nabla, (\nabla_\tau, m_\tau)_{\tau \in T}\}$ where ∇ is a set of mappings $e : \tau \longrightarrow e(\tau) \in \nabla_\tau$ for P-almost every $\tau \in T$ such that

1. $\{\nabla, (\nabla_\tau, m_\tau)_{\tau \in T}\}$ is a measurable field of metric space.
2. If $e, g \in \nabla$, $h(\tau) = e(\tau) \vee g(\tau)$ for P-almost every $\tau \in T$, then $h \in \nabla$.
3. If $e \in \nabla, g(\tau) = (\hat{1} - e)(\tau)$ for P-almost every $\tau \in T$, then $g \in \nabla$.

***Definition 2.8* : [9]**

A measurable field of Boolean algebra $\{\nabla, (\nabla_\tau, m_\tau)_{\tau \in T}\}$ is said to be saturated, if for any $e \in \nabla, \mathcal{A} \in \mathrm{A}$ the function $g(\tau) = e(\tau)\, \chi_{\mathcal{A}}(\tau)$ belongs to ∇, where

$$\chi_{\mathcal{A}(\tau)} = \begin{cases} 1 \;, & \tau \in \mathcal{A} \\ 0 \;, & \tau \notin \mathcal{A} \end{cases}$$

***Definition 2.9* : [15]**

A Boolean algebra with a countable dense subset is called separable.

***Definition 2.10* : [9]**

A Boolean algebra ∇ with a S^*-valued measure m is said to be S^* -non-atomic if for any $e \in \nabla$ and $0 \leq \alpha \leq m(e), \alpha \in S^*$,there exists $g \in \nabla$ such that $g \leq e$ and $m(g) = \alpha$.

***Definition 2.11* :[2,3,4]**

The collection B of Borel sets of a topological space X is the smallest σ-algebra containing all open sets of X.

***Definition 2.12* : [12,13]**

A Borel mapping is a mapping such that the inverse image of every Borel set is Borel. It is called Borel function.

***Definition 2.13* : [5]**

Let ∇ be a Boolean algebra, and let

$$\Delta = \{e \in \nabla: if\ 0 \leq e \leq q\ then\ either\ e = 0\ or\ q = e\}.$$

The elements of Δ are called atoms.

If $\Delta = \Phi$, then ∇ is said to be a non-atomic Boolean algebra.

***Remark 2.14* : [9]**

A strictly positive measure $m : \nabla \longrightarrow S^*$ has the following moduleness property:

$$m(e\, g) = e\, m(g) \qquad \text{for all } e \in \nabla_1, g \in \nabla.$$

3. The Main Result

In this section, we investigate the important result concerning with the characteristic separable S^*-non-atomic Boolean algebra.

Firstly, we need the following information .

***Definition 3.1* :**

A non-zero element $e \in \widetilde{\nabla}$ is called a S^*-atom if for any $g \in \widetilde{\nabla}$, $g \leq e$, the equality

$$P(\{m(g) = m(e)\} \vee \{m(g) = 0\}) = 1,$$

satisfies, where $\widetilde{\nabla}$ is σ-complete Boolean algebra.

***Remark 3.2* :**

Any atom from $\widetilde{\nabla}$ is a S^*-atom.

In general, the converse of the above remark does not hold. The next example will be show this.

***Example 3.3* :**

Let $\widetilde{\nabla} = \nabla_1$, and let $m : \widetilde{\nabla} \longrightarrow S^*$ be given by the formula

$$m(e) = e\,, \quad e \in \widetilde{\nabla}.$$

Then m is a strictly positive measure on $\widetilde{\nabla}$ with values in S^* and m has the moduleness property.

For any $e, g \in \widetilde{\nabla}, e \neq 0, g \leq e$, we have

$$P(\{m(g) = m(e)\} \vee \{m(g) = 0\}) = P(g \vee (\hat{1} - g)) = P\big(\hat{1}\big) = 1.$$

Thus, any non-zero element of $\widetilde{\nabla}$ is a S^*-atom.

At the same time algebra $\widetilde{\nabla}$ is non-atomic (it has no atoms).

In this example, we note also, that the algebra $\widetilde{\nabla}$ is separable with respect to the S^*-metric, $d(e, g) = m(e\Delta g)$, where

$$e \,\Delta\, g = \big(e \wedge (\hat{1} - g)\big) \vee \big((\hat{1} - e)\big).$$

***Definition 3.4*:[16]**

A set in a Hausdorff space is called Souslin if it is the image of a complete separable metric space under a continuous mapping.

A Souslin space is a Hausdorff space that is a Souslin set.

***Theorem (Lusin-Yankov) 3.5* :[16]**

Let X and Y be Souslin spaces and let $F: X \longrightarrow Y$ be a Borel mapping such that $F(X) = Y$. Then, one can find a mapping $G: Y \longrightarrow X$ such that $F\big(G(y)\big) = y$ for all $y \in Y$and G is measurable with respect to the σ-algebra generated by all Souslin subsets in Y. In addition, the set $G(Y)$ belongs to the σ-algebra generated by Souslin sets in X.

***Remark 3.6* :**

If $\big(\widetilde{\nabla}, m\big)$ is a complete separable Boolean algebra with a S^*-valued measure m which has the moduleness property, then there exists a saturated measurable field of Boolean algebra $(\{\nabla_i\}_{i\in T}\,, \nabla)$ such that the Boolean algebra $\widehat{\nabla}$ constructed by this measurable field is S^*-isometrically isomorphic to $\widetilde{\nabla}$.

***Theorem 3.7* :**

Let $\big(\widehat{\nabla}\,, \widehat{m}\big)$ be a complete separable Boolean algebra with a strictly positive S^*-valued measure $\widehat{m}$, and let $((\nabla_\tau\,, m_\tau)_{\tau\in T})$ be a measurable field of Boolean algebra generating $\big(\widehat{\nabla}\,, \widehat{m}\big)$.

The following conditions are the equivalent :

(1) $\big(\widehat{\nabla}\,, \widehat{m}\big)$ is a S^*-non-atomic Boolean algebra.

(2) $\big(\widehat{\nabla}\,, \widehat{m}\big)$ has no S^*-atoms.

(3) $(\nabla_\tau\,, m_\tau)$ is a non-atomic Boolean algebra for P-almost every $\tau \in T$.

Proof :

(1) $\Longrightarrow$ (2) :

Let e^* be a S^*-atom in $(\widehat{\nabla}, \widehat{m})$. Since $\widehat{\nabla}$ is S^*-non-atomic, then there exists $g^* \in \widehat{\nabla}$, $g^* \leq e^*$ such that

$$\widehat{m}(g^*) = 2^{-1}\, \widehat{m}(e^*)\,,$$

this implies, that

$$\{\widehat{m}(g^*) = \widehat{m}(e^*) = \hat{1} - S(\widehat{m}(e^*)) = \widehat{m}(g^*) = 0\}.$$

Since e^* is a S^*-atom, then

$$1 = P(\{\widehat{m}(g^*) = \widehat{m}(e^*)\} \vee \{\widehat{m}(g^*) = 0\}) = P(\{\widehat{m}(g^*) = 0\}),$$

i.e. $\widehat{m}(g^*) = 0$, therefore $\widehat{m}(e^*) = 0$ which is not the case.

Therefore, $(\widehat{\nabla}, \widehat{m})$ has no S^*-atom.

(2) $\Longrightarrow$ (3) :

Let (U, d) be the universal separable metric space of Uryson, and let $\widehat{\nabla}$ be S^*-isometrically imbedded into $S^*(T, U)$, where $S^*(T, U)$ represented the set of all measurable mapping from T into U. Moreover , let $\{g_n\}_{n=1}^{\infty}$ be a dense subset of $(\widehat{\nabla}, \hat{d})$ where

$$\hat{d}(e^*, g^*) = \widehat{m}(e^* \Delta g^*),$$

Such that $\{g_n(\tau)\}_{n=1}^{\infty}$ is dense in ∇_τ with respect to

$$d_\tau(e_\tau, g_\tau) = m_\tau(e_\tau \,\Delta\, g_\tau) = d(e_\tau, g_\tau)$$

for almost every $\tau \in T$.

In this connection, we may consider that $g_n(\tau)$ are chosen in such a way that $g_n(\tau)$ are Borel functions from T into U. We define two sets A_n , $B_n \subset T \times$ U as follows:

$$A_n = \{(\tau, e) : d(g_n(\tau), e) = d(g_n(\tau), 0) - d(e, 0)\}$$
$$B_n = \{(\tau, e) : d(g_n(\tau), e) = d(g_n(\tau), 0) + d(e, 0)\}.$$

Since $g_n(\tau)$ is a Borel function and the metric d is continuous by the totality of variables, then A_n and B_n are Borel sets in $T \times$ U for all $n = 1,2, \dots$. Then the set

$$C = A \cap (\bigcap_{n=1}^{\infty} (A_n \cup B_n))$$

is also a Borel set in $T \times$ U, where

$$A = \{(\tau, e) : d(e, 0) > 0\}.$$

It follows from [8] that $\{(\nabla_\tau, d_\tau)\}_{\tau \in T}$ is a measurable field of closed sets in $S^*(T, U)$, hence (see [14]), we may consider without any loss of generality that $D = \{(\tau, e) : e \in \nabla_\tau\}$ is a Borel set in $T \times$ U.

Now, set $E = C \cap D$, then E is a Borel set in $T \times$ U. Futhermore, it follows from the definition of C and from the inclusion $E \subset D$ that

$$E = \{\{\tau, e\} : e \in \nabla_\tau, g_n(\tau) \geq e \text{ } or \text{ } g_n(\tau) \wedge e = 0, n = 1,2, \dots \}$$
$$= \{(\tau, e) : e \text{ } is \text{ } an \text{ } atom \text{ } of \text{ } \nabla_\tau\}.$$

By Lusin-Yankov theorem in the first place, the set

$$T_1 = \{\tau \in T : \text{ } there \text{ } exists \text{ } e \in \nabla_\tau \text{ } such \text{ } that \text{ } (\tau, e) \in E\}$$

is a measurable subset of T and secondly, there exists a measurable mapping $\tilde{e}(\tau) \in S^*(T, U)$, such that $(\tau, \tilde{e}(\tau)) \in E$ for almost every $\tau \in T_1$.

Suppose that $P(T_1) > 0$ and consider $e^* \in \widehat{\nabla}$ with the representative $\{e(\tau)\}_{\tau \in T}$ where $e(\tau) = e^*(\tau)$ for almost every $\tau \in T_1$ and $e(\tau) = 0$ for $\tau \notin T_1$.

Now, we must show that e^* is an S^*-atom in $\widehat{\nabla}$.

Let $g^* \in \widehat{\nabla}$ with the representative $\{g(\tau)\}_{\tau \in T}$ and let $g^* \leq e^*$. Since $e(\tau)$ is an atom in ∇_τ for almost every $\tau \in T_1$, then for almost every $\tau \in T$ we have either $g(\tau) = e(\tau)$ or $g(\tau) = 0$.

It means that;

$$P(\{\widehat{m}(g^*) = \widehat{m}(e^*)\} \vee \{\widehat{m}(g^*) = 0\}) = P(T) = 1.$$

Therefore e^* is an S^*-atom in $\widehat{\nabla}$, which contradicts the assumption. Thus $P(T_1) = 0$, in other words, for P-almost every $\tau \in T$ the Boolean algebra (∇_τ , m_τ) is non-atomic.

(3) ⟹ (1) :

Let $e^* \in \widehat{\nabla}$ and $0 \leq \alpha \leq \widehat{m}(e^*), \alpha^* \in S^*$,and let $e(\tau) , \alpha(\tau)$ be Borel representatives of e^* and of α respectively. Set

$$A = \{(d(e,0) , e) : d(e,0) = \alpha(\tau)\}$$
$$B = \{(\tau , e) : d(e(\tau),0) - d(e,0) = d(e(\tau), e)\} .$$

By the same ways as in the proof of the implication ((2) ⟹ (3)), we get that the sets A and B are Borel subsets of $T \times$ U. Hence, the space $C = A \cap B \cap D$ is the same set as in the proof of the implication

((2) ⟹ (3)). It is clear that

$$C = \{(\tau , e) : \ e \in \nabla_\tau , e \leq e(\tau), m_\tau(e) = \alpha(\tau)\}$$

By Lusin-Yankov theorem, the set

$$T_1 = (\{\tau \in T : \exists \ e \in \nabla_\tau \ \text{ such that } (\tau , e) \in C \})$$

is a measurable subset of T, and there exists a measurable mapping $g(\tau) \in S^*(T, U)$ such that $(\tau , g(\tau)) \in C$ for almost every $\tau \in T_1$.

Since $\alpha(\tau) \leq m_\tau(e(\tau))$ and $(\nabla_\tau , \mathrm{m}_\tau)$ is a non- atomic Boolean algebra for almost every $\tau \in T$, we have $P(T_1) = 1$.

Let us consider the element $g^* \in S^*(T, U)$ with the representative $\{g(\tau)\}_{\tau \in T}$.

It is clear that $g^* \in \widehat{\nabla}$ and $\widehat{m}(g^*) = \alpha$.

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