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ON THEMEASURABLE FAMILIES OF Ŝ- BANACH LATTICE

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ABSTRACT:

The aim of this paper is to study a measurable families of$\hat{S}$- Banach lattice and their decomposition of a separable Banach$\hat{S}$- vector lattice into measurable field of Banach lattice . Moreover there exists a Freudenthal unit in $\hat{X}$ if and only if it exists in X_τ .

المستخلص :-

يهدف هذا البحث الى دراسة الفصائل القابلة للقياس للحزم المتجه البناخية $\hat{S}$-
وتحللها من الحزم البناخية المتجه المنفصلة - $\hat{S}$ الى الحقول القياسية من تلك الحزم .أضافة الى ذلك وجود الوحدة الفريدنتالية في $\hat{X}$ اذا وفقط كانت في X_τ .

1.Introduction :-

In this paper , we introduce the important definitions and some information about the measurable field of metric space and of Boolean algebras that we needed it in this work . We get the main results about the decomposition of separable Banach$\hat{S}$- vector lattice into a measurable field of Banach lattice .

2.The preliminaries

We recall the basic definitions and information which are needed in our work .

2.1 Definition: [11]

A mapping $\rho : X \times X \longrightarrow \hat{S}$ is called a ***metric*** on a set X with values in $\hat{S}$ if

1. $\rho(x,y) \geq o$ for any $x,y \in X$ and $\rho(x,y) = 0$ if and only if $x=y$.
2. $\rho(x,y) = \rho(y,x)$ for any $x ,y \in X$.
3. $\rho(x,y) \leq \rho(x,z) + \rho(z,y)$ for any $x,y,z \in X$.

2.2 Remark:[8]

Suppose that T is the interval [0,1] , let A be the σ- algebra of the Lebesgue measurable subset of T , and P be the Lebesgue measure on T .

We shall denote by $\hat{S}$ the ring of all real measurable functions on *[0,1]*.

2.3 Definition: [11]

The space *(X, ρ)* is called ***separable*** , if there exists a countable subset $M \subset X$ such that for any $x \in X$, there is $\{z_n\}_{n=1}^{\infty} \subset M$ for which $\rho(x, z_n) \xrightarrow{(t)} 0$.
Suppose that (X_τ, ρ_τ) be a complete separable metric space defined for P-almost every $\tau \in T$.

2.4 Definition: [11]

A ***measurable field of metric space*** is a pair $\{X, \{X_\tau\}_{\tau\in T}\}$ where X is a totality of functions $x: \tau \to x(\tau) \in X_\tau$ for P-almost every $\tau \in T$ such that :

1. There exists a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ such that $\{x_n(\tau)\}_{n=1}^{\infty}$ is dense in (X_τ, ρ_τ) for P-almot every $\tau \in T$.
2. The function $\tau \to \rho_\tau(x(\tau), y(\tau))$ is measurable on (T,$\mathcal{A}$,P) for all $x,y \in X$.
3. If $\{y_n\}_{n=1}^{\infty} \subset X, y : \tau \to y(\tau) \in X_\tau$ for P-almost every $\tau \in T$ and $\rho_\tau(y_n(\tau), y(\tau)) \longrightarrow 0$ $as\ n \longrightarrow \infty$ for P-almost every $\tau \in T$, then $y \in X$.

2.5 Remark:[9,10]

Let $\{X, \{X_\tau\}_{\tau\in T}\}$ be a measurable field of metric space and let $\hat{X}$ be the set of all classes for P-almost everywhere coinciding elements from X.

For any $\hat{x}, \hat{y} \in \hat{X}$, we denote by $\hat{\rho}(\hat{x}, \hat{y})$ the class from $\hat{S}$containing the function $\rho_\tau(x(\tau), y(\tau))$, where$x(\tau)$ and $y(\tau)$ are representatives of the classes $\hat{x}$ and $\hat{y}$, respectively . Then $(\hat{X}, \hat{\rho})$ is a complete separable space with a $\hat{S}$ - valued metric .

2.6 Definition: [11]

A measurable field of metric space is said to be ***saturated*** if the following two conditions are satisfied :

1. $y: \tau \to y(\tau) \in X_\tau$ for P-almost every $\tau \in T$.
2. The function $\tau \to \rho_\tau(x(\tau), y(\tau))$ is measurable on (T, $\mathcal{A}$,P) for any $x \in X$, imply $y \in X$.

2.7 Remark: [11]

Suppose that in the definition of saturated measurable field of metric space all X_τ coincide and are equal to (Z, ρ_τ) .
We shall denoted by $S(T, Z)$ the set of measurable mapping from $(T, \mathcal{A}, P)$ into Z and by $\hat{S}(T, Z)$the set of the classes of P-almost everywhere coinciding function from $S(T, Z)$.
A $\hat{S}$- valued metric $\hat{\rho}(\hat{x}, \hat{y}) = \rho_\tau(\widehat{x(\tau), y(\tau)})$ is naturally defined on $\hat{S}(T, Z)$. In [8] it is shown that $\hat{S}(T, Z)$is $\hat{S}$– isometric to$\hat{X}$.

2.8 Remark: [11]

Let (U, d) be a universal metric space of Uryson (one can consider that U=C[0,1] with the uniform metric .
In [9] it is shown that any complete separable space X with an $\hat{S}$-valued metric is $\hat{S}$-isometric to some closed subset of $\hat{S}(T, Z)$.
Now , let $\{X, \{X_\tau\}_{\tau\in T}\}$ be a measurable field of metric space , and let A_τ be closed subset of X_τ for P-almost every $\tau \in T$.

2.9 Definition: [9]

A ***measurable field of closed sets*** is a pair $\{A, \{A_\tau\}_{\tau\in T}\}$ where $A \in X$ and

1. If $x \in A$, then $x(\tau) \in A_\tau$ for P-almost every $\tau \in T$.
2. There exists $\{x_n\}_{n=1}^{\infty} \subset A$ such that $\{x_n(\tau)\}_{n=1}^{\infty}$is dense in A_τ for P-almost every $\tau \in T$.
3. If $\{y_n\}_{n=1}^{\infty} \subset A$, $y \in X$ and $\rho_\tau\big(y_n(\tau), y(\tau)\big) \to 0$ as$n \to \infty$ for$P$- almost every $\tau \in T$, then $y \in A$.

2.10 Remark:[10]

We say that X be a bimodule over $\hat{S} = [0,1]$, i.e.X is abelian group with respect to addition operation (+) and right and left multiplication by element from $\hat{S}$are defined on X having the properties:

1.$\lambda(x+y) = \lambda x + \lambda y$, $(x+y)\lambda = x\lambda + y\lambda$
2.$(\lambda+\mu)x = \lambda x + \mu x$, $x(\lambda+\mu) = x\lambda + x\mu$
3.$\lambda(\mu x) = (\lambda\mu)x$, $(x\lambda)\mu = x(\lambda\mu)$
4.$\hat{\imath} . x = x . \hat{\imath}$,for all $x, y \in X, \lambda , \mu \in \hat{S}$

2.11 Definition: [2, 4, 7]

The collection B of Borel sets of a topological space X is the smallest σ- algebra containing all open sets of X . That is , in addition to containing open sets, B must be closed under complement and countable intersections (and , thus , is also closed under countable unions) .

2.12 Definition: [12, 13]

A ***Borel mapping*** is a mapping such that the inverse image of every Borel set is Borel . It is called Borel function .

2.13 Definition:[3]

A set C $\subseteq R^n$ is called ***convex*** if for all x, y ∈ C, we have **λx+(1−λ)y ∈ C** for all $\lambda \in$ [0, 1].

2.14 Definition:[5]

A ***Banach space*** is a normed linear space in which every Cauchy sequence is convergent.

We cite now example of ***S^*-vector lattice***s

2.15 Example:[10]

Let P be the Lebesgue measure on $\nabla(S^*), L_1(\nabla(S^*), P)$ be the Banach space of all P- integrable functions on $([0,1], p), \nabla_1$ be a σ- subalgebra of $\nabla(S^*)$ isomorphic to $\nabla(S^*)$,
$E: L_1(\nabla(S^*), P) \to L_1(\nabla_1, m)$ be the conditional expectation .

Then $\mu(e) = E(e)$ is a strictly positive measure on $\nabla(S^*)$ with values in the σ- complete sublattice S_1 of all functions from S^*which are measurable with respect to ∇_1 (it is clear that S_1 can be identified with S^* , since ∇_1 is isomorphic to $\nabla(S^*)$.
In addition , $\mu(eg) = e\mu(g)$ for any e $\in \nabla_1$, $g \in \nabla(S^*)$.

It is clear that S^* is a normal S_1 - module .

Denote by $\hat{\mu}$ the integral constructed with respect to the measure μ . For every number $P \geq 1$ we set

$$L_P(\nabla(S^*), \mu) = \{x \in S^* : \hat{\mu}(|x|^P) \in S\}$$

And

$$\|x\|_P = (\hat{\mu}\,|x|^P)^{\frac{1}{P}}$$

It is shown in [3 ,4] that $L_P(\nabla(S^*), \mu, \|.\|_P)$ is a Banach S_1- vector lattice .

3. The Main Results :

This section is devoted to the main results concerning the measurable families of Banach lattice and decomposition of separable Banach$\hat{S}$-vector lattice of ordinary Banach lattices .

We begin with the useful information that used in our paper .

3.1 Definition : [10]

A measurable field X of Banach spaces $(X_\tau, \|.\|_\tau)$, $\tau \epsilon$ T = ([0,1]) , P) is a set of functions $x: \tau \to x(\tau) \in X_\tau$ defined p- almost everywhere on T and such that

1. The function $\tau \to \|x_\tau\|_\tau$ is measurable on T for all $x \in X$;
2. If x,y ϵX , $\propto, \beta \in \hat{S}$,then the function $\propto(\tau)x(\tau) + \beta(\tau)y(\tau)$ also belongs to X .
3. There exists a sequence $\{x_n\} \subset X$ such that the set $\{x_n(\tau)\}$ is dense in X_τ for p – almost every $\tau \epsilon$T .
4. If $\{y_n\} \subset X$ and $y: \tau \to y(\tau) \in X_\tau$ is a function and $\|y_n(\tau) - y(\tau)\|_\tau \to 0$ as $n \to \infty$ for p – almost every $\tau \epsilon$T , then $y \in X$.

It is clear that a measurable field Banach space X is also a measurable field metric space (X_τ, d_τ) , $\tau \in T$, where

$$d_\tau\big(X(\tau), y(\tau)\big) = \|x(\tau) - y(\tau)\|_\tau,\ \ x(\tau), y(\tau) \in X_\tau\ .$$

3.2Note:[10]

A measurable field Banach space $(X_\tau, \|.\|_\tau)$ with property (1,2) of definition (2.1) is said to be ***saturated***.

In the sequel the record $\{X, \{X_\tau\}_{\tau \in T}\}$ will mean that X is a measurable field Banach space $(X_\tau, \|.\|_\tau)$.

Now , for any measurable field Banach space X denoted by $\hat{X}$ the set of all classes $\hat{x}$ of p- almost everywhere equal functions $x: \tau \to x(\tau)$from X .

The pointwise operations :-

$$x + y: \tau \rightarrow x(\tau) + y(\tau),$$

$$\alpha x: \tau \rightarrow \alpha(\tau)x(\tau),$$

$$x\alpha: \tau \rightarrow x(\tau)\alpha(\tau),$$

$x, y \in X$, $\alpha \in \hat{S}$ and the function $\|.\|: \tau \rightarrow \|x(\tau)\|_\tau$, $x \in X$, determine on $\hat{X}$ a structure of a bimodule over $\hat{S}$ with the $\hat{S}$ – valued norm $\|.\|_{\hat{x}}$ .In addition , $(\hat{X}, \|.\|_{\hat{x}})$ is a separable bimodule , i.e. there exists a countable set $\{\hat{X}_n\} \subset \hat{X}$such that , for any

$\hat{x} \in \hat{X}$ there exists a subsequence $\{ \hat{X}_{n_k}\}$ for which $\left\|\hat{x} - \hat{x}_{n_k}\right\|_{\hat{x}} \overset{(o)}{\rightarrow} 0$.

3.3 Remark :[12]

Suppose $\hat{X}$ be a separable Banach $\hat{S}$ – module with partial order in which represents a lattice coincide with the norm in $\hat{X}$.
From $|\hat{x}| \leq |\hat{y}|$imply that $\|\hat{x}\| \leq \|\hat{y}\|$ for each $\hat{x}, \hat{y} \in \hat{X}$.

3.4 Definition :[6]

A vector lattice X is ***Archimedean*** whenever the relation $0 \leq nx \leq y$, $n \in N$, imply that $x = 0$.

3.5 Proposition:

Let $\hat{X}$ be the set of all classes for P- almost everywhere coinciding elements from X ,The Archimedean condition is satisfied in $\hat{X}$.

Proof :-

Let $\hat{x} \geq 0$ and $n\hat{x} \leq \hat{y}$, $n = \overline{1, \infty}$

Then$\|n\hat{x}\| \leq \|\hat{y}\|$, or $\|\hat{x}\| \leq \frac{1}{n}\|\hat{y}\| \rightarrow 0 \;\; as \;\; n \rightarrow \infty$

i.e.$\hat{x} = 0$

Therefore

$$|x \vee z - y \vee z| \leq |x - y| \;\; ....(1)$$

□

3.6 Definition:[14]

Let X be a vector space over the real field R. A nonempty convex subset P of X is called a ***cone*** if λ **P⊂P** for all $\lambda \geq$**0.**

3.7 Theorem:

Let $\{X, \{X_\tau\}_{\tau \in T}\}$ be a measurable field Banach space generating $\hat{X}$,then for almost everywhere $\tau \in T$ we can define a Banach Lattice on X_τ such that , the order in X_τ for almost everywhere $\tau \in T$ induces the order in $\hat{X}$.

Proof :

let $\hat{G}$ be a countable dense subset of $\hat{K}$ such that :

1- $\hat{G} + \hat{G} \leq \hat{G}$.
2- $r\hat{G} \leq \hat{G}$ for all rational numbers $r \geq 0$.
3- $\hat{G} \cap (-\hat{G}) = 0$.

Now , suppose that $\{G_\tau\}_{\tau \in T}$ generated$\hat{G}$, such that , for almost everywhere countable dense subset G_τ of $\hat{K}_\tau$.
Since $\hat{G}$ and G_τ are countable for almost everywhere $\tau \in T$,then the properties (1 - 3) are satisfied in G_τ for almost everywhere $\tau \in T$.
Thus , K_τ is a cone for each $\tau \in T$.
In fact , we need to prove $\propto K_\tau \subseteq K_\tau$for each $\propto \in R_+$but , that is clear when we use the property (2) and consider the fact that K_τ is closed .
Thus , for almost everywhere $\tau \in T$, we can define a partial order induced the same order in $\hat{X}$.
Now , we need to prove that $X_\tau$is a Banach lattice for that order, i.e. there exist $X_\tau \vee y_\tau$ for any$x_\tau, y_\tau \in X_\tau$.
From $|x_\tau| \leq |y_\tau|$ we have$\|x_\tau\|_\tau \leq \|y_\tau\|_\tau$.
Let $\hat{\Gamma}$ be a countably densely subset of $\hat{X}$ such that $\hat{\Gamma} \vee \hat{\Gamma} \subseteq \hat{\Gamma}$ and let $\{\Gamma_\tau\}_{\tau \in T}$ be a family of a countable dense set in X_τ, for almost everywhere $\tau \in T$ generating $\hat{\Gamma}$.

Therefore almost everywhere $\tau \in T$ and for all $x_\tau, y_\tau \in \Gamma_\tau$, we can define $x_\tau \vee y_\tau$ such that $\Gamma_\tau \vee \Gamma_\tau \subseteq \Gamma_\tau$ and therefore the relation (1) and the order conditions are satisfied on Γ_τ .

Let$y_\tau \in \Gamma_\tau$,$x_\tau \in X_\tau$and $\{x_n^\tau\}_{n=1}^\infty \subseteq \Gamma_\tau$, $x_{n=1}^\tau \to x_\tau \ as \ n \to \infty$

Then

$$|x_n^\tau \vee y_\tau - x_m^\tau \vee y_\tau| \leq |x_n^\tau - x_m^t|$$

i.e.$\{x_n^\tau \vee y_\tau\}_{n=1}^\infty$ is a fundamental sequence and it is easy to prove that is convergent to $x_\tau \vee y_\tau$.

Thus , the operations will be defined for all pair x_τ, y_τ such that $x_\tau \in X_\tau$, $y_\tau \in \Gamma_\tau$,
In the similar way this operation can be extended to all X_τ , and there is no difficult to show that the order condition can be extended with Γ_τ on X_τ. □

3.8 Definition :[1]

An element from X_+ is called a ***Freudenthal*** unit and denoted by î , if it follows from $x \in X$, $x \wedge \hat{\imath} = 0$, that $x = 0$.

3.9 Proposition :

There exists a Freudenthal unit in $\hat{X}$ if and only if it exists in X_τ almost everywhere $\tau \in T$.

Proof :

Let $\hat{e} \in \hat{X}$ be a Freudenthal unit . Then from the equality $\hat{e} \wedge \hat{x} = 0$, we deduce that $\hat{x} = 0$. Moreover for all $\hat{x} \in \hat{G}$, we have $e(\tau) \wedge x_\tau = 0$, implay that $x_\tau = 0$ for all $x_\tau \in G_\tau$.
Since G_τ is dense in K_τ , then , we get ,that is true for all $x_\tau \geq 0$. It means that $e(\tau)$ is a Freudenthal unit in X_τ for almost everywhere $\tau \in T$.

Conversely , let $\{X_\tau\}_{\tau \in T} \subseteq S^*(T, U)$, where U is the Uneversal space, $\{X_n(\tau)\}_{n=1}^{\infty} = G_\tau$ for almost everywhere $\tau \in T$ and $x_n(\tau)$ be a Borel representation of elements in $\hat{G}$, Then the set

$$A = \bigcap_{n=1}^{\infty} \{ (e, \tau) : \|x_n(\tau) \wedge e\|_\tau = 0 \text{ and } \|x_n(\tau)\|_\tau = 0\}$$

is a Borelset, so since hypothesis

$$A_\tau = \{e : e \text{ is a freudenthal unit in } X_\tau\} = \{e : (e, \tau) \in A\} \neq \emptyset$$

for almost everywhere $\tau \in T$, then , there exist $e(\tau)$ such that $(e(\tau), \tau) \in A$ for almost everywhere $\tau \in T$. It is clear that , $e = e(\tau)^n$ is a freudenthal unit in $\hat{X}$. □

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