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Fuzzy Compact and Coercive Mappings

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Abstract

In this paper we introduce the concept of fuzzy compact and fuzzy coercive mappings in fuzzy topological spaces. Several characterizations and some interesting properties of these mappings are discussed. Also, we explain the relation between fuzzy compact mapping and fuzzy coercive mapping.

الخلاصة

قدمنا في هذا البحث مفهومين التطبيق المتراص الضبابي والتطبيق الاضطراري الضبابي في الفضاء التبولوجي الضبابي. العديد من الصفات والخصائص المهمة قد نوقشت. كذلك وضحنا العلاقة بين التطبيق المتراص الضبابي والتطبيق الاضطراري الضبابي

Introduction

The concept of fuzzy set and fuzzy set operations were first introduced by Zadeh [11] in 1965. Several other authors applied fuzzy sets to various branches of mathematics. One of these objects is a topological space. At the first time in 1968, Chang in [2] formulated the natural definition of fuzzy topology on a set and investigated how some of the basic ideas and theorems of point-set topology behave in generalized setting. Chang's definition on a fuzzy topology is very similar to the general topology by exchange all subsets of a universal set by fuzzy subset but this definition is not investigate some properties if we comparison with the general topology. For example in a general topology, any constant mapping is continuous while this idea is not true in Chang's definition on fuzzy topology. One of the very important concepts in fuzzy topology is the concept of mapping. There are several types of mapping. The purpose of this paper is to introduced and study the concept of fuzzy compact and fuzzy coercive mappings in fuzzy setting and explain the relation between them.

§ 1. Preliminaries

First, we present some fundamental definitions and propositions which are needed in the next sections.

Definition 1.1 [11]

Let X be a non-empty set and let I be the unit closed interval, i.e., $I = [0,1]$. A fuzzy set A in X is a function from X into the unit closed interval I, (i.e., $A: X \longrightarrow I$ be a function). $A(x)$ is interpreted as the degree of membership of element x in a fuzzy set A for each $x \in X$. A fuzzy set A in X can be represented by the set of pairs: A={($x, A(x)$): $x \in X$}. The family of all fuzzy sets in X isdenotedbyI^X.

Definition 1.2 [2]

Let A and B be any two fuzzy sets in X. Then we put:

(a) $A = B$ if and only if $A(x) = B(x)$, for all $x \in X$.
(b) $A \leq B$ if and only if $A(x) \leq B(x)$, for all $x \in X$.
(c) $Z = A \vee B$ if and only if $Z(x) = \max\{ A(x), B(x)\}$, for all $x \in X$.
(d) $D = A \wedge B$ if and only if $D(x) = \min\{ A(x), B(x)\}$, for all $x \in X$.
(e) $E = A^c$ (the complement of A) if and only if $E(x) = 1 - A(x)$, for all $x \in X$.

Remark 1.3

Let X be a non-empty set. Then:
(a) For any fuzzy sets A and B in X, $A \leq B$ if and only if $B^c \leq A^c$.
(b) $(0_X)^c = 1_X$, $(1_X)^c = 0_X$.

Theorem 1.4 [9]

Let A, B and C be fuzzy sets in a set X, the following statements are holds:
(a) (Commutatively): $A \vee B = B \vee A$, $A \wedge B = B \wedge A$.
(b) (Associatively): $(A \vee B) \vee C = A \vee (B \vee C)$, $(A \wedge B) \wedge C = A \wedge (B \wedge C)$.
(c) (Idempotency): $A \vee A = A$, $A \wedge A = A$.
(d) (Destributivity): $A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$, $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$.
(e) (Absorption): $A \vee 0_X = A$, $A \wedge 1_X = A$.
(f) (De Morgan's law): $(A \vee B)^c = A^c \wedge B^c$, $(A \wedge B)^c = A^c \vee B^c$.
(g) (Involution): $(A^c)^c = A$.
(h) (Equivalence formula): $(A^c \vee B) \wedge (A \vee B^c) = (A^c \wedge B^c) \vee (A \wedge B)$.
(i)(Symmetrical difference formula):$(A^c \wedge B) \vee (A \wedge B^c)$ =$(A^c \vee B^c) \wedge (A \vee B)$.
(j) (Difference): $(A - B) = A \wedge B^c$.

Definition 1.5 [1]

Let X and Y be two non-empty sets, $f: X \to Y$ be a mapping. For a fuzzy set B in Y with membership function $B(y)$. Then the inverse image of B under f, written as $f^{-1}(B)$, is a fuzzy set in X whose membership function is defined by:
$f^{-1}(B)(x) = B(f(x))$ for all $x \in X$. (i.e., $f^{-1}(B) = B \circ f$).

Conversely, let A be a fuzzy set in X with membership function $A(x)$. The image of A under f, written as $f(A)$, is a fuzzy set in Y whose membership function is given by:

$$f(A)(y) = \begin{cases} \sup_{z \in f^{-1}(y)} A(z), & if\ f^{-1}(y) \neq \emptyset \\ 0, & if\ f^{-1}(y) = \emptyset \end{cases}$$

for all $y \in Y$, where $f^{-1}(y) = \{x: f(x) = y\}$.

Proposition 1.6 [2]

Let X, Y and Z are non-empty sets and $f: X \to Y$, $g: Y \to Z$ be mappings, then the following statements are holds:
(a) $f^{-1}(B^c) = (f^{-1}(B))^c$, for any fuzzy set B in Y.
(b) $(f(A))^c \leq f(A^c)$, for any fuzzy set A in X.
(c) If $B_1 \leq B_2$, then $f^{-1}(B_1) \leq f^{-1}(B_2)$, where B_1 and B_2 are fuzzy sets in Y.
(d) If $A_1 \leq A_2$, then $f(A_1) \leq f(A_2)$, where A_1 and A_2 are fuzzy sets in X.
(e) For any fuzzy set A in X:
  (1) $A \leq f^{-1}(f(A))$;
  (2) $A = f^{-1}(f(A))$, if f is an injective mapping.
(f) For any fuzzy set B in Y:
  (1) $f(f^{-1}(B)) \leq B$;
  (2) $f\left(f^{-1}(B)\right) = B$, if f is a surjective mapping.
(g) If f is bijective, then $(\ f(A)\ )^c = \ f(A^c)$.
(h) If $g \circ f: X \to Z$ is the composition mapping between g and f, then:
  (1)$(g \circ f)(\ A) = g(f(A))$, for any fuzzy set A in X.
  (2) $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$, for any fuzzy set C in Z.

Definition 1.7 [6]

Let A be a fuzzy set in X. Then the set $\{\ x \in X: A(x) > 0\}$ is called the support of A and denoted by $S(A)$.

Definition 1.8 [3,8]

A fuzzy point (singleton) in X is a fuzzy set in X defined by:

$$x_r(y)=\begin{cases} r, & if \quad y=x, \\ 0, & otherwise, \end{cases}$$

For each $x \in X$, the single point x is called the support of x_r and $r \in (0,1]$ its value. We denote the class of all fuzzy points in X by $\mathrm{FP}(X)$. A fuzzy point x_r is said to be contained in a fuzzy set A or x_r belongs to A, i.e., $x_r \in A$ if and only if $r \leq A(x)$. Two fuzzy points x_r and y_s in X are said to be distinct if and only if their supports are distinct.

Theorem 1.9 [8]

Let A be a fuzzy set in X. Then A is the union of all its fuzzy points, i.e., $A = \bigvee_{x_r \in A} x_r$.

Definition 1.10 [7]

A fuzzy point x_r in X is said to be quasi-coincident (in short q-coincident) with a fuzzy set A in X, denoted by $x_r qA$ if and only if $r + A(x) > 1$.

Definition 1.11 [7]

A fuzzy set A in X is called q-coincident with a fuzzy set B in X, denoted by AqB if and only if $A(x) + B(x) > 1$, for some $x \in X$. Otherwise if $A(x) + B(x) \leq 1$, for every $x \in X$, then A is called not q-coincident with B and it is denoted by $A\tilde{q}B$.

Remark 1.12

Let $x_r \in \mathrm{FP}(X)$ and A be a fuzzy subset of a set X. Then:

(a) $x_r qA$ if and only if $r > A^c(x)$.

(b) $x_r \tilde{q}A$ if and only if $r \leq A^c(x)$.

Lemma 1.13 [3]

Let x_r be a fuzzy point in X and A be any fuzzy subset of X. Then $x_r \in A$ if and only if $x_r \tilde{q} A^c$.

Proposition 1.14 [3,4]

Let A, B and C be fuzzy sets in X, and x_r be a fuzzy point in X. Then:

(a) If $A \wedge B = 0_X$, then $A\tilde{q}B$.

(b) $A\tilde{q}B$ if and only if $A \leq B^c$.

(c) $A\tilde{q}A^c$.

(d) $A\tilde{q}B$ and $C \leq B$, then $A\tilde{q}C$.

(e) $A \leq B$ if and only if $x_r qB$ for every $x_r qA$.

Definition 1.15 [2]

Let X be a set. A fuzzy topology on X is a family T of fuzzy sets in X, which satisfies the following conditions:

(1) $0_X, 1_X \in T$.

(2) If $A, B \in T$, then $A \wedge B \in T$.

(3) If $A_i \in T$ for each $i \in I$, then $\bigvee_{i \in I} A_i \in T$.

T is called a fuzzy topology for X, and the pair (X, T) is called a fuzzy topological space (in short fts). Every member of T is called a fuzzy open set. A fuzzy set is fuzzy closed if and only if its complementisfuzzyopen.

Examples 1.16 [2,6]

Let X be a set. Then:

(1) $T = \{0_X, 1_X\}$ is called the fuzzy indiscrete topology on a set X.

(2) The collection of all fuzzy sets in X is called the fuzzy discrete topology.

(3) If X is an infinite set, then $T_\infty = \{A \in I^X : S(A^c)$ is a finite set$\} \vee \{0_X\}$ is called the fuzzy co-finite topology on X.

Theorem 1.17 [10]

Let (X, T) be an fts and let Y be a subset of X. Then the family $T_Y = \{ 1_Y \wedge A : A \in T\}$ is a fuzzy topology on Y.

Definition 1.18 [10]

The fuzzy topology T_Y is called the relative fuzzy topology on Y or the fuzzy topology on Y induced by the fuzzy topology T on X. Also, (Y, T_Y) is called a fuzzy subspace of (X, T).

Definition 1.19 [7]

A fuzzy set A in an fts (X, T) is called a fuzzy quasi-neighborhood (in short q-nhd) of a fuzzy point x_r in X if and only if there exist $B \in T$ such that $x_r q B$ and $B \leq A$. The family of all fuzzy q-nbds of x_r is called the system of fuzzy q-nbds of x_r and it is denoted by $N_{x_r}^{Q}$.

Proposition 1.20 [7]

Let (X, T) be an fts. Then for each x_r ∈FP(X), $N_{x_r}^{Q}$ satisfies the followings:

(1) x_r is a quasi-coincident with A, for each $A \in N_{x_r}^{Q}$.

(2) If $A, B \in N_{x_r}^{Q}$, then $A \wedge B \in N_{x_r}^{Q}$.

(3) If $A \in N_{x_r}^{Q}$ and $A \leq B$, then $B \in N_{x_r}^{Q}$.

Conversely, for each fuzzy point x_r in X, if $N_{x_r}^{Q}$ is the family of fuzzy sets in X satisfying the conditions (1), (2) and (3), then the family T of all fuzzy sets A such that $A \in N_{x_r}^{Q}$ whenever $x_r q A$ is a fuzzy topology for X.

Definition 1.21 [6]

An fts (X, T) is called fuzzy Hausdorff or fuzzy T_2-space if and only if for any pair of distinct fuzzy points x_r, y_s in X, there exists $A \in N_{x_r}^{Q}$, $B \in N_{y_s}^{Q}$ such that $A \wedge B = 0_X$.

Definition 1.22 [2]

Let A be a fuzzy set in an fts (X, T). Then the fuzzy interior of A is denoted by A° and defined as the union of all fuzzy open subsets of X which are contained in A, i.e., $A^{\circ} = \vee\{ B \in I^X : B \leq A, B \in T\}$.

Definition 1.23 [2]

Let A be a fuzzy set in an fts (X, T). Then the fuzzy closure of A is denoted by $\overline{A}$ and defined as the intersection of all fuzzy closed subsets of X which are containing A, i.e., $\overline{A} = \wedge\{B \in I^X : A \leq B, B^{c} \in T\}$.

Proposition 1.24 [9]

Let x_r be a fuzzy point in X and A be any fuzzy set in an fts (X, T), then $x_r \in \overline{A}$ if and only if BqA for every fuzzy q-nbd B of x_r.

Theorem 1.25 [6]

Let A and B be fuzzy sets in an fts (X, T). Then:

(a) If B is a fuzzy open set, then BqA if and only if $Bq\overline{A}$.

(b) A is fuzzy open if and only if for each $x_r q A$, there exists $B \in N_{x_r}^{Q}$, such that $B \leq A$.

Proposition 1.26 [9]

For fuzzy sets A and B in an fts (X, T), we have the following:

(a) A is fuzzy open (resp. fuzzy closed) if and only if $A^{\circ} = A$ (resp. $\overline{A} = A$).

(b) If $A \leq B$, then $A^{\circ} \leq B^{\circ}$ (resp. $\overline{A} \leq \overline{B}$).

(c) $A^{\circ\circ} = A^{\circ}$; $(\overline{\overline{A}} = \overline{A})$.

(d) $A^{\circ} \wedge B^{\circ} = (A \wedge B)^{\circ}$; $A^{\circ} \vee B^{\circ} \leq (A \vee B)^{\circ}$.

(e) $\overline{(A \wedge B)} \leq \overline{A} \wedge \overline{B}$; $\overline{A} \vee \overline{B} = \overline{(A \vee B)}$.

(f) $A^{c\circ} = (\overline{A})^{c}$; $\overline{A^{c}} = (A^{\circ})^{c}$.

Definition 1.27 [2]

Let X and Y be fts's. A mapping $f: X \longrightarrow Y$ is called fuzzy continuous if the inverse image of each fuzzy open set in Y is a fuzzy open set in X.

Example 1.28

Let (X,T) be an fts, then the identity mapping $id_X: (X,T) \rightarrow (X,T)$ is fuzzy continuous.

Remark 1.29 [5]

Some constant mapping from one fuzzy topological space to another fail to be fuzzy continuous.

Proposition 1.30 [5]

Let (X,T) be an fts. Then every constant mapping from (X,T) into another fts is fuzzy continuous if and only if T contains all constant fuzzy sets in X.

Proposition 1.31 [2,5]

A composition of fuzzy continuous mappings is fuzzy continuous.

Proposition 1.32 [2,8]

Let $f: X \rightarrow Y$ be a mapping, then f is fuzzy continuous if and only if the inverse image of each fuzzy closed set in Y is a fuzzy closed set in X.

§ 2. Fuzzy Compact Space

In this section, we introduce the definition of fuzzy compact space and give some propositions and theorems about it.

Definition 2.1

A set D is called a directed set if there is a relation $\geqslant$ on D satisfying:

(1) $n \geqslant n$ for each $n \in D$.

(2) If $n_1 \geqslant n_2$ and $n_2 \geqslant n_3$, then $n_1 \geqslant n_3$.

(3) If $n_1, n_2 \in D$, then there is some $n_3 \in D$, with $n_3 \geqslant n_1$ and $n_3 \geqslant n_2$.

Definition 2.2 [6]

A mapping $\mathfrak{F}: D \rightarrow \text{FP}(X)$ is called a fuzzy net in X and is denoted by $\{\mathcal{S}(n): n \in D\}$, where D is a directed set. If $\mathcal{S}(n) = x^n_{r_n}$ for each $n \in D$, where $x \in X$, $n \in D$ and $r_n \in (0,1]$, then the fuzzy net $\mathfrak{F}$ is denoted as $\{ x^n_{r_n}: n \in D\}$ or simply $\{x^n_{r_n}\}_{n\in D}$.

Definition 2.3 [6]

A fuzzy net $\mathfrak{L} = \{y^m_{r_m}: m \in E\}$ in X is called a fuzzy subnet of a fuzzy net $\mathfrak{F} = \{ x^n_{r_n}: n \in D\}$ if and only if there exists a mapping $f: E \rightarrow D$ such that :

(1) $\mathfrak{L} = \mathfrak{F} \circ f$, that is, $\mathfrak{L}_i = \mathfrak{F}_{f(i)}$ for each $i \in E$.

(2) For each $n \in D$ there exists some $m \in E$ such that, if $p \in E$ with $p \geq m$, then $f(p) \geq n$.

We shall denoted a fuzzy subnet of a fuzzy net $\{ x^n_{r_n}: n \in D\}$ by $\{x^{f(m)}_{r_{f(m)}}: m \in E\}$.

Definition 2.4

Let (X,T) be an fts and $\mathfrak{F} = \{x^n_{r_n}: n \in D\}$ be a fuzzy net in X and A be a fuzzy set in X. Then $\mathfrak{F}$ is said to be:

(1) Q-eventually with A if $\exists\, m \in D$, such that $x^n_{r_n} qA$, $\forall\, n \geq m$.

(2) Q-frequently with A if $\forall\, n \in D$, $\exists\, m \in D$, with $m \geq n$, such that $x^m_{r_m} qA$.

Definition 2.5 [6]

Let (X,T) be an fts, $x_r \in \text{FP}(X)$ and let $\mathfrak{F} = \{ x^n_{r_n}: n \in D\}$ be a fuzzy net in X. Then x_r is called a Q-adherent point of a fuzzy net $\mathfrak{F}$, denoted by $\mathfrak{F} \propto^Q x_r$, if for each $A \in N^Q_{x_r}$, $\mathfrak{F}$ is Q-frequently with A.

Definition 2.6 [6]

Let (X,T) be an fts, $\mathfrak{F} = \{x^n_{r_n}: n \in D\}$ be a fuzzy net in X and $x_r \in \text{FP}(X)$. Then $\mathfrak{F}$ is said to be Q-convergent to x_r and x_r is called the Q-limit point of $\mathfrak{F}$, denoted by $\mathfrak{F} \xrightarrow{Q} x_r$, if for each $A \in N^Q_{x_r}$, $\mathfrak{F}$ is Q-eventually with A.

Remark 2.7

It is clear from the definition that if $\mathfrak{F} \xrightarrow{Q} x_r$, then $\mathfrak{F} \propto^Q x_r$.

The converse of remark 2.7, need not be true as the following example.

Example 2.8

Let $X=\{a\}$, $A(a)=\frac{2}{3}$ be a fuzzy set in X, and $T=\{0_X, 1_X, A(a)\}$ be a fuzzy topology on X. Define a fuzzy net $\mathfrak{F}: (N, \geq) \to \mathrm{FP}(X)$ as follows:
$\mathfrak{F}=\{a^1_{\frac{1}{2}}, a^2_{\frac{1}{3}}, a^3_{\frac{1}{2}}, a^4_{\frac{1}{3}}, \ldots\ldots\ldots\}$, then $\mathfrak{F} \propto^Q a_{\frac{1}{2}}$, because $a^n_{\frac{1}{2}} qA$, when $n=1, 3, 5, \ldots$. But $a^n_{\frac{1}{3}} \tilde{q} A$, when $n=2,4,6,\ldots$. Thus $\mathfrak{F}$ is not Q-convergent to $a_{\frac{1}{2}}$

Proposition 2.9

Let (X,T) be an fts, $x_r \in \mathrm{FP}(X)$ and $\mathfrak{F}=\{x^n_{r_n}: n \in D\}$ be a fuzzy net in X. If $\mathfrak{F} \overset{Q}{\to} x_r$, then every fuzzy subnet of $\mathfrak{F}$ is also Q-convergent to x_r.

Proof :

Let $\mathfrak{F}=\{x^n_{r_n}: n \in D\}$ be a fuzzy net in X, such that $\mathfrak{F} \overset{Q}{\to} x_r$ and let $\mathfrak{L}=\{ y^m_{r_m}: m \in E\}$ be a fuzzy subnet of $\mathfrak{F}$. Let $B \in N^Q_{x_r}$. Then there exists $m \in D$, such that $x^n_{r_n} qB$, $\forall\, n \geq m$. By definition of $\mathfrak{L}$, for this m there exists $l \in E$, such that for each $p \in E$, $p \geq l$, we have $f(p) \geq m$, where $f: E \to D$. Now, $y^p_{r_p} = x^{f(p)}_{r_{f(p)}}$. Then $y^p_{r_p} qB$ for all $p \geq l$ and so $\mathfrak{L} \overset{Q}{\to} x_r$.

Proposition 2.10 [6]

Let (X,T) be an fts, A fuzzy point x_r is Q-adherent point of a fuzzy net $\{x^n_{r_n} : n \in D\}$, if and only if it has a fuzzy subnet which is Q-convergent to x_r.

Theorem 2.11

Let (X,T) be an fts, $x_r \in \mathrm{FP}(X)$ and A be a fuzzy set in X. Then $x_r \in \overline{A}$ if and only if there exists a fuzzy net $\mathfrak{F}=\{x^n_{r_n}: n \in D\}$in A, such that it's Q-convergent to x_r.

Proof :

$\Longrightarrow$ Let $x_r \in \overline{A}$, then BqA for each $B \in N^Q_{x_r}$. That is, there exists $r_B \in (0,1]$, such that $x^B_{r_B} \in A$ and $x^B_{r_B} qB$. Let $D = N^Q_{x_r}$. Then $(D, \geqslant)$ is a directed set under the inclusion relation so we can define a fuzzy net $\mathfrak{F}: N^Q_{x_r} \longrightarrow \mathrm{FP}(X)$ given by $\mathfrak{F} = x^B_{r_B}$, $\forall\, B \in N^Q_{x_r}$. Then $\mathfrak{F}$ is a fuzzy net in A. Now, let $P \in N^Q_{x_r}$, such that $P \geqslant B$ iff $P \leq B$, so there exists a fuzzy net $\{x^P_{r_P}\}_{P \in N^Q_{x_r}}$ such that $x^P_{r_P} qP$. Then $x^P_{r_P} qB$. So $\mathfrak{F} \overset{Q}{\to} x_r$.

$\Longleftarrow$ Let $\mathfrak{F}$ be a fuzzy net in A, such that $\mathfrak{F} \overset{Q}{\to} x_r$, and let $B \in N^Q_{x_r}$, then $\forall\, n \in D$, $\exists\, m \in D$ such that $x^n_{r_n} qB$ with $n \geq m$ and so BqA for each $B \in N^Q_{x_r}$. Thus by proposition 1.24, $x_r \in \overline{A}$.

Proposition 2.12

An fts X is fuzzy T_2-space if and only if every Q-converges fuzzy net $\mathfrak{F}$ on X has a unique Q-limit point.

Proof :

$\Longrightarrow$ Let $\mathfrak{F}$ be a fuzzy net in X, such that $\mathfrak{F} \overset{Q}{\to} x_r$ and $\mathfrak{F} \overset{Q}{\to} y_s$, such that $x \neq y$. Since $\mathfrak{F} \overset{Q}{\to} x_r$, we have for each $A \in N^Q_{x_r}$, there exists $m \in D$, such that $x^n_{r_n} qA$, $\forall\, n \geq m$. Also, since $\mathfrak{F} \overset{Q}{\to} y_s$, we have for each $B \in N^Q_{y_s}$, there exists $k \in D$, such that $x^n_{r_n} qB$, $\forall\, n \geq k$. Since $(D, \geqslant)$ is a directed set, then there exists $p \in D$, such that $p \geq m$ and $p \geq k$, then $x^n_{r_n} qA$, $\forall\, n \geq p$ and $x^n_{r_n} qB$, $\forall\, n \geq p$, therefore $A \wedge B \neq 0_X$ for each $A \in N^Q_{x_r}$ and for each $B \in N^Q_{y_s}$, thus X is not fuzzy T_2-space.

$\Longleftarrow$ Let X be a not fuzzy T_2-space, then there exists $x_r, y_s \in \mathrm{FP}(X)$, such that $x \neq y$ and $A \wedge B \neq 0_X$, $\forall\, A \in N^Q_{x_r}$ and $\forall\, B \in N^Q_{y_s}$.
Put $N^Q_{x_r, y_s} = \{ A \wedge B : A \in N^Q_{x_r}$ and $B \in N^Q_{y_s}\}$.

Thus $\forall\ F \in N^Q_{x_r,y_s}$, $\exists x^F_{r_F} q D$. Then $\{x^F_{r_F}\}_{F\in N^Q_{x_r,y_s}}$ is a fuzzy net in X. To prove $x^F_{r_F} \xrightarrow{Q} x_r$ and $x^F_{r_F} \xrightarrow{Q} y_s$. Let $E \in N^Q_{x_r}$, then $E \in N^Q_{x_r,y_s}$ $(E = E \wedge 1_X)$. Thus $x^F_{r_F} q E$, $\forall\ F \leq E$, hence $x^F_{r_F} \xrightarrow{Q} x_r$. Also $x^F_{r_F} \xrightarrow{Q} y_s$, so $\{x^F_{r_F}\}_{F\in N^Q_{x_r,y_s}}$ has two Q-limit points.

Definition 2.13 [2]

A family Ω of fuzzy sets is a cover of a fuzzy set A if and only if $A \leq \vee\{G_i : G_i \in \Omega\}$ and it is called a fuzzy open cover if and only if Ω is a cover of A and each member of Ω is a fuzzy open set. A subcover of Ω is a subfamily of Ω which is also a cover of A.

Definition 2.14

Let (X, T) be an fts and A be a fuzzy set in X. Then A is said to be fuzzy compact if for every open cover of A has a finite subcover of A. An fts (X, T) is called fuzzy compact if and only if each fuzzy open cover of X has a finite subcover.

Example 2.15

The indiscrete fuzzy topological space is fuzzy compact.

Proposition 2.16

The fuzzy continuous image of a fuzzy compact set is fuzzy compact.

Proof:

It is clear.

Definition 2.17 [2]

A family Ω of fuzzy sets has the finite intersection property if and only if the intersection of the members of each finite subfamily of Ω is nonempty.

Theorem 2.18 [2]

An fts is fuzzy compact if and only if each family of fuzzy closed sets which has the finite intersection property has a non-empty intersection.

Theorem 2.19

An fts X is fuzzy compact if and only if every fuzzy net in X has a Q-adherent point.

Proof :

$\Longrightarrow$ Let X be a fuzzy compact space. If possible, let $\{x^n_{r_n} : n \in D\}$, be a fuzzy net in X which has no Q-adherent point. For each fuzzy point x_r, there is a fuzzy q-nbd V_{x_r} of x_r and an $n_{V_{x_r}} \in D$, such that $x^m_{r_m} \tilde{q} V_{x_r}$ for all $m \in D$ with $m \geq n_{V_{x_r}}$. Let $\mathcal{V}$ denote the collection of all fuzzy q-nbds of x_r such V_{x_r}, where x_r runs over all fuzzy points in X. Now, we will prove that the collection $\mathcal{W} = \{1 - V_{x_r} : V_{x_r} \in \mathcal{V}\}$ is a family of fuzzy closed sets in X possessing finite intersection property. In fact, let $\mathcal{W}_0 = \{1 - V_{x^i_{r_i}} : i = 1,2,\ldots,m\}$ be a finite subfamily of $\mathcal{W}$. Then there exists $k \in D$, such that $k \geq n_{V_{x^i_{r_i}}}, \ldots, n_{V_{x^m_{r_m}}}$ and so $x^p_{r_p} \tilde{q} V_{x^i_{r_i}}$ for $i = 1,2,\ldots,m$ and for all $p \geq k$ $(p \in D)$, i.e., $x^p_{r_p} \in 1 - \vee^m_{i=1} V_{x^i_{r_i}} = \wedge^m_{i=1}(1 - V_{x^i_{r_i}})$ for all $p \geq k$. Hence $\wedge \mathcal{W}_0 \neq 0_X$. Since X is fuzzy compact, by theorem 2.18, there exists a fuzzy point y_s in X, such that $y_s \in \wedge\{1 - V_{x_r} : V_{x_r} \in \mathcal{V}\} = 1 - \vee\{V_{x_r} : V_{x_r} \in \mathcal{V}\}$. Thus $y_s \in 1 - V_{x_r}$, for all $V_{x_r} \in \mathcal{V}$, and hence in particular, $y_s \in 1 - V_{y_s}$ i.e., $y_s \tilde{q} V_{y_s}$. But by construction, for each fuzzy point x_r there exists a $V_{x_r} \in \mathcal{V}$, such that $x_r q V_{x_r}$, and we arrive a contradiction.

$\Longleftarrow$ Let $\mathcal{A} = \{A_j : j \in J\}$ be a family of fuzzy closed sets having finite intersection property. Let $D = \{\wedge_{j\in J_0} A_j : J_0 \subseteq J$ and J_0 is finite$\}$. Then $\mathcal{A} \subseteq D$. For each $\lambda_j \in D$ let us choose a fuzzy point $x^{\lambda_j}_{r_{\lambda_j}}$ and consider the fuzzy net $\mathfrak{F} = \{x^{\lambda_j}_{r_{\lambda_j}} : \lambda_j \in D\}$ with the directed set $(D, \geqslant)$, where for $\lambda_1, \lambda_2 \in D$, $\lambda_1 \geqslant \lambda_2$ iff $\lambda_1 \leq \lambda_2$. By hypothesis, $\mathfrak{F}$ has a Q-adherente point x_r. Let $B \in N^Q_{x_r}$ and $A_j \in \mathcal{A}$.

Since $A_j \in D$, there is $\lambda \in D$ with $\lambda \geqslant A_j$ (that is $\lambda \leq A_j$) such that $x_\lambda qB$. As $x_\lambda \leq \lambda \leq A_j$ and hence $A_j qB$. Thus $x_r \in \overline{A_j} = A_j$, for each $j \in J$ and so $\wedge_{j\in J} A_j \neq 0_X$. Hence by theorem 2.18, X is fuzzy compact.

Proposition 2.20

An fts X is fuzzy compact if and only if each fuzzy net in X has a Q-convergent fuzzy subnet.

Proof :

It follows from theorem 2.19 and proposition 2.10.

Proposition 2.21

In any fts X, the intersection of any fuzzy closed set with any fuzzy compact set is fuzzy compact.

Proof :

Let A be a fuzzy closed set and B be a fuzzy compact set in X. Supposed that $\{x^n_{r_n}\}_{n\in D}$ be a fuzzy net in $A \wedge B$, then $\{x^n_{r_n}\}_{n\in D}$ in A and B. Since B is fuzzy compact, then by proposition 2.20 , $\{x^n_{r_n}\}_{n\in D}$ has a Q-convergent fuzzy subnet. Since A is a fuzzy closed set in X, then by theorem 2.11, $x_r \in \overline{A} = A$, therefore $\{x^n_{r_n}\}_{n\in D}$ be a fuzzy net in $A \wedge B$ which has a Q-convergent fuzzy subnet, hence by proposition 2.20, $A \wedge B$ is fuzzy compact.

Proposition 2.22

Let Y be a fuzzy subspace of an fts X, and A be a fuzzy set in Y. Then A is a fuzzy compact set in X if and only if A is a fuzzy compact set in Y.

Proof :

$\Longrightarrow$ Let A be a fuzzy compact set in X. To prove that A is a fuzzy compact set in Y. Let $\{V_\lambda\}_{\lambda\in\Lambda}$ be a fuzzy open cover of A in Y. Then there exists fuzzy open sets G_λ in X, such that $V_\lambda = 1_Y \wedge G_\lambda$ for each $\lambda \in \Lambda$. Then $\{G_\lambda\}_{\lambda\in\Lambda}$ be a fuzzy open cover of A in X. Since A is a fuzzy compact set in X, there exists $\{\lambda_1, \lambda_2, \ldots\ldots, \lambda_n\}$, such that $A \leq \vee_{j=1}^{n} G_{\lambda_j}$. Since $A \leq 1_Y$, we have $A \leq 1_Y \wedge \{G_{\lambda_1} \vee G_{\lambda_2} \vee \ldots\ldots\ldots \vee G_{\lambda_n}\}$=$\{(1_Y \wedge G_{\lambda_1}) \vee (1_Y \wedge G_{\lambda_2}) \vee \ldots\ldots\ldots\ldots \vee (1_Y \wedge G_{\lambda_n})\}$. Since $1_Y \wedge G_{\lambda_j} = V_{\lambda_j}$, ($j$ =1, 2,..., n), we obtain $A \leq \vee_{j=1}^{n} V_{\lambda_j}$. This show that A is a fuzzy compact set in Y.

$\Longleftarrow$ Let A be a fuzzy compact set in Y. To prove that A is a fuzzy compact set in X. Let $\{G_\lambda\}_{\lambda\in\Lambda}$ be a fuzzy open cover of A in X so that $A \leq \vee_{\lambda\in\Lambda} G_\lambda$. Since $A \leq 1_Y$, then $A \leq 1_Y \wedge [\vee\{G_\lambda : \lambda \in \Lambda\}] = \vee\{1_Y \wedge G_\lambda : \lambda \in \Lambda\}$. Since $1_Y \wedge G_\lambda$ is fuzzy open cover of A in Y and A is fuzzy compact in Y, we must have $A \leq (1_Y \wedge G_{\lambda_1}) \vee (1_Y \wedge G_{\lambda_2}) \vee \ldots\ldots\ldots\ldots\ldots \vee (1_Y \wedge G_{\lambda_n})$, therefore $A \leq G_{\lambda_1} \vee \ldots\ldots \vee G_{\lambda_n}$. Thus A is fuzzy compact in X.

Definition 2.23

Let X be an fts. A fuzzy subset V of X is called fuzzy compactly closed if for every fuzzy compact set K in X, $V \wedge K$ is fuzzy compact.

Example 2.24

Every fuzzy subset of a fuzzy indiscrete space is fuzzy compactly closed.

Proposition 2.25

Every fuzzy closed subset of an fts X is fuzzy compactly closed.

Proof :

Let A be a fuzzy closed subset of an fts X and K be a fuzzy compact set in X. Then by proposition 2.21, $A \wedge K$ be a fuzzy compact set. Thus A is fuzzy compactly closed.

Proposition 2.26

A fuzzy compact subset of a fuzzy T_2-space is fuzzy closed.

Proof :

Suppose that A be a fuzzy compact subset of a fuzzy T_2-space. We must show that A is a fuzzy closed set. If $x_r \in \overline{A}$, then there exists a fuzzy net $\{x_{r_n}^n\}_{n \in D}$ in A such that $x_{r_n}^n \overset{Q}{\rightarrow} x_r$. But since A is fuzzy compact, then by theorem 2.19, $\{x_{r_n}^n\}_{n \in D}$ has a Q-adherent point y_s in A and thus there is a fuzzy subnet in A which is Q-converges to y_s. Since X is a fuzzy T_2-space, then by proposition 2.12, $x_r = y_s$. Thus $x_r \in A$, showing that A is fuzzy closed.

Proposition 2.27

A fuzzy closed subset of a fuzzy compact space is fuzzy compact.

Proof :

Let A be a fuzzy closed set of a fuzzy compact space (X,T). Since $A \wedge 1_X = A$. Then by proposition 2.21, A is a fuzzy compact set in X.

§ 3. Fuzzy Compact Mapping

In this section, we investigate the important result concerning with the characteristic fuzzy compact mapping.

Definition 3.1

Let X and Y be fts's. We say that the mapping $f: X \to Y$ is fuzzy compact if the inverse image of each fuzzy compact set in Y, is a fuzzy compact set in X.

Proposition 3.2

Let A be a fuzzy subspace of an fts X. Then A is fuzzy compactly closed if and only if the inclusion mapping $i_A: A \to X$ is fuzzy compact.

Proof :

$\Longrightarrow$ Let B be a fuzzy compact set in X, then $A \wedge B$ is a fuzzy compact set in X, thus by proposition 2.22, $A \wedge B$ is a fuzzy compact set in A. But $i_A^{-1}(B) = A \wedge B$, then $i_A^{-1}(B)$ is a fuzzy compact set in A. Hence $i_A: A \longrightarrow X$ is a fuzzy compact mapping.

$\Longleftarrow$ Let B be a fuzzy compact set in X. Since $i_A: A \longrightarrow X$ is a fuzzy compact mapping, then $i_A^{-1}(B)$ is a fuzzy compact set in A. Thus by proposition 2.22, $i_A^{-1}(B)$ is a fuzzy compact set in X. But $i_A^{-1}(B) = A \wedge B$ is a fuzzy compact set in X, for every fuzzy compact set B in X. Therefore A is a fuzzy compactly closed set in X.

Corollary 3.3

For any fuzzy closed subset A of an fts X, the inclusion mapping $i_A: A \to X$ is fuzzy compact.

Proof :

Let A be a fuzzy closed set in X, then by proposition 2.25, A is a fuzzy compactly closed set in X, hence by proposition 3.2, the inclusion mapping $i_A: A \to X$ is fuzzy compact.

Proposition 3.4

Let X and Y and Z be fts's, and $f: X \to Y$, $g: Y \to Z$ be mappings. Then:

(a) If f and g are fuzzy compact mappings, then $g \circ f$ is a fuzzy compact mapping.

(b) If $g \circ f$ is a fuzzy compact mapping, f is onto and fuzzy continuous, then g is fuzzy compact.

(c) If $g \circ f$ is a fuzzy compact mapping, g is fuzzy continuous and one to one, then f is fuzzy compact.

Proof :

(a) Let A be a fuzzy compact set in Z, then $g^{-1}(A)$ is a fuzzy compact set in Y, and so $f^{-1}(g^{-1}(A)) = (g \circ f)^{-1}(A)$ is a fuzzy compact set in X. Hence $g \circ f: X \to Z$ is a fuzzy compact mapping.

(b) Let A be a fuzzy compact set in Z, then $(g \circ f)^{-1}(A)$ is a fuzzy compact set in X and then $(f(g \circ f)^{-1})(A)$ is a fuzzy compact set in Y. Now, since f is onto, then $(f(g \circ f)^{-1})(A) = g^{-1}(A)$, hence $g^{-1}(A)$ is a fuzzy compact set in Y. Therefore g is a fuzzy compact mapping.

(c) Let A be a fuzzy compact set in Y, then $g(A)$ is a fuzzy compact set in Z, thus $(g \circ f)^{-1}(g(A))$ is a fuzzy compact set in X. Since g is one to one, then $(g \circ f)^{-1}(g(A)) = f^{-1}(A)$, hence $f^{-1}(A)$ is a fuzzy compact set in X. Thus f is a fuzzy compact mapping.

Proposition 3.5

If $f: X \to Y$ is a fuzzy compact mapping and A is a fuzzy closed subset of X, then the restriction mapping $f_{|A}: A \to Y$ is fuzzy compact.

Proof :

Since A is a fuzzy closed subset of X, then by corollary 3.3, the inclusion mapping $i_A: A \to X$ is fuzzy compact. But $f_{|A} \equiv f \circ i_A$, then by proposition 3.4,a, $f_{|A}$ is a fuzzy compact mapping.

§ 4. Fuzzy Coercive Mapping

In this section, we investigate the important result concerning with the characteristic fuzzy coercive mapping.

Definition 4.1

Let X and Y be fts's. A mapping $f: X \to Y$ is called fuzzy coercive if for every fuzzy compact set B in Y, there exists a fuzzy compact set A in X such that $f(1_X - A) \leq (1_Y - B)$.

Example 4.2

If X is a fuzzy compact space, then the mapping $f: X \to Y$ is fuzzy coercive. Let B be a fuzzy compact set in Y. Since X is fuzzy compact and $f(1_X - 1_X) = f(0_X) = 0_X \leq (1_Y - B)$, then f is a fuzzy coercive mapping.

Proposition 4.3

Every fuzzy compact mapping is fuzzy coercive.

Proof :

Let $f: X \to Y$ be a fuzzy compact mapping. To prove that f is a fuzzy coercive mapping. Let A be a fuzzy compact set in Y. Since f is fuzzy compact, then $f^{-1}(A)$ is a fuzzy compact set in X. Thus $f(1_X - f^{-1}(A)) \leq 1_Y - A$. Hence $f: X \to Y$ is a fuzzy coercive mapping.

Proposition 4.4

If $f: X \to Y$ is a fuzzy continuous mapping, such that Y is a fuzzy T_2-space. Then f is fuzzy compact if and only if it is fuzzy coercive.

Proof :

$\Longrightarrow$ By proposition 4.3

$\Longleftarrow$ Let B be a fuzzy compact set in Y. To prove that $f^{-1}(B)$ is a fuzzy compact set in X. Since Y is a fuzzy T_2-space and f is a fuzzy continuous mapping, then by proposition 1.32 and proposition 2.26, $f^{-1}(B)$ is a fuzzy closed set in X. Since f is a fuzzy coercive mapping, then there exists a fuzzy compact set A in X, such that $f(1_X - A) \leq 1_Y - B$. Then $f(A^c) \leq B^c$, therefore $f^{-1}(B) \leq A$, then by proposition 2.27, $f^{-1}(B)$ is a fuzzy compact set in X. Hence f is a fuzzy compact mapping.

Proposition 4.5

Let $f: X \to Y$ and $g: Y \to Z$ be fuzzy coercive mappings, then $g \circ f: X \to Z$ is a fuzzy coercive mapping.

Proof :

Let C be a fuzzy compact set in Z. Since g is a fuzzy coercive mapping, then there exists a fuzzy compact set B in Y, such that $g(1_Y - B) \leq 1_Z - C$. Since f is a fuzzy coercive mapping, then there exists a fuzzy compact set A in X, such that $f(1_X - A) \leq (1_Y - B)$, then $g(f(1_X - A)) \leq g(1_Y - B) \leq 1_Z - C$, hence $(g \circ f)(1_X - A) \leq 1_Z - C$. Thus $g \circ f$ is a fuzzy coercive mapping.

Proposition 4.6

Let X and Y be fts's, and $f: X \to Y$ be a mapping. Then:

(a) If f be fuzzy coercive, such that A is a fuzzy closed subset of X, then the restriction mapping $f_{|A}: A \to Y$ is fuzzy coercive.

(b) If X is a fuzzy compact space and A is a fuzzy closed set in X, then $f_{|A}: A \rightarrow Y$ is a fuzzy coercive mapping.

Proof :

(a) Since A is a fuzzy closed subset of X, then by corollary 3.3 and proposition 4.3, the inclusion mapping $i_A: A \rightarrow X$ is fuzzy coercive. But $f_{|A} = f \circ i_A$, then by proposition 4.5, $f_{|A}$ is a fuzzy coercive mapping.

(b) Since X is a fuzzy compact space, then f is a fuzzy coercive mapping. Since A is a fuzzy closed set in X, then by (a), $f_{|A}$ is a fuzzy coercive mapping.

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