A Numerical Method for Solving the Multi-Dimensional Hyperbolic Equations With Nonlocal Non-Linear Conditions

Dr.Ahlam Jameel Khaleel and Jehan Adnan Khtan
Department of Mathematics and Computer Applications,
College of science, Al-Nahrain University

Abstract:

In this work we used numerical method for solving the initial value problem that consists of the multi-dimensional hyperbolic equation with (2m) nonlocal non-linear integral boundary conditions. This method depends on Crank-Niklson finite difference scheme and Taylor's expansion.In this method the (2m) integrals in the (2m) nonlocal non-linear boundary conditions are approximated by using the composite Simpson 1/3 rule. Some examples are presented to illustrate the applicability of this method.

المستخلص:

في هذا العمل استعملنا طريقة عددية لحل مشكلة القيمة الابتدائية التي تَشْمل معادلة القطع الزائد المتعددة الأبعاد مَع (2m) شروط حدودية تكاملية غير محلية لاخطية . تعتمد هذه الطريقة على طريقة كرانك نيكولسون للفروقات المنتهية وتوسيع تايلر . في هذه ألطريقة التكاملات (2m)في الشروط الحدودية (2m) الغير محلية لاخطية مقَرَّبة باستعمال طريقة سيمبسن 1/3 . وسنقدم بَعْض الأمثلة لتَصوير تطبيق هذه الطريقة.

1-Introduction:

The nonlocal conditions appear when values of the function on the boundary are connected to values inside the domain, such boundary value problems are known as nonlocal problems. Nonlocal problems have been a major research area in modern physics, biology, chemistry, and engineering when it is impossible to determine the boundary values of the unknown function. Many complex physical phenomena are frequently described and modeled by partial differential equations with the nonlinear boundary conditions. Theoretical studies on the existence and uniqueness and the behaviors of solutions for problems governed by wave equation with nonlocal conditions have received considerable attention in the literature ([5], [7] and [8]), [1] used finite difference scheme for solving multi-dimensional hyperbolic equation with one nonlocal linear integral boundary condition, [2] used finite difference scheme to solve the one-dimensional heat equation with two nonlocal non-linear integral boundary conditions, [3] used the shifted Legendre tau technique for solving the one-dimensional wave equation with one nonlocal linear integral boundary condition, [6], [11] used the variational iteration method and the homotopy analysis method for approximating solutions of the one-dimensional wave equation with one nonlocal linear integral condition respectively, [9] gave a numerical method which depends on the properties of Chebyshev polynomials of the second kind for solving the one-dimensional wave equation subject to one nonlocal linear integral boundary condition and [10] described a numerical technique based on an integro-differential equation and local interpolating functions for solving the one-dimensional wave equation with one nonlocal linear integral boundary condition. In this paper we concerned with the numerical solution of the multi-dimensional hyperbolic equation:

$$\frac{\partial^2 u(x_1,x_2,\dots,x_m,t)}{\partial t^2}-\sum_{i=1}^{m}\frac{\partial}{\partial x_i}\left[a_i(x_1,x_2,\dots,x_m)\,\frac{\partial u(x_1,x_2,\dots,x_m,t)}{\partial x_i}\right]=f(x_1,x_2,\dots,x_m,t)$$

$$x_i\in(a_i,b_i), i=1,2,\dots,m,\qquad t\in[0,T]\quad\dots\quad(1.1)$$

together with the initial conditions

$$u(x_1,x_2,\dots,x_m,0)=d(x_1,x_2,\dots,x_m),\qquad x_i\in(a_i,b_i),\quad i=1,2,\dots,m\quad\dots\quad(1.2)$$

$$u_t(x_1,x_2,\dots,x_m,0)=r(x_1,x_2,\dots,x_m),\qquad x_i\in(a_i,b_i),\quad i=1,2,\dots,m\quad\dots\quad(1.3)$$

and the 2m nonlocal non-linear integral boundary conditions:

$$u(x_1,x_2,\dots,x_{i-1},a_i,x_{i+1},\dots,x_m,t)=\int_{a_i}^{b_i} w_{0,i}(x_1,x_2,\dots,x_m)\,u^{p_i}(x_1,x_2,\dots,x_m,t)\,dx_i + g_{0,i}(t),\qquad i=1,2,\dots,m,\ t\in[0,T] \qquad \dots \quad (1.4)$$

$$u(x_1,x_2,\dots,x_{i-1},b_i,x_{i+1},\dots,x_m,t)=\int_{a_i}^{b_i} w_{1,i}(x_1,x_2,\dots,x_m)\,u^{q_i}(x_1,x_2,\dots,x_m,t)\,dx_i + g_{1,i}(t),\qquad i=1,2,\dots,m,\ t\in[0,T] \qquad \dots \quad (1.5)$$

where $p_i \geq 1, q_i \geq 1$ are known constants, m is a positive integer, $w_{0,i}, w_{1,i}, g_{0,i}, g_{1,i}$, $i=1,2,\dots,m$, f, d and r are known functions that must satisfy the compatibility conditions:

(1) $d(x_1,x_2,\dots,x_{i-1},a_i,x_{i+1},\dots,x_m)=\int_{a_i}^{b_i} w_{0,i}(x_1,x_2,\dots,x_m)\big(d(x_1,x_2,\dots,x_m)\big)^{p_i}dx_i + g_{0,i}(0)$

(2) $d(x_1,x_2\dots,,x_{i-1},b_i,x_{i+1},\dots,x_m)=\int_{a_i}^{b_i} w_{1,i}(x_1,x_2,\dots,x_m)\big(d(x_1,x_2,\dots,x_m)\big)^{q_i}dx_i + g_{1,i}(0)$

(3) $r(x_1,x_2,\dots,x_{i-1},a_i,x_{i+1},\dots,x_m)=$
$$\int_{a_i}^{b_i} w_{0,i}(x_1,x_2,\dots,x_m)p_i\big(d(x_1,x_2,\dots,x_m)\big)^{p_i-1}r(x_1,x_2,\dots,x_m)dx_i+\left.\frac{dg_{0,i}}{dt}\right|_{t=0}$$

(4) $r(x_1,x_2,\dots,x_{i-1},b_i,x_{i+1},\dots,x_m)=$
$$\int_{a_i}^{b_i} w_{1,i}(x_1,x_2,\dots,x_m)q_i\big(d(x_1,x_2,\dots,x_m)\big)^{q_i-1}r(x_1,x_2,\dots,x_m)dx_i+\left.\frac{dg_{1,i}}{dt}\right|_{t=0}$$

2- Solutions of nonlocal problem (1.1)-(1.5):

In this section, we used Crank-Niklson finite difference scheme for finding the solutions of the nonlocal problem given by equations (1.1)-(1.5). To do this, we divide the domain $[a_1, b_1]\times[a_2, b_2]\times\dots\times[a_m,b_m]\times[0,,T]$ into $N_1\times N_2\times\dots\times N_m\times M$ mesh with spatial step size $h_i=\frac{b_i-a_i}{N_i}$ in x_i-direction and the time step size $k=\frac{T}{M}$ respectively, where M and N_i are positive integer. The grid points are given by

$$x_i^{s_i}=a_i+s_ih_i,\qquad s_i=0,1,\dots,N_i,\quad i=1,2,\dots,m,$$
$$t_j=jk,\qquad j=0,1,\dots,M.$$

We define the following difference operators:

$$\delta_{x_i}u\big(x_1^{s_1},x_2^{s_2},\dots,x_m^{s_m},t_j\big)=u_{s_1,s_2,\dots,s_i+1,s_{i+1},\dots,s_m,j}-u_{s_1,s_2,\dots,s_m,j},\qquad s_i=0,1,\dots,N_i,\ j=0,1,\dots,M,$$

$$\delta_{x_i}^2u\big(x_1^{s_1},x_2^{s_2},\dots,x_m^{s_m},t_j\big)=u_{s_1,s_2,\dots,s_i-1,s_{i+1},\dots,s_m,j}-2u_{s_1,s_2,\dots,s_m,j}+u_{s_1,s_2,\dots,s_i+1,s_{i+1},\dots,s_m,j},\qquad s_i=1,2,\dots,N_i,\ j=0,1,\dots,M,$$

$$\delta_t^2u\big(x_1^{s_1},x_2^{s_2},\dots,x_m^{s_m},t_j\big)=u_{s_1,s_2,\dots,s_m,j+1}-2u_{s_1,s_2,\dots,s_m,j}+u_{s_1,s_2,\dots,s_m,j-1},\qquad s_i=0,1,\dots,N_i,\ j=1,2,\dots,M,$$

where $u_{s_1,s_2,\dots,s_m,j}$ is the numerical solution of the I.V.P (1.1)-(1.5) at the point $\big(x_1^{s_1},x_2^{s_2},\dots,x_m^{s_m},t_j\big)$.

We replaced $\frac{\partial^2u}{\partial {x_i}^2}$ by the mean of its finite difference representation on the (j+1) and jth time rows:

$$\frac{\partial^2u\big(x_1^{s_1},x_2^{s_2},\dots,x_m^{s_m},t_j\big)}{\partial {x_i}^2}=\frac{1}{2h_i^2}\big(\delta_{x_i}^2u\big(x_1^{s_1},x_2^{s_2},\dots,x_m^{s_m},t_j\big)+\delta_{x_i}^2u\big(x_1^{s_1},x_2^{s_2},\dots,x_m^{s_m},t_{j+1}\big),\qquad s_i=1,2,\dots,N_i,\ j=1,2,\dots,M.$$

Then we approximate equation (1.1) by

$$\frac{1}{k^2}\delta_t^2 u_{s_1,s_2,\dots,s_m,j} = \sum_{i=1}^{m}\left[\frac{\frac{\partial}{\partial x_i}a_i\left(x_1^{s_1},x_2^{s_2},\dots,x_m^{s_m}\right)}{h_i}\delta_{x_i}u_{s_1,s_2,\dots,s_m,j}\right]$$
$$+\sum_{i=1}^{m}\left[\frac{a_i\left(x_1^{s_1},x_2^{s_2},\dots,x_m^{s_m}\right)}{2h_i^2}\delta_{x_i}^2\left(u_{s_1,s_2,\dots,s_m,j+1}+u_{s_1,s_2,\dots,s_m,j}\right)\right]+f\left(x_1^{s_1},x_2^{s_2},\dots,x_m^{s_m},t_j\right),$$
$$j=1,2,\dots,M.$$

After simple computations, one can have

$$-\sum_{i=1}^{m}a_i\left(x_1^{s_1},x_2^{s_2},\dots,x_m^{s_m}\right)r_i^2u_{s_1,s_2,\dots,s_i-1,s_{i+1},\dots,s_m,j+1}$$
$$+2\left[1+\sum_{i=1}^{m}a_i\left(x_1^{s_1},x_2^{s_2},\dots,x_m^{s_m}\right)r_i^2\right]u_{s_1,s_2,\dots,s_m,j+1}$$
$$-\sum_{i=1}^{m}a_i\left(x_1^{s_1},x_2^{s_2},\dots,x_m^{s_m}\right)r_i^2u_{s_1,s_2,\dots,s_i+1,s_{i+1},\dots,s_m,j+1}=M_{s_1,s_2,\dots,s_m,j}\qquad j=1,2,\dots,M\ \dots \qquad (2.1)$$

where $M_{s_1,s_2,\dots,s_m,j} = \sum_{i=1}^{m}a_i\left(x_1^{s_1},x_2^{s_2},\dots,x_m^{s_m}\right)r_i^2u_{s_1,s_2,\dots,s_i-1,s_{i+1},\dots,s_m,j}$

$$+2\left[2-\sum_{i=1}^{m}\left(a_i\left(x_1^{s_1},x_2^{s_2},\dots,x_m^{s_m}\right)r_i^2+k^2\frac{\frac{\partial a_i\left(x_1^{s_1},x_2^{s_2},\dots,x_m^{s_m}\right)}{\partial x_i}}{h_i}\right)\right]u_{s_1,s_2,\dots,s_m,j}$$
$$+\sum_{i=1}^{m}\left[a_i\left(x_1^{s_1},x_2^{s_2},\dots,x_m^{s_m}\right)r_i^2+2\,k^2\frac{\frac{\partial a_i\left(x_1^{s_1},x_2^{s_2},\dots,x_m^{s_m}\right)}{\partial x_i}}{h_i}\right]u_{s_1,s_2,\dots,s_i+1,s_{i+1},\dots,s_m,j}$$
$$-2u_{s_1,s_2,\dots,s_m,j-1}+2k^2f\left(x_1^{s_1},x_2^{s_2},\dots,x_m^{s_m},t_j\right),\ s_i=1,2,\dots,N_i\,,i=1,2,\dots,m,j=1,2,\dots,M$$

and $r_i=\frac{k}{h_i}$, *i=1,2,...,m*

By substituting j=0 in equation (1.2), we obtain

$$u_{s_1,s_2,\dots,s_m,0}=d\left(x_1^{s_1},x_2^{s_2},\dots,x_m^{s_m}\right),\qquad s_i=1,2,\dots,N_i\,,i=1,2,\dots,m$$

then we approximate equation (1.3) by using forward finite difference formula to get

$$u_{s_1,s_2,\dots,s_m,1}=kr\left(x_1^{s_1},x_2^{s_2},\dots,x_m^{s_m}\right)+u_{s_1,s_2,\dots,s_m,0}\,,\qquad s_i=1,2,\dots,N_i\,,i=1,2,\dots,m\ \dots\ (2.2)$$

Now, by using Taylor's expansion for the nonlinear functions $u^{p_i}\left(x_1^{s_1},x_2^{s_2},\dots,x_m^{s_m},t_{j+1}\right)$ and $u^{q_i}\left(x_1^{s_1},x_2^{s_2},\dots,x_m^{s_m},t_{j+1}\right)$ about the point $\left(x_1^{s_1},x_2^{s_2},\dots,x_m^{s_m},t_j\right)$, one can get

$$u_{s_1,s_2,\dots,s_m,j+1}^{p_i}=u_{s_1,s_2,\dots,s_m,j}^{p_i}+kp_iu_{s_1,s_2,\dots,s_m,j}^{p_i-1}\frac{\partial u_{s_1,s_2,\dots,s_m j}}{\partial t}+o(k^2)$$

and

$$u_{s_1,s_2,\dots,s_m,j+1}^{q_i}=u_{s_1,s_2,\dots,s_m,j}^{q_i}+kq_iu_{s_1,s_2,\dots,s_m,j}^{q_i-1}\frac{\partial u_{s_1,s_2,\dots,s_m,j}}{\partial t}+o(k^2)$$

respectively, where $o(k^2)$ denotes terms second and higher powers of k . But

$$\frac{\partial u_{s_1,s_2,\dots,s_m,j}}{\partial t}=\frac{\left(u_{s_1,s_2,\dots,s_m,j+1}-u_{s_1,s_2,\dots,s_m,j}\right)}{k}$$

Therefore, one can get

$$\left.\begin{aligned} u^{p_i}_{s_1,s_2,\dots,s_m,j+1} &= (1-p_i)u^{p_i}_{s_1,s_2,\dots,s_m,j} + q_i u^{p_i}_{s_1,s_2,\dots,s_m,j} u_{\,s_1,s_2,\dots,s_m,j+1} \\ u^{q_i}_{s_1,s_2,\dots,s_m,j+1} &= (1-q_i)u^{q_i}_{s_1,s_2,\dots,s_m,j} + q_i u^{q_i}_{s_1,s_2,\dots,s_m,j} u_{\,s_1,s_2,\dots,s_m,j+1} \end{aligned}\right\} \quad \dots \quad (2.3)$$

the integrals in equations (1.4)-(1.5) can be approximated by using quadrature rules say Simpson 1/3 rule, to obtain

$$\begin{aligned} u_{s_1,s_2,\dots,s_{i-1},a_i,s_{i+1},\dots,s_m,j+1} = \tfrac{h_i}{3}\big[& w_{0,i}\big(x_1^{s_1},x_2^{s_2},\dots,x_i^{a_i},x_{i+1}^{s_{i+1}},\dots,x_m^{s_m}\big)u^{p_i}_{s_1,s_2,\dots,s_{i-1},a_i,s_{i+1},\dots,s_m,j+1} + \\ & 4\,w_{0,i}\big(x_1^{s_1},x_2^{s_2},\dots,x_i^{s_i+1},x_{i+1}^{s_{i+1}},\dots,x_m^{s_m}\big)u^{p_i}_{s_1,s_2,\dots,s_{i-1},s_i+1,s_{i+1},\dots,s_m,j+1} + \\ & 2w_{0,i}\big(x_1^{s_1},x_2^{s_2},\dots,x_i^{s_i+2},x_{i+1}^{s_{i+1}},\dots,x_m^{s_m}\big)u^{p_i}_{s_1,s_2,\dots,s_{i-1},s_i+2,s_{i+1},\dots,s_m,j+1} + \dots + \\ & w_{0,i}\big(x_1^{s_1},x_2^{s_2},\dots,x_i^{N_i},x_{i+1}^{s_{i+1}},\dots,x_m^{s_m}\big)u^{p_i}_{s_1,s_2,\dots,s_{i-1},N_i,s_{i+1},\dots,s_m,j+1}\big] + g_{0,i}(t_{j+1}), \\ & j = 1,2,\dots,M\,,\; i = 1,2,\dots,m \qquad \dots \qquad (2.4) \end{aligned}$$

and

$$\begin{aligned} u_{s_1,s_2,\dots,s_{i-1},b_i,s_{i+1},\dots,s_m,j+1} = \tfrac{h_i}{3}\big[& w_{1,i}\big(x_1^{s_1},x_2^{s_2},\dots,x_i^{a_i},x_{i+1}^{s_{i+1}},\dots,x_m^{s_m}\big)u^{q_i}_{s_1,s_2,\dots,s_{i-1},a_i,s_{i+1},\dots,s_m,j+1} + \\ & 4\,w_{1,i}\big(x_1^{s_1},x_2^{s_2},\dots,x_i^{s_i+1},x_{i+1}^{s_{i+1}},\dots,x_m^{s_m}\big)u^{q_i}_{s_1,s_2,\dots,s_{i-1},s_i+1,s_{i+1},\dots,s_m,j+1} + \\ & 2w_{1,i}\big(x_1^{s_1},x_2^{s_2},\dots,x_i^{s_i+2},x_{i+1}^{s_{i+1}},\dots,x_m^{s_m}\big)u^{q_i}_{s_1,s_2,\dots,s_{i-1},s_i+2,s_{i+1},\dots,s_m,j+1} + \dots + \\ & w_{1,i}\big(x_1^{s_1},x_2^{s_2},\dots,x_i^{N_i},x_{i+1}^{s_{i+1}},\dots,x_m^{s_m}\big)u^{q_i}_{s_1,s_2,\dots,s_{i-1},N_i,s_{i+1},\dots,s_m,j+1}\big] + g_{1,i}(t_{j+1}) \\ & j = 0,1,\dots,M\,,\; i = 1,2,\dots,m \qquad \dots \qquad (2.5). \end{aligned}$$

By substituting equations (2.3) in equations (2.4) and (2.5), we obtain

$$\begin{aligned} & a_{s_1,s_2,\dots,s_{i-1},a_i,s_{i+1},\dots,s_m,j}u_{s_1,s_2,\dots,s_{i-1},a_i,s_{i+1},\dots,s_m,j+1} \\ & \quad + a_{s_1,s_2,\dots,s_{i-1},s_i+1,s_{i+1},\dots,s_m,j}\,u_{s_1,s_2,\dots,s_{i-1},s_i+1,s_{i+1},\dots,s_m,j+1} \\ & \quad + \dots + a_{s_1,s_2,\dots,s_{i-1},N_i,s_{i+1},\dots,s_m,j}\,u_{s_1,s_2,\dots,s_{i-1},N_i,s_{i+1},\dots,s_m,j+1)} = L_{s_1,s_2,\dots,s_{i-1},N_i,s_{i+1},\dots,s_m,j} \\ & \qquad j = 1,2,\dots,M, i = 1,2,\dots,m \qquad \dots \qquad (2.6) \end{aligned}$$

and

$$\begin{aligned} & b_{s_1,s_2,\dots,s_{i-1},a_i,s_{i+1},\dots,s_m,j}u_{s_1,s_2,\dots,s_{i-1},a_i,s_{i+1},\dots,s_m,j+1} \\ & \quad + b_{s_1,s_2,\dots,s_{i-1},s_i+1,s_{i+1},\dots,s_m,j}\,u_{s_1,s_2,\dots,s_{i-1},s_i+1,s_{i+1},\dots,s_m,j+1} \\ & \quad + \dots + b_{s_1,s_2,\dots,s_{i-1},N_i,s_{i+1},\dots,s_m,j}\,u_{s_1,s_2,\dots,s_{i-1},N_i,s_{i+1},\dots,s_m,j+1)} = Q_{s_1,s_2,\dots,s_{i-1},N_i,s_{i+1},\dots,s_m,j} \\ & \qquad j = 1,2,\dots,M, i = 1,2,\dots,m, \qquad \dots \qquad (2.7) \end{aligned}$$

where

$$a_{s_1,s_2,\dots,s_{i-1},a_i,s_{i+1},\dots,s_m,j} = p_i h_i\, w_{0,i}\big(x_1^{s_1},x_2^{s_2},\dots,x_i^{a_i},x_{i+1}^{s_{i+1}},\dots,x_m^{s_m}\big)u^{p_i-1}_{s_1,s_2,\dots,s_{i-1},a_i,s_{i+1},\dots,s_m,j} - 3\,,$$
$$j = 1,2,\dots,M, i = 1,2,\dots,m,$$

$$b_{s_1,s_2,\dots,s_{i-1},a_i,s_{i+1},\dots,s_m,j} = q_i h_i\, w_{1,i}\big(x_1^{s_1},x_2^{s_2},\dots,x_i^{a_i},x_{i+1}^{s_{i+1}},\dots,x_m^{s_m}\big)u^{q_i-1}_{s_1,s_2,\dots,s_{i-1},a_i,s_{i+1},\dots,s_m,j},$$
$$j = 1,2,\dots,M, i = 1,2,\dots,m,$$

$$a_{s_1,s_2,\dots,s_{i-1},2r_i+1,s_{i+1},\dots,s_m,j} = 4p_i h_i\, w_{0,i}\big(x_1^{s_1},x_2^{s_2},\dots,x_i^{2r_i+1},x_{i+1}^{s_{i+1}},\dots,x_m^{s_m}\big)u^{p_i-1}_{s_1,s_2,\dots,s_{i-1},2r_i+1,s_{i+1},\dots,s_m,j},$$
$$r_i = 0,1,\dots,\frac{N_i}{2}-1, \qquad j = 1,2,\dots,M, i = 1,2,\dots,m,$$

$$b_{s_1,s_2,\dots,s_{i-1},2r_i+1,s_{i+1},\dots,s_m,j} = 4q_i h_i\, w_{1,i}\big(x_1^{s_1},x_2^{s_2},\dots,x_i^{2r_i+1},x_{i+1}^{s_{i+1}},\dots,x_m^{s_m}\big)u^{q_i-1}_{s_1,s_2,\dots,s_{i-1},2r_i+1,s_{i+1},\dots,s_m,j},$$
$$r_i = 0,1,\dots,\frac{N_i}{2}-1, \qquad j = 1,2,\dots,M, i = 1,2,\dots,m,$$

$$a_{s_1,s_2,\dots,s_{i-1},2r_i,s_{i+1},\dots,s_m,j} = 2p_i h_i\, w_{0,i}\big(x_1^{s_1},x_2^{s_2},\dots,x_i^{2r_i},x_{i+1}^{s_{i+1}},\dots,x_m^{s_m}\big)u^{p_i-1}_{s_1,s_2,\dots,s_{i-1},2r_i,s_{i+1},\dots,s_m,j},$$
$$r_i = 1,2,\dots,\frac{N_i}{2}-1, \qquad j = 1,2,\dots,M, i = 1,2,\dots,m,$$

$$b_{s_1,s_2,\dots,s_{i-1},2r_i,s_{i+1},\dots,s_m,j} = 2q_i h_i\, w_{1,i}\big(x_1^{s_1},x_2^{s_2},\dots,x_i^{2r_i},x_{i+1}^{s_{i+1}},\dots,x_m^{s_m}\big)u^{q_i-1}_{s_1,s_2,\dots,s_{i-1},2r_i,s_{i+1},\dots,s_m,j},$$

$$r_i = 1,2,\dots,\frac{N_i}{2}-1, \qquad j = 1,2,\dots,M, i = 1,2,\dots,m,$$

$$a_{s_1,s_2,\dots,s_{i-1},N_i,s_{i+1},\dots,s_m,j} = p_i h_i\, w_{0,i}\left(x_1^{s_1},x_2^{s_2},\dots,x_i^{N_i},x_{i+1}^{s_{i+1}},\dots,x_m^{s_m}\right)u_{s_1,s_2,\dots,s_{i-1},N_i,s_{i+1},\dots,s_m,j}^{p_i-1},$$
$$j = 1,2,\dots,M, i = 1,2,\dots,m,$$

$$b_{s_1,s_2,\dots,s_{i-1},N_i,s_{i+1},\dots,s_m,j} = q_i h_i\, w_{1,i}\left(x_1^{s_1},x_2^{s_2},\dots,x_i^{N_i},x_{i+1}^{s_{i+1}},\dots,x_m^{s_m}\right)u_{s_1,s_2,\dots,s_{i-1},N_i,s_{i+1},\dots,s_m,j}^{q_i-1} - 3,$$
$$j = 1,2,\dots,M, i = 1,2,\dots,m,$$

$$L_{s_1,s_2,\dots,s_{i-1},N_i,s_{i+1},\dots,s_m,j} = (p_i-1)h_i\, w_{0,i}\left(x_1^{s_1},x_2^{s_2},\dots,x_i^{a_i},x_{i+1}^{s_{i+1}},\dots,x_m^{s_m}\right)u_{s_1,s_2,\dots,s_{i-1},a_i,s_{i+1},\dots,s_m,j}^{p_i}$$
$$+4(p_i-1)h_i w_{0,i}\left(x_1^{s_1},x_2^{s_2},\dots,x_i^{s_i+1},x_{i+1}^{s_{i+1}},\dots,x_m^{s_m}\right)u_{s_1,s_2,\dots,s_{i-1},s_i+1,s_{i+1},\dots,s_m,j}^{p_i}$$
$$+2(p_i-1)h_i w_{0,i}\left(x_1^{s_1},x_2^{s_2},\dots,x_i^{s_i+2},x_{i+1}^{s_{i+1}},\dots,x_m^{s_m}\right)u_{s_1,s_2,\dots,s_{i-1},s_i+2,s_{i+1},\dots,s_m,j}^{p_i}$$
$$+\cdots+(p_i-1)h_i w_{0,i}\left(x_1^{s_1},x_2^{s_2},\dots,x_i^{N_i},x_{i+1}^{s_{i+1}},\dots,x_m^{s_m}\right)u_{s_1,s_2,\dots,s_{i-1},N_i,s_{i+1},\dots,s_m,j}^{p_i}$$
$$-3\, g_{0,i}(t_{j+1}), \qquad j = 1,2,\dots,M, i = 1,2,\dots,m$$

and

$$Q_{s_1,s_2,\dots,s_{i-1},N_i,s_{i+1},\dots,s_m,j} = (q_i-1)h_i w_{1,i}\left(x_1^{s_1},x_2^{s_2},\dots,x_i^{a_i},x_{i+1}^{s_{i+1}},\dots,x_m^{s_m}\right)u_{s_1,s_2,\dots,s_{i-1},a_i,s_{i+1},\dots,s_m,j}^{q_i}$$
$$+4\,(q_i-1)h_i w_{1,i}\left(x_1^{s_1},x_2^{s_2},\dots,x_i^{s_i+1},x_{i+1}^{s_{i+1}},\dots,x_m^{s_m}\right)u_{s_1,s_2,\dots,s_{i-1},s_i+1,s_{i+1},\dots,s_m,j}^{q_i}$$
$$+2(q_i-1)h_i w_{1,i}\left(x_1^{s_1},x_2^{s_2},\dots,x_i^{s_i+2},x_{i+1}^{s_{i+1}},\dots,x_m^{s_m}\right)u_{s_1,s_2,\dots,s_{i-1},s_i+2,s_{i+1},\dots,s_m,j}^{q_i}$$
$$+\cdots+(q_i-1)h_i w_{1,i}\left(x_1^{s_1},x_2^{s_2},\dots,x_i^{N_i},x_{i+1}^{s_{i+1}},\dots,x_m^{s_m}\right)u_{s_1,s_2,\dots,s_{i-1},N_i,s_{i+1},\dots,s_m,j}^{q_i}$$
$$-3\, g_{1,i}(t_{j+1}), \qquad j = 1,2,\dots,M, i = 1,2,\dots,m$$

Therefore equations (2.1), (2.6) and (2.7) can be written in the matrix form:

$$\left[\begin{array}{ccccccccccc}
a_{a_1,s_2,\dots,s_m,j} & a_{s_1+1,s_2,\dots,s_m,j} & a_{s_1+2,s_2,\dots,s_m,j} & \cdots & a_{N_1,s_2,\dots,s_m,j} & 0 & & 0 & 0 & & 0 \\
\alpha_1 & \beta & \alpha_1 & & 0 & & & & & & \\
0 & \alpha_1 & \beta & & 0 & & & & & & \\
\vdots & \vdots & \vdots & & \ddots & & \cdots & & & & \\
0 & 0 & 0 & & \alpha_1 & & & & & & \\
b_{a_1,s_2,\dots,s_m,j} & b_{s_1+1,s_2,\dots,s_m,j} & b_{s_1+2,s_2,\dots,s_m,j} & & b_{N_1,s_2,\dots,s_m,j} & 0 & & 0 & 0 & & 0 \\
0 & 0 & 0 & & \cdots & 0 & & & & & \\
 & & \vdots & & & & \ddots & & \vdots & & \\
0 & 0 & 0 & & & & & a_{s_1,s_2,s_3,\dots,a_m,j} & a_{s_1,s_2,\dots,s_m+1,j} & \cdots & a_{s_1,s_2,\dots,s_{m-1},N_m,j} \\
 & & & & & & & \alpha_m & 0 & \cdots & 0 \\
 & & & & & & & 0 & \alpha_m & 0 & 0 \\
 & & & & & & \cdots & \vdots & \vdots & & \vdots \\
 & & & & & & & & & \ddots & \\
 & & & & & & & 0 & 0 & & \alpha_m \\
0 & 0 & 0 & & & & & b_{s_1,s_2,s_3,\dots,a_m,j} & b_{s_1,s_2,\dots,s_m+1,j} & \cdots & b_{s_1,s_2,\dots,s_{m-1},N_m,j}
\end{array}\right].$$

$$\begin{bmatrix} u_{a_1,s_2,\dots,s_m,j+1} \\ u_{s_1+1,s_2,\dots,s_m,j+1} \\ \vdots \\ u_{N_1-1,s_2,\dots,s_m,j+1} \\ u_{N_1,s_2,\dots,s_m,j+1} \\ \vdots \\ u_{s_1,s_2,s_3,\dots,a_m,j+1} \\ u_{s_1,s_2,\dots,s_m+1,j+1} \\ \vdots \\ u_{s_1,s_2,\dots,s_{m-1},N_{m-1},j+1} \\ u_{s_1,s_2,\dots,s_{m-1},N_m,j+1} \end{bmatrix} = \begin{bmatrix} L_{N_1,s_2,\dots,s_m,j} \\ M_{s_1+1,s_2,\dots,s_m,j} \\ \vdots \\ M_{s_1+N_1-1,s_2,\dots,s_m,j} \\ Q_{N_1,s_2,\dots,s_m,j} \\ \vdots \\ L_{s_1,s_2,\dots,N_m,j} \\ M_{s_1,s_2,\dots,s_m+1,j} \\ \vdots \\ M_{s_1,s_2,\dots,s_{m-1},N_m-1,j} \\ Q_{s_1,s_2,\dots,N_m,j} \end{bmatrix} \qquad \dots \qquad (2.8)$$

where, $\alpha_i = -a_i(x_1^{s_1}, x_2^{s_2}, \dots, x_m^{s_m})r_i^2$, $\beta = 2[1 + \sum_{i=1}^{m} a_i(x_1^{s_1}, x_2^{s_2}, \dots, x_m^{s_m})r_i^2]$*. This linear system can be solved by using any suitable method to find the numerical solutions* $u_{s_1,s_2,\dots,s_m,j}$, $s_i = 0,1,\dots,N_i$, $i = 1,2,\dots,m$, $j = 1,2,\dots,M$, *of the nonlocal problem given by equations (1.1)-(1.5).*

3- NUMERIACAL EXAMPLES:

In this section we present two examples that can be solve by using the above method.

Example (1):

Consider the one-dimensional wave equation:

$$u_{tt} - u_{xx} = -2, \qquad x \in (0,1)\,, t \in [0,1] \qquad \dots \qquad (3.1)$$

together with the initial conditions

$$u(x,0) = x^2\,, \qquad x \in [0,1]\,, \qquad \dots \qquad (3.2)$$

$$u_t(x,0) = 1\,, \qquad x \in [0,1]\,, \qquad \dots \qquad (3.3)$$

and the nonlocal non-linear integral boundary conditions

$$u(0,t) = \int_0^1 x\,u^2(x,t)\,dx + \frac{6t - (1+t)^3 + t^3}{6}\,, \qquad t \in [0,1] \qquad \dots \qquad (3.4)$$

$$u(1,t) = \int_0^1 \frac{x}{2} u^3(x,t)\,dx + \frac{16(1+t) - (1+t)^4 + t^4}{16}\,, \quad t \in [0,1] \qquad \dots \qquad (3.5)$$

This example is constructed such that the exact solution is

$$u(x,t) = x^2 + t.$$

Let N= M=4, then we get h=k=0.25 and r=1. From equation (3.2) one get the solution at j=0, $u_{0,0} = 0$, $u_{1,0} = 0.0625$, $u_{2,0} = 0.2500$, $u_{3,0} = 0.5625$, $u_{4,0} = 1$.
From equation (2.2) and by using equation (3.3), one can get
$u_{0,1} = 0.2500$, $u_{1,1} = 0.3125$, $u_{2,1} = 0.5000$, $u_{3,1} = 0.8125$, $u_{4,1} = 1.2500$
In this case, equation (2.8) takes the form

$$\begin{bmatrix} -3 & 0.1563 & 0.2500 & 1.2188 & 0.6250 \\ -1 & 4 & -1 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & -1 & 4 & -1 \\ 0 & 0.0366 & 0.0938 & 0.7427 & -2.4141 \end{bmatrix} \begin{bmatrix} u_{0,2} \\ u_{1,2} \\ u_{2,2} \\ u_{3,2} \\ u_{4,2} \end{bmatrix} = \begin{bmatrix} 1.0977 \\ 1 \\ 1.375 \\ 2 \\ -2.6331 \end{bmatrix}.$$

This system can be easily solved by using any suitable method to find the numerical solution at j=2, to get the results that are tabulated down in table (1).

Table (1), represent the solutions of example (1) for h=k=0.25.

x_i	$u(x_i,t_0)$	$u_{i,0}$	$u(x_i,t_1)$	$u_{i,1}$	$u(x_i,t_2)$	$u_{i,2}$
0	0	0	0.2500	0.2500	0.5000	0.4492
0.25	0.0625	0.0625	0.3125	0.3125	0.5625	0.5478
0.5	0.2500	0.2500	0.5000	0.5000	0.7500	0.7421
0.75	0.5625	0.5625	0.8125	0.8125	1.0625	1.0455
1	1	1	1.2500	1.2500	1.5000	1.4398

Now if we take N=20 and M=40, then we get h=0.05, k=0.025, r=0.5. The results when $x \in [0,0.5]$ that are tabulated down in Table (2).

Table (2), represents the solutions of example (1) for h=0.05, k=0.025.

x_i	$u(x_i,t_0)$	$u_{i,0}$	$u(x_i,t_1)$	$u_{i,1}$	$u(x_i,t_2)$	$u_{i,2}$
0.0000	0.0000	0.0000	0.0250	0.0250	0.0500	0.0497
0.0500	0.0025	0.0025	0.0275	0.0275	0. 0525	0. 0525
0.1000	0.0100	0.0100	0.0350	0.0350	0. 0600	0. 0600
0.1500	0.0225	0.0225	0.0475	0.0475	0.0725	0.0725
0.2000	0.0400	0.0400	0.0650	0.0650	0.0900	0.0900
0.2500	0.0625	0.0625	0.0875	0.0875	0.1125	0.1125
0.3000	0.0900	0.0900	0.1150	0.1150	0.1400	0.1400
0.3500	0.1225	0.1225	0.1475	0.1475	0.1725	0.1725
0.4000	0.1600	0.1600	0.1850	0.1850	0.2100	0.2100
0.4500	0.2025	0.2025	0.2275	0.2275	0.3525	0.3525
0.5000	0.2500	0.2500	0.3275	0.3275	0.3000	0.3000

In table (3) the absolute errors at specific points in the domain D={$(x,t)|x \in (0,1), t \in (0,1)$} for different values of h and k are presented

Table (3), represents the absolute errors at specific points for different values for h and k for example (1)

k	h	r	x_i	absolute errors
0.25	0.25	1	0.25 0.75 1	0.0147 0.017 0.0602
0.05	0.01	5	0.25 0.75 1	1.1532×10^{-6} 9.837×10^{-7} 0.0011
0.025	0.005	5	0.25 0.75 1	2.4293×10^{-10} 1.9455×10^{-10} 2.5739×10^{-4}

Example (2):

Consider the one-dimensional wave equation

$$u_{tt} - u_{xx} = 0\,, \qquad x \in (0,1)\,, t \in [0,1] \qquad \ldots \qquad (3.6)$$

together with the initial conditions

$$u(x,0) = e^x\,, \qquad x \in [0,1] \qquad \ldots \qquad (3.7)$$

$$u_t(x,0) = e^x\,, \qquad x \in [0,1] \qquad \ldots \qquad (3.8)$$

and the nonlocal non-linear boundary conditions

$$u(0,t) = \int_0^1 x\,u^2(x,t)\,dx + \frac{4e^t - e^{2t} - e^{2t+2}}{4}\,, \qquad t \in [0,1] \qquad \ldots \qquad (3.9)$$

$$u(1,t) = \int_0^1 x\,u^2(x,t)\,dx + \frac{4e^{t+1} - e^{2t} - e^{2t+2}}{4}\,, \qquad t \in [0,1] \qquad \ldots \qquad (3.10)$$

This example is constructed such that the exact solution is

$$u(x,t) = e^{x+t}$$

We take N=20 and M=40, then we get h=0.05, k=0.025, r=0.5. The results when x∈[0,0.5] that are tabulated down in Table (4).

Table (4), represents the solutions of example (2) for h=0.05, k=0.025.

x_i	$u(x_i,t_0)$	$u_{i,0}$	$u(x_i,t_1)$	$u_{i,1}$	$u(x_i,t_2)$	$u_{i,2}$
0.0000	1	1	1.0253	1.0250	1.1052	1.0468
0.0500	1.0513	1.0513	1.0779	1.0776	1.1052	1.1041
0.1000	1.1052	1.1052	1.1331	1.1328	1.1618	1.1611
0.1500	1.1618	1.1618	1.1912	1.1909	1.2214	1.2207
0.2000	1.2214	1.2214	1.2523	1.2519	1.2840	1.2833
0.2500	1.2840	1.2840	1.3165	1.3161	1.3499	1.3491
0.3000	1.3499	1.3499	1.3840	1.3836	1.4191	1.4182
0.3500	1.4191	1.4191	1.4550	1.4545	1.4918	1.4909
0.4000	1.4918	1.4918	1.5296	1.5291	1.5683	1.5674
0.4500	1.5683	1.5683	1.6080	1.6075	1.6487	1.6477
0.5000	1.6487	1.6487	1.6905	1.6899	1.7333	1.7322

In table (5) the absolute errors at specific points in the domain $D=\{(x,t)|x \in (0,1), t \in (0,1)\}$ for different values of h and k are presented

Table (5) represents the absolute errors at specific points for different values for h and k for example (2)

k	*h*	*r*	x_i	*absolute errors*
0.025	0.05	0.5	0.05 0.50 1	0.001 0.001 0.0045
0.0025	0.01	0.25	0.05 0.50 1	6.5677×10^{-6} 1.0300×10^{-5} 3.9969×10^{-5}
0.001	0.005	0.2	0.05 0.50 1	1.0511×10^{-6} 1.6484×10^{-6} 6.3386×10^{-6}

Conclusion:

In this paper, we solved the initial value problem that consists of the multi-dimensional hyperbolic equation with 2m nonlocal non-linear integral boundary conditions. This method is based on Crank-Niklson finite difference scheme and Taylor's expansion. It's an easy way for transforming the nonlinear equations to linear system that can be solved easily by using any suitable method. Also, this numerical method gives acceptable results, and we found through the numerical results that the numerical solution was closed to the exact ones.

REFERENCES:

[1] A. Ashyralyev and N. Aggez," Finite difference method for hyperbolic equation with the nonlocal integral conditions", Discrete Dynamics in Nature and Society, vol. 2011, Article ID 562385, (2011), 1–15.

[2] A. Borhanifar, A. H. Pour and M. M. Kabir, "A numerical method for solution of the heat equation with nonlocal nonlinear condition", World Applied Sciences Journal, vol.13, (2011), 2405-2409.

[3] A. Saadatmandi and M. Dehghan, "Numerical solution of the one-dimensional wave equation with integral condition", Wiley Inter Science, vol. 23, (2006), 282–292.

[4] G. D. Smith, "Numerical solution of partial differential equations", Clarendon press, Oxford (1978).

[5] L. P. Ngoc and T. M. Thuyet, "On a nonlocal nonlinear wave equation with a nonlocal boundary condition", Acta Mathematical Vietnamica, vol. 36, (2011), 345–374.

[6] M. Dehghan and A. Saadatmandi, "Variational iteration method for solving the wave equation subject to an integral conservation condition", Chaos, Solitons and Fractals, vol. 41, (2009), 1448–1453.

[7] S. A. Beilin, "Existence of solution for one-dimensional wave equations with non-local conditions", Electron J. Differential Equations, vol. 76, (2001), 1-8.

[8] S. Mesloub and A. Bouziani, "On a class of singular hyperbolic equation with a weighted integral condition", Inter. J. Math. and Math. Sciences, vol. 22, (1999), 511–519.

[9] S. Nemati and Y. Ordokhani, "A second kind chebyshev polynomial approach for the wave equation subject to an integral conservation condition", Journal of Information and Computing Science, vol. 7, (2012), 163–171.

[10] W. T. Ang, "A numerical method for wave equation subject to a nonlocal conservation condition", Applied Numerical Mathematics, vol. 56, (2006), 1054–1060.

[11] X. Li, "Numerical solution of an initial-boundary value problem with nonlocal condition for the wave equation", Mathematical Science, vol. 2, (2008), 281–292.