Some results on 3-normed spaces and fuzzy 3-normed spaces

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Abstract:-

In this paper we give some definitions and basic concepts related with 3-normed space like we give definitions of closed subset, closure subset, bounded subset and equivalent norms. Moreover, we prove every Cauchy sequence in 3-normed space is bounded and a Cauchy sequence is convergent in an 3-normed space if and only if it has a convergent subsequence. Thereafter, we generalize this facts to fuzzy 3-normed space.

بعض النتائج عن الفضاءات 3-المعيارية و الفضاءات الضبابية 3- المعيارية

الخلاصة :-

في هذا البحث قدمنا بعض التعاريف والحقائق الأساسية المتعلقة بفضاء 3-المعياري مثلا قدمنا تعار يف المجموعة الجزيئية المغلقة ، انغلاق المجموعة الجزيئية ، والمجموعة الجزيئية المقيدة وتكافؤ المعايير. أكثر من هذا فلقد تم إثبات كل متتابعة كوشية في فضاء 3- المعياري تكون مقيدة وكل متتابعة كوشية في الفضاء 3- المعياري تكون متقاربة إذا امتلكت متتابعة جزئية متقاربة. بعد ذلك قمنا بأعمام هذه الحقائق على الفضاء الضبابي 3- المعياري.

1. Introduction:-

In 1964, the theory of 2-normed space was investigated by Gahler [8]. While the theory of an n-normed spaces can be found in [4]. Different authors introduced different definitions of fuzzy normed space (see [2],[3],[5],[7],[11]). The notation of fuzzy n-normed linear space is introduced in [1], [9]. Since fuzzy 3-normed space can be applied in fuzzy operations research specific on fuzzy scheduling then in this paper we give some properties for 3-normed and then generalized to fuzzy 3-normed this properties important in the future work in fuzzy operations research.

Throughout this work, we assume X to be a real linear space of dimension $d \geq 3$.

2. Preliminaries:-

In this section, we give some basic concepts that we needed then later.

Definition (2.1), [4]:-

Let X be a real linear space of dimension $d \geq 3$. **A function** $\|.,.,.\| : X \times X \times X \longrightarrow R^{+} \cup \{0\}$ **which satisfy the following axioms:**

(N1) $\|x_1, x_2, x_3\| = 0$ **if and only if** x_1, x_2, x_3 **are linearly dependent.**

(N2) $\|x_1, x_2, x_3\|$ **is an invariant under any permutation of** x_1, x_2, x_3.

(N3) $\|x_1, x_2, cx_3\| = |c|\|x_1, x_2, x_3\|$ **for any** $c \in R$,

(N4) $\|x_1, x_2, y+z\| \leq \|x_1, x_2, y\| + \|x_1, x_2, z\|$,

is said to be a 3-norm on X and the pair $(X, \|.,.,.\|)$ **is called an 3-normed space.**

Definition (2.2), [4]:-

Let $(X,\|.,.,.\|)$ be an 3-normed space. A sequence $\{x_n\}$ in X is said to be convergent if there exists an element $x \in X$ such that $\lim_{n\to\infty}\|x_1, x_2, x_n - x\| = 0$ for all $x_1, x_2 \in X$. In this case x is said to be the limit of the sequence $\{x_n\}$ and we denote it by lim x_n. Otherwise the sequence is divergent.

Definition (2.3), [4]:-

Let $(X,\|.,.,.\|)$ be an 3-normed space. A sequence $\{x_n\}$ of X is said to be Cauchy sequence in case $\lim_{n\to\infty}\|x_1, x_2, x_{n+p} - x_n\| = 0$ for all $x_1, x_2 \in X$ and p=1,2,…

Definition (2.4), [1]:-

A fuzzy subset N of $X^3 \times R$ is said to be a fuzzy 3-norm on the real linear space X in case the following axioms hold:

(FN1) $N(x_1, x_2, x_3, t) = 0$ **for each** $t \le 0$.

(FN2) $N(x_1, x_2, x_3, t) = 1$ **for each t>0 if and only if** x_1, x_2, x_3 **are linearly dependent.**

(FN3) $N(x_1, x_2, x_3, t)$ **is an invariant under any permutation of** x_1, x_2, x_3.

(FN4) If $0 \ne c \in R$ then $N(x_1, x_2, cx_3, t) = N(x_1, x_2, x_3, \frac{t}{|c|})$ for each $t > 0$.

(FN5) $N(x_1, x_2, x + y, s + t) \ge \min\{N(x_1, x_2, x, s), N(x_1, x_2, y, t)\}$ **for each** $s, t \in R$.

(FN6) $N(x_1, x_2, x_3, .)$ **is a nondecreasing function of R and** $\lim_{t\to\infty} N(x_1, x_2, x_3, t) = 1$.

The pair (X,N) will be referred to as a fuzzy 3-normed linear space.

Now, the question arises: can one generate an 3-norm from a fuzzy 3-norm ?.
To answer this question, see the following theorem.

Theorem (2.5), [1]:-

Let (X,N) be a fuzzy 3-normed linear space. Assume further that for each t>0, $N(x_1, x_2, x_3, t) > 0$ **implies** x_1, x_2, x_3 **are linearly dependent. For each** $x_1, x_2, x_3 \in X$, **define** $\|x_1, x_2, x_3\|_\alpha = \inf\{t : N(x_1, x_2, x_3, t) \ge \alpha\}, \quad \alpha \in (0,1)$.

Then for each $\alpha \in (0,1)$, $\|.,.,.\|_\alpha$ **is an 3-norm on X and** $\{\|.,.,.\|_\alpha | \alpha \in (0,1)\}$ **is an ascending family of 3-norms on X.**

Theorem (2.6), [10]:-

Let (X,N) be a fuzzy 3-normed space satisfying the following conditions

(1) For each t>0, $N(x_1, x_2, x_3, t) > 0$ **implies** x_1, x_2, x_3 **are linearly dependent.**

(2)For x_1, x_2, x_3 **are linearly independent,** $N(x_1, x_2, x_3, t)$ **is a continuous of** $t \in R$ **and strictly increasing in the subset** $\{t : 0 < N(x_1, x_2, x_3, t) < 1\}$ **of R.**

Let $\|x_1, x_2, x_3\|_\alpha = \inf\{t : N(x_1, x_2, x_3, t) \ge \alpha\}, \alpha \in (0,1)$ **and** $N' : X^3 \times R \to [0,1]$ **is defined by**

$$N'(x_1, x_2, x_3, t) = \begin{cases} \sup\{\alpha \in (0,1) : \|x_1, x_2, x_3\|_\alpha \le t\} & \text{when } x_1, x_2, x_3 \text{ are linearly independent}, t \ne 0 \\ 0 & \text{Otherwise} \end{cases}$$

Then

(a) $\{\|.,.,.\|_\alpha | \alpha \in (0,1)\}$ **is an ascending family of** α **-3-norms corresponding to the fuzzy 3-**

normed space (X,N).

(b) (X, N') is a fuzzy 3-normed space.

(c) N'=N.

Definition (2.7), [9]:-

Let (X,N) be a fuzzy 3-normed linear space, a sequence {x_n} in X is said to be convergent if there exists an element $x \in X$ such that $\lim_{n\to\infty} N(x_1, x_2, x_n - x, t) = 1$ for each $x_1, x_2 \in X$ and for each t>0. In this case x is said to be the limit of the sequence {x_n}. Otherwise the sequence is divergent.

Definition (2.8), [9]:-

Let (X,N) be a fuzzy 3-normed linear space, a sequence {x_n} of X is said to be Cauchy sequence in case $\lim_{n\to\infty} N(x_1, x_2, x_{n+p} - x_n, t) = 1$ for each $x_1, x_2 \in X$, t>0 and p=1,2,….

3. Some Results in 3-Normed Spaces:-

In this section we give some results in 3-normed spaces. We start with the following theorem. This theorem shows that the limit of a convergent sequence in an 3-normed space is unique. This theorem is used in [4] without proof, here we give its proof for the sake of completeness.

Theorem (3.1):-

Let $(X, \|.,.,.\|)$ be an 3-normed space and {x_n} be a sequence in X. If $\lim x_n = x$ and $\lim x_n = y$ then x=y.

Proof:-

For each $x_1, x_2 \in X$

$$\|x_1, x_2, x - y\| = \lim_{n\to\infty} \|x_1, x_2, x - x_n + x_n - y\|$$

$$\leq \lim_{n\to\infty} \|x_1, x_2, x - x_n\| + \lim_{n\to\infty} \|x_1, x_2, x_n - y\|$$

$$= \lim_{n\to\infty} \|x_1, x_2, x_n - x\| + \lim_{n\to\infty} \|x_1, x_2, x_n - y\|$$

=0

Hence $\|x_1, x_2, x - y\| = 0$ for each $x_1, x_2 \in X$. Then x=y.

Next, the following proposition illustrates that every subsequence of a convergent sequence converges.

Proposition (3.2):-

Let $(X, \|.,.,.\|)$ be an 3-normed space and $\lim x_n = x$. Then $\lim x_{n_k} = x$ for every subsequence $\{x_{n_k}\}$ of the sequence {x_n}.

Proof:-

Since $\lim x_n = x$, then $\lim_{n\to\infty} \|x_1, x_2, x_n - x\| = 0$ for each $x_1, x_2 \in X$.

Fixed $x_1, x_2 \in X$, Then $\lim_{n\to\infty} \|x_1, x_2, x_n - x\| = 0$. Hence $\lim_{k\to\infty} \|x_1, x_2, x_{n_k} - x\| = 0$. Therefore, for each $x_1, x_2 \in X$

$\lim_{k\to\infty} \|x_1, x_2, x_{n_k} - x\| = 0$. **Then** $\lim x_{n_k} = x$.

Proposition (3.3):-

Let $(X,\|.,.,.\|)$ be an 3-normed space and $\lim x_n = x$, $\lim y_n = y$. Then $\lim(\alpha x_n + \beta y_n) = \alpha x + \beta y, \alpha, \beta \in R$.

Proof:-

Since $\lim x_n = x$ and $\lim y_n = y$ then $\lim\limits_{n\to\infty}\|x_1, x_2, x_n - x\| = 0$

$\lim\limits_{n\to\infty}\|x_1, x_2, y_n - y\| = 0$ **for each** $x_1, x_2 \in X$. **Hence,**

$$\lim_{n\to\infty}\|x_1, x_2, (\alpha x_n + \beta y_n) - (\alpha x + \beta y)\| = \lim_{n\to\infty}\|x_1, x_2, \alpha x_n - \alpha x + \beta y_n - \beta y\|$$

$$\leq \lim_{n\to\infty}\|x_1, x_2, \alpha x_n - \alpha x\| + \lim_{n\to\infty}\|x_1, x_2, \beta y_n - \beta y\|$$

Therefore, $\lim(\alpha x_n + \beta y_n) = \alpha x + \beta y$.

Next, the following theorem illustrates that every convergent sequence is Cauchy sequence. This is used in [4] without proof, here we give its proof for the sake of completeness.

Theorem (3.4):-

In an 3-normed space $(X,\|.,.,.\|)$, every convergent sequence is Cauchy sequence.

Proof:-

Suppose that for each $x_1, x_2 \in X$, $\lim\limits_{n\to\infty}\|x_1, x_2, x_n - x\| = 0$.

Then , for p=1,2,…, one can have

$$\lim_{n\to\infty}\|x_1, x_2, x_{n+p} - x_n\| = \lim_{n\to\infty}\|x_1, x_2, x_{n+p} - x + x - x_n\|$$

$$\leq \lim_{n\to\infty}\|x_1, x_2, x_{n+p} - x\| + \lim_{n\to\infty}\|x_1, x_2, x_n - x\|$$

By using proposition (3.2) one can get $\lim\limits_{n\to\infty}\|x_1, x_2, x_{n+p} - x\| = 0$. **Thus**

$\lim\limits_{n\to\infty}\|x_1, x_2, x_{n+p} - x_n\| = 0$ **for each** $x_1, x_2 \in X$ **and p=1,2,…. Therefore $\{x_n\}$ is a Cauchy sequence in** $(X,\|.,.,.\|)$.

The question now arises: does every Cauchy sequence in an 3-normed space is convergent?. The following example gives an answer.

Example (3.5):-

Let X be a real linear space of finitely nonzero sequences. Let

$$\|x,y,z\|_S = \left(\begin{vmatrix} \sum_{i=1}^{\infty}|x_i|^2 & \sum_{i=1}^{\infty}x_i y_i^* & \sum_{i=1}^{\infty}x_i z_i^* \\ \sum_{i=1}^{\infty}y_i x_i^* & \sum_{i=1}^{\infty}|y_i|^2 & \sum_{i=1}^{\infty}y_i z_i^* \\ \sum_{i=1}^{\infty}z_i x_i^* & \sum_{i=1}^{\infty}z_i y_i^* & \sum_{i=1}^{\infty}|z_i|^2 \end{vmatrix}\right)^{1/2}$$

Then, $(X,\|.,.,.\|_S)$ **is an 3-normed space. There exist a sequence $\{x_n\}$ defined by**

$$x_n = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, \ldots\}$$

such that x_n is Cauchy but not converges in X.

Next, in [6] gave the definitions of closed subset, closure subset, bounded subset and compact subset in 2-normed space. Here we give the same definitions, but for the an 3-normed space due to [4].

Definition (3.6):-

Let $(X,\|.,.,.\|)$ be an 3-normed space. A subset U of X is said to be closed in case for any sequence $\{x_n\}$ in U such that $\lim_{n\to\infty}\|x_1,x_2,x_n-x\|=0$ for each $x_1,x_2\in X$, implies $x\in U$.

Definition (3.7):-

Let $(X,\|.,.,.\|)$ be an 3-normed space. A subset V of X is said to be the closure of a subset U of X in case for any $x\in V$, there exists a sequence $\{x_n\}$ in U such that $\lim_{n\to\infty}\|x_1,x_2,x_n-x\|=0$ for each $x_1,x_2\in X$. We denote the set V by $\overline{U}$.

Definition (3.8):-

Let $(X,\|.,.,.\|)$ be an 3-normed space. A subset U of X is said to be bounded in case there exists two independent vectors z_1,z_2 in X and M>0 such that $\|z_1,z_2,x\|<M$ for each $x\in U$

Definition (3.9):-

Let $(X,\|.,.,.\|)$ be an 3-normed space. A subset U of X is said to be compact in case every sequence $\{x_n\}$ in U has subsequence $\{x_{n_k}\}$ such that there exists $x\in U$ and $\lim_{k\to\infty}\|x_1,x_2,x_{n_k}-x\|=0$ for each $x_1,x_2\in X$.

Proposition (3.10):-

Every compact subset U of an 3-normed space $(X,\|.,.,.\|)$ is closed and bounded.

Proof:-

Suppose U is compact subset of an 3-normed space and $\{x_n\}$ be a sequence in U such that $\lim_{n\to\infty}\|x_1,x_2,x_n-x\|=0$ for each $x_1,x_2\in X$. Since U is compact then there exists subsequence $\{x_{n_k}\}$ of sequence $\{x_n\}$ converges to a point in U. Again $\lim x_n=x$ and $\lim x_{n_k}=x$ by proposition (3.2) then $x\in U$. If U is not bounded, then would contain a sequence $\{y_n\}$ such that $\|z_1,z_2,y_n\|>n$, for any fixed independent vectors z_1 and z_2. Now this sequence could not have a convergent subsequence because if $\{y_{n_k}\}$ were a convergent subsequence to y then $\lim_{k\to\infty}\|z_1,z_2,y_{n_k}-y\|=0$ and for ε there would exist a positive integer N such that $\|z_1,z_2,y_{n_k}\|-\|z_1,z_2,y\|\le\|z_1,z_2,y_{n_k}-y\|\le\varepsilon$ for each $k>N$ which is a contradiction.

The following example shows that the converse of proposition (3.10) is not true.

Example (3.11):-

Let $(R^3,\|.,.,.\|_E)$ be an 3-normed space where an 3-norm defined as follows:

$$\|x_1,x_2,x_3\|_E = \text{abs}\begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix}$$ **. The set** $U = \{x \in R^3 \mid \|(1,0,0),(0,1,0),x\|_E \le 1\}$

is not compact set. Because the sequence $\{(n,0,0)\}$ **has no convergent subsequence. Suppose on the contrary that** $\{(n_k,0,0)\}$ convergent (a,b,c) **then we have** $\lim_{k\to\infty}\|(0,1,0),(0,0,1),(n_k,0,0)-(a,b,c)\|_E = 0$

That is $|n_k - a| \to 0$ **which is a contradiction.**

Proposition (3.12):-

Every Cauchy sequence in an 3-normed space $(X,\|.,.,.\|)$ **is bounded.**

Proof:-

Let $\{x_n\}$ **be Cauchy sequence in an 3-normed space** $(X,\|.,.,.\|)$**. Then** $\lim_{n\to\infty}\|x_1,x_2,x_{n+p}-x_n\| = 0$ **for each** $x_1,x_2 \in X$, **p=1,2,…. Let** z_1,z_2 **be independent vectors in X. Then** $\lim_{n\to\infty}\|z_1,z_2,x_{n+p}-x_n\| = 0$ **p=1,2,…. Let** $\varepsilon > 0$ **then there exists N>0 such that** $\|z_1,z_2,x_{n+p}-x_n\| < \varepsilon$ for each $n \ge N, p = 1,2,\ldots$ **In particular,**

$\|z_1,z_2,x_N - x_n\| < \varepsilon$ for each $n \ge N$. **Let**

$r = \max\{\varepsilon, \|z_1,z_2,x_N - x_1\|, \|z_1,z_2,x_N - x_2\|, \ldots, \|z_1,z_2,x_N - x_{N-1}\|\}$

Therefore for all $n = 1,2,\ldots$, $\|z_1,z_2,x_N - x_n\| < r$**. Hence,**

$$\|z_1,z_2,x_n\| = \|z_1,z_2,x_N - x_N + x_n\| \le \|z_1,z_2,x_N\| + \|z_1,z_2,-(x_N - x_n)\|$$
$$= \|z_1,z_2,x_N\| + \|z_1,z_2,x_N - x_n\|$$
$$\le \|z_1,z_2,x_N\| + r$$

Replacing r by r* > r. Then

$\|z_1,z_2,x_n\| < \|z_1,z_2,x_N\| + r^*$ for each n

Therefore $\{x_n\}$ **is bounded.**

Proposition (3.13):-

Let $(X,\|.,.,.\|)$ **an 3-normed space. A Cauchy sequence is convergent in an 3-normed space** $(X,\|.,.,.\|)$ **if and only if it has a convergent subsequence**

Proof:-

Suppose $\{x_n\}$ **is a Cauchy sequence in** $(X,\|.,.,.\|)$ **which is also convergent in it. Then, every subsequence of it will be convergent in X by proposition (3.2).**

For the converse, assume that $\{x_{n_k}\}$ **is a subsequence of** $\{x_n\}$ **which converges to x** $\in$**X. Then** $\lim_{n\to\infty}\|x_1,x_2,x_{n_k} - x\| = 0$ **for each** $x_1,x_2 \in X$**. Since** $\{x_n\}$ **is Cauchy sequence then** $\lim_{n\to\infty}\|x_1,x_2,x_{n+p}-x_n\| = 0$ **for each** $x_1,x_2 \in X$, **p=1,2,….**

Hence for each $x_1, x_2 \in X$,

$$\|x_1, x_2, x_n - x\| = \|x_1, x_2, x_n - x_{n_k} + x_{n_k} - x\|$$
$$\leq \|x_1, x_2, x_n - x_{n_k}\| + \|x_1, x_2, x_{n_k} - x\|$$

Hence, $\lim_{n\to\infty} \|x_1, x_2, x_n - x\| = 0$ **for each** $x_1, x_2 \in X$. **Therefore {** x_n **} is convergent.**

Definition (3.14):-

An 3-norm $\|.,.,.\|_1$ **on a linear space X is said to be equivalent to an 3-norm** $\|.,.,.\|_2$ **on X (denoted by** $\|.,.,.\|_1 \sim \|.,.,.\|_2$**) if there exist positive numbers a and b such that**

$a\|x_1, x_2, x_3\|_2 \leq \|x_1, x_2, x_3\|_1 \leq b\|x_1, x_2, x_3\|_2$, **for each** $x_1, x_2, x_3 \in X$

Proposition (3.15):-

The relation ~ defined as above is an equivalence relation.

Proof:-

(1) The relation ~ is reflexive, since

$$1\|x_1, x_2, x_3\|_1 \leq \|x_1, x_2, x_3\|_1 \leq 1\|x_1, x_2, x_3\|_1$$

(2) To prove ~ symmetric, we assume that

$$a\|x_1, x_2, x_3\|_2 \leq \|x_1, x_2, x_3\|_1 \leq b\|x_1, x_2, x_3\|_2$$

hold and we have to show that there exist two positive number c and d such that

$$c\|x_1, x_2, x_3\|_1 \leq \|x_1, x_2, x_3\|_2 \leq d\|x_1, x_2, x_3\|_1$$

Since $a\|x_1, x_2, x_3\|_2 \leq \|x_1, x_2, x_3\|_1$ **and** $\|x_1, x_2, x_3\|_1 \leq b\|x_1, x_2, x_3\|_2$ **then**

$$\|x_1, x_2, x_3\|_2 \leq \frac{1}{a}\|x_1, x_2, x_3\|_1 \text{ and } \frac{1}{b}\|x_1, x_2, x_3\|_1 \leq \|x_1, x_2, x_3\|_2$$

Hence $\frac{1}{b}\|x_1, x_2, x_3\|_1 \leq \|x_1, x_2, x_3\|_2 \leq \frac{1}{a}\|x_1, x_2, x_3\|_1$

Let c=$\frac{1}{b}$ **and d=**$\frac{1}{a}$ **then** $c\|x_1, x_2, x_3\|_1 \leq \|x_1, x_2, x_3\|_2 \leq d\|x_1, x_2, x_3\|_1$

(3) To prove ~ is transitive, we assume $a\|x_1, x_2, x_3\|_0 \leq \|x_1, x_2, x_3\| \leq b\|x_1, x_2, x_3\|_0$

and $c\|x_1, x_2, x_3\|_1 \leq \|x_1, x_2, x_3\|_0 \leq d\|x_1, x_2, x_3\|_1$

then we have to show there exist two positive number e and f such that

$$e\|x_1, x_2, x_3\|_1 \leq \|x_1, x_2, x_3\| \leq f\|x_1, x_2, x_3\|_1$$

Since $a\|x_1, x_2, x_3\|_0 \leq \|x_1, x_2, x_3\|$ **and** $c\|x_1, x_2, x_3\|_1 \leq \|x_1, x_2, x_3\|_0$

then $\|x_1, x_2, x_3\|_0 \leq \frac{1}{a}\|x_1, x_2, x_3\|$ **and** $c\|x_1, x_2, x_3\|_1 \leq \frac{1}{a}\|x_1, x_2, x_3\|$

Hence, $ac\|x_1, x_2, x_3\|_1 \leq \|x_1, x_2, x_3\|$

On the other hand,

$\|x_1, x_2, x_3\| \leq b\|x_1, x_2, x_3\|_0$ **and** $\|x_1, x_2, x_3\|_0 \leq d\|x_1, x_2, x_3\|_1$

Then, $\frac{1}{b}\|x_1, x_2, x_3\| \leq \|x_1, x_2, x_3\|_0$ **and** $\frac{1}{b}\|x_1, x_2, x_3\| \leq d\|x_1, x_2, x_3\|_1$

$\|x_1,x_2,x_3\| \le bd\|x_1,x_2,x_3\|_1$

Therefore, $ac\|x_1,x_2,x_3\|_1 \le \|x_1,x_2,x_3\| \le bd\|x_1,x_2,x_3\|_1$

Let ac=e and bd=f

$e\|x_1,x_2,x_3\|_1 \le \|x_1,x_2,x_3\| \le f\|x_1,x_2,x_3\|_1$.

4. Some Results in fuzzy 3-normed spaces:-

In this section we give some results in fuzzy 3-normed spaces. We start with the following theorem. This theorem shows that the limit of a convergent sequence in a fuzzy-3-normed space is unique. This theorem is used in [1] without proof, here we give its proof for the sake of completeness.

Theorem (4.1):-

Let (X,N) **be a fuzzy 3-normed space and {** x_n **} be a sequence in X. If lim** $x_n = x$ **and lim** $x_n = y$ **then x=y.**

Proof:-

For each $x_1,x_2 \in X$ **and for each s, t >0 one can have**

$$\begin{aligned} N(x_1,x_2,x-y,s+t) &= N(x_1,x_2,x-x_n+x_n-y,s+t) \\ &\ge \min\{N(x_1,x_2,x-x_n,s),N(x_1,x_2,x_n-y,t)\} \\ &= \min\{N(x_1,x_2,x_n-x,s),N(x_1,x_2,x_n-y,t)\} \end{aligned}$$

Therefore,

$$N(x_1,x_2,x-y,s+t) \ge \min\{\lim_{n\to\infty} N(x_1,x_2,x_n-x,s), \lim_{n\to\infty} N(x_1,x_2,x_n-y,t)\} = 1$$

Hence, for each $x_1,x_2 \in X$

$N(x_1,x_2,x-y,s+t) = 1$, **for each s,t >0**

Hence, one can get x=y.

Next, the following proposition illustrates that every subsequence of a convergent sequence converges in fuzzy 3-normed space.

Proposition (4.2):-

Let (X,N) **be a fuzzy 3-normed space and lim** $x_n = x$. **Then** $\lim x_{n_k} = x$ **for every subsequence** $\{x_{n_k}\}$ **of sequence {** x_n **}.**

Proof:-

Suppose lim $x_n = x$

Then $\lim_{n\to\infty} N(x_1,x_2,x_n-x,t) = 1$ **for each** $x_1,x_2 \in X$ **and for each t>0.**

Fixed $x_1,x_2 \in X$ **and t>0. Then,** $\lim_{n\to\infty} N(x_1,x_2,x_n-x,t) = 1$.

Hence, $\lim_{k\to\infty} N(x_1,x_2,x_{n_k}-x,t) = 1$. **Therefore, for each** $x_1,x_2 \in X$ **and for each t>0,**

$\lim_{k\to\infty} N(x_1,x_2,x_{n_k}-x,t) = 1$

Then, $\lim x_{n_k} = x$.

Proposition (4.3):-

Let (X,N) **be a fuzzy 3-normed space and lim** $x_n = x$ **and lim** $y_n = y$. **Then** $\lim(\alpha x_n + \beta y_n) = \alpha x + \beta y, \alpha, \beta \in R$.

Proof:-

Since lim $x_n = x$ **and lim** $y_n = y$

Then $\lim_{n\to\infty} N(x_1, x_2, x_n - x, s) = 1$, $\lim_{n\to\infty} N(x_1, x_2, y_n - y, t) = 1$ **for each** $x_1, x_2 \in X$ **and for each s,t>0**

Hence, for each $x_1, x_2 \in X$ **and for each s,t>0**

$$N(x_1, x_2, (\alpha x_n + \beta y_n) - (\alpha x + \beta y), s + t) = N(x_1, x_2, (\alpha x_n - \alpha x) + (\beta y_n - \beta y), s + t)$$
$$\geq \min\{N(x_1, x_2, \alpha x_n - \alpha x, s), N(x_1, x_2, \beta y_n - \beta y, t)\}$$

Then, $\lim_{n\to\infty} N(x_1, x_2, (\alpha x_n + \beta y_n) - (\alpha x + \beta y), s + t) = 1$ **for each** $x_1, x_2 \in X$ **and for each s, t>0**

Therefore, $\lim(\alpha x_n + \beta y_n) = \alpha x + \beta y$.

Next, in [9] proved that every convergent sequence is Cauchy sequence in special types of fuzzy 3-normed space. Here we prove the same result, but for the fuzzy 3-normed due to [1].

Theorem (4.4):-

Let (X,N) **be a fuzzy 3-normed space, every convergent sequence is Cauchy sequence.**

Proof:-

Suppose $\{x_n\}$ **be a sequence in X and** $\lim_{n\to\infty} N(x_1, x_2, x_n - x, t) = 1$ **for each** $x_1, x_2 \in X$ **and for each t>0.**

For $x_1, x_2 \in X$**, s, t>0 and p=1,2,… we have**

$$\lim_{n\to\infty} N(x_1, x_2, x_{n+p} - x_n, s + t) = \lim_{n\to\infty} N(x_1, x_2, x_{n+p} - x + x - x_n, s + t)$$
$$\geq \min\left\{\lim_{n\to\infty} N(x_1, x_2, x_{n+p} - x, s), \lim_{n\to\infty} N(x_1, x_2, x - x_n, t)\right\}$$

By using proposition (4.2) we have $\lim_{n\to\infty} N(x_1, x_2, x_{n+p} - x, s) = 1$**. Thus** $\lim_{n\to\infty} N(x_1, x_2, x_{n+p} - x_n, s + t) = 1$ **for each** $x_1, x_2 \in X$**, s,t>0 and p=1,2,…. Therefore** $\{x_n\}$ **is a Cauchy sequence in** (X,N)**.**

The question now arises: does every Cauchy sequence convergent in a fuzzy 3-normed linear space?. The following example gives an answer.

Example (4.5):-

Let X be a real linear space of finitely nonzero sequences. Let

$$N_f(x,y,z,t) = \begin{cases} \dfrac{t}{t + \|x,y,z\|_S} & \text{for } t > 0 \\ 0 & \text{for } t \leq 0 \end{cases}$$

where $\|.,.,\|_S$ standard an 3-norm defined in example (3.5), then (X, N_f) is a fuzzy 3-normed linear space which has Cauchy sequence not converges.

Next, in [2] gave the definitions of closed subset, closure subset, bounded subset and compact subset in fuzzy 1-normed space. Here we give the same definitions, but for the fuzzy 3-normed space due to [1].

Definition (4.6):-

Let (X, N) be a fuzzy 3-normed space. A subset U of X is said to be closed in case for any sequence $\{x_n\}$ in U such that $\lim_{n\to\infty} N(x_1, x_2, x_n - x, t) = 1$ for each $x_1, x_2 \in X$ and for each t>0, implies $x \in U$.

Definition (4.7):-

Let (X, N) be a fuzzy 3-normed space. A subset V of X is said to be the closure of a subset U of X in case for any $x \in V$, there exists a sequence $\{x_n\}$ in U such that $\lim_{n\to\infty} N(x_1, x_2, x_n - x, t) = 1$ for each $x_1, x_2 \in X$ and for each t>0. We denote the set V by $\overline{U}$.

Definition (4.8):-

Let (X, N) be a fuzzy 3-normed space. A subset U of X is said to be bounded in case there exists independent two vectors z_1, z_2 in X, t>0 and 0<r<1 such that $N(z_1, z_2, x, t) > 1 - r$, for each $x \in U$.

Definition (4.9):-

Let (X, N) be a fuzzy 3-normed space. A subset U of X is said to be compact in case every sequence $\{x_n\}$ in U has subsequence $\{x_{n_k}\}$ such that there exists $x \in U$ and $\lim_{k\to\infty} N(x_1, x_2, x_{n_k} - x, t) = 1$ for each $x_1, x_2 \in X$ and for each t>0.

Proposition (4.10):-

Every compact subset U of a fuzzy 3-normed space (X, N) is closed and bounded.

Proof:-

Suppose U is compact of a fuzzy 3-normed space (X, N) and $\{x_n\}$ be a sequence in U such that $\lim_{n\to\infty} N(x_1, x_2, x_n - x, t) = 1$ for each $x_1, x_2 \in X$ and t>0, since U is compact then there exists subsequence $\{x_{n_k}\}$ of sequence $\{x_n\}$ converges to a point in U. Again $\lim x_n = x$ and $\lim x_{n_k} = x$ by proposition (4.2) then $x \in U$. Then U is close. Now, we show that U is bounded. If U were not bounded, it would contain a sequence $\{y_n\}$ such that $N(z_1, z_2, y_n, n) \le 1 - r_\circ$ for any fixed independent vectors z_1, z_2 and for any fixed $r_\circ$ where $0 < r_\circ < 1$. Since U is compact, there exist a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ converging to element $y \in U$, therefore

$$\lim_{i\to\infty} N(z_1, z_2, y_{n_i} - y, t) = 1 \text{ for each } t > 0$$

$$\text{Also } N(z_1, z_2, y_{n_i}, n_i) \le 1 - r_\circ$$

Now,

$$1-r_\circ \geq N(z_1,z_2,y_{n_i},n_i) = N(z_1,z_2,y_{n_i}-y+y,n_i-t+t) \text{ where } t>0$$
$$\geq \min\left\{N(z_1,z_2,y_{n_i}-y,t), N(z_1,z_2,y,n_i-t)\right\}$$
$$\geq \min\{\lim_{i\to\infty} N(z_1,z_2,y_{n_i}-y,t), \lim_{i\to\infty} N(z_1,z_2,y,n_i-t)\}$$

This implies that $r_\circ \leq 0$ which is a contradiction

Hence, U is bounded.

The following example shows that the converse of proposition (4.10) is not true.

Example (4.11):-

Let $(R^3, \|.,.,.\|_E)$ **be an 3-normed space. For each** $x_1,x_2,x_3 \in R^3$**. Define**

$$N_f(x_1,x_2,x_3,t) = \begin{cases} \dfrac{t}{t+\|x_1,x_2,x_3\|_E} & \text{for } t>0 \\ 0 & \text{for } t \leq 0 \end{cases}$$

Let U be the set defined by $U = \left\{x \in R^3 \mid N_f((1,0,0),(0,1,0),x,1) \geq 0.5\right\}$ **. It is easy to check U= U" where** $U'' = \left\{x \in R^3 \,\|\, \|(1,0,0),(0,1,0),x\|_E \leq 1\right\}$

Assume U is a compact set. Then each sequence $\{x_n\}$ **in U has a convergent subsequence** $\{x_{n_k}\}$**. Say** $x_{n_k} \to x$ where $x \in U$. Thus

$$\lim_{k\to\infty} N_f(x_1,x_2,x_{n_k}-x,t) = \lim_{k\to\infty} \frac{t}{t+\|x_1,x_2,x_{n_k}-x\|_E} = 1$$

for each $x_1,x_2 \in R^3$ and for each $t>0$. **This implies that** $\lim_{k\to\infty}\|x_1,x_2,x_{n_k}-x\|_E = 0$ **for each** $x_1,x_2 \in R^3$**. Therefore U" is a compact set which is a contradiction for example (3.11)**

Proposition (4.12):-

Every Cauchy sequence in a fuzzy 3-normed space (X,N) **is bounded.**

Proof:-

Let $\{x_n\}$ **be a Cauchy sequence in a fuzzy 3-normed space. Then** $\lim_{n\to\infty} N(x_1,x_2,x_{n+p}-x_n,t) = 1$ **for each** $x_1,x_2 \in X$ **,t>0 and p=1,2,…. Let** z_1 and z_2 **be independent vectors in X. Then** $\lim_{n\to\infty} N(z_1,z_2,x_{n+p}-x_n,t) = 1$**, for p=1,2,… and t>0. Choose a fixed** $\alpha_\circ$, $0<\alpha_\circ<1$. **Then we have** $\lim_{n\to\infty} N(z_1,z_2,x_{n+p}-x_n,t) = 1 > \alpha_\circ$. **For t'>0, There exists** $n_\circ$ **such that** $N(z_1,z_2,x_{n+p}-x_n,t') > \alpha_\circ$ for each $n \geq n_\circ$, $p=1,2,\ldots.$

Since $\lim_{t\to\infty} N(z_1, z_2, x, t) = 1$, there exist t_i such that $N(z_1, z_2, x_i, t_i) > \alpha_\circ$

for each $t \geq t_i, i = 1,2,..n_\circ$

let $t_\circ = t' + \max\{t_1, t_2, ..., t_{n_\circ}\}$

Then $N(z_1, z_2, x_n, t_\circ) > \alpha_\circ$ for each $n = 1,2,...,n_\circ$

$$N(z_1, z_2, x_n, t_\circ) \geq N(z_1, z_2, x_n, t' + t_{n_\circ})$$
$$= N(z_1, z_2, x_n - x_{n_\circ} + x_{n_\circ}, t' + t_{n_\circ})$$
$$\geq \min\{N(z_1, z_2, x_n - x_{n_\circ}, t'), N(z_1, z_2, x_{n_\circ}, t_{n_\circ})\}$$

Therefore, $N(z_1, z_2, x_n, t_\circ) \geq \{\alpha_\circ, \alpha_\circ\} = \alpha_\circ$ *for each* $n \geq n_\circ$

Also $N(z_1, z_2, x_n, t_\circ) \geq N(z_1, z_2, x_n, t_n) \geq \alpha_\circ$ *for each* $n = 1,2,..., n_\circ$

Hence, $N(z_1, z_2, x_n, t_\circ) \geq \alpha_\circ$ *for each*

Then there exist $\alpha_1 \in (0,1)$ *such that* $\alpha_\circ > \alpha_1$

Therefore $\{x_n\}$ *is bounded.*

Next, in [9] proved that every Cauchy sequence is convergent sequence in special types of a fuzzy 3-normed space iff it has a convergent subsequence. Here we prove the same result, but for the fuzzy 3-normed due to [1].

Proposition (4.13):-

Let (X, N) be a fuzzy 3-normed space. A Cauchy sequence is convergent in a fuzzy 3-normed space (X, N) if and only if it has a convergent subsequence.

Proof:-

Suppose $\{x_n\}$ is a Cauchy sequence in (X, N) which is also convergent in it. Then, by using proposition (4.2)every subsequence of it will be convergent in X.

conversely, assume that $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ which converges to x $\in$X. Then $\lim_{k\to\infty} N(x_1, x_2, x_{n_k} - x, t) = 1$ for each $x_1, x_2 \in X$ and t>0. Since $\{x_n\}$ is Cauchy sequence then $\lim_{n\to\infty} N(x_1, x_2, x_{n+p} - x_n, s) = 1$ for each $x_1, x_2 \in X$, s>0 and p=1,2,….

Hence for each $x_1, x_2 \in X$

$$N(x_1, x_2, x_n - x, s + t) = N(x_1, x_2, x_n - x_{n_k} + x_{n_k} - x, s + t)$$
$$\geq \min\{N(x_1, x_2, x_n - x_{n_k}, s), N(x_1, x_2, x_{n_k} - x, t)\}$$

Hence, $\lim_{n\to\infty} N(x_1, x_2, x_n - x, s + t) = 1$ for each $x_1, x_2 \in X$ and s>0,t>0

Therefore $\{x_n\}$ is convergent.

Definition (4.14):-

A fuzzy 3-norm N_1 on a linear space X is said to be equivalent to a fuzzy 3-norm N_2 on X (denoted by $N_1 \sim N_2$) if there exist positive numbers a and b such that

$N_2(x_1, x_2, ax_3, t) \leq N_1(x_1, x_2, x_3, t) \leq N_2(x_1, x_2, bx_3, t)$, **for each $t \in R$.**

Proposition (4.15):-

The relation ~ defined as above is an equivalent relation.

<u>Proof:-</u>

(1) The relation ~ is reflexive, since

$$N_1(x_1,x_2,1.x_3,t) \le N_1(x_1,x_2,x_3,t) \le N_1(x_1,x_2,1.x_3,t)$$

(2) To prove ~ is symmetric, we assuming that

$$N_2(x_1,x_2,ax_3,t) \le N_1(x_1,x_2,x_3,t) \le N_2(x_1,x_2,bx_3,t)$$

holds and we have to show that there are two positive integer c and d such that

$$N_1(x_1,x_2,cx_3,t) \le N_2(x_1,x_2,x_3,t) \le N_1(x_1,x_2,dx_3,t)$$

we have $N_2(x_1,x_2,ax_3,t) \le N_1(x_1,x_2,x_3,t)$

$$N_2(x_1,x_2,x_3,\frac{t}{a}) \le N_1(x_1,x_2,x_3,t)$$

putting $s = \frac{t}{a} \Rightarrow as = t$, we get $N_2(x_1,x_2,x_3,s) \le N_1(x_1,x_2,x_3,as)$

$$= N_1\left(x_1,x_2,\frac{1}{a}x_3,s\right)$$

therefore

$$N_2(x_1,x_2,x_3,s) \le N_1\left(x_1,x_2,\frac{1}{a}x_3,s\right)\text{..........................} \quad \textbf{(4.1)}$$

Again, $N_1(x_1,x_2,x_3,t) \le N_2(x_1,x_2,bx_3,t)$

$$= N_2\left(x_1,x_2,x_3,\frac{t}{b}\right)$$

putting $\frac{bt}{a}$ for t, **we get** $N_1(x_1,x_2,x_3,\frac{bt}{a}) \le N_2\left(x_1,x_2,x_3,\frac{t}{a}\right)$

or $N_1(x_1,x_2,x_3,bs) \le N_2(x_1,x_2,x_3,s)$

or $N_1(x_1,x_2,\frac{1}{b}x_3,s) \le N_2(x_1,x_2,x_3,s)$..(4.2)

Combing ineq. (4.1) and ineq.(4.2) we get

$$N_1(x_1,x_2,\frac{1}{b}x_3,s) \le N_2(x_1,x_2,x_3,s) \le N_1(x_1,x_2,\frac{1}{a}x_3,s)$$

then $N_1(x_1,x_2,cx_3,s) \le N_2(x_1,x_2,x_3,s) \le N_1(x_1,x_2,dx_3,s)$

where $c = \frac{1}{b}$ and $d = \frac{1}{a}$

(3)To prove ~ transitive, let $N_0(x_1,x_2,ax_3,t) \le N(x_1,x_2,x_3,t) \le N_0(x_1,x_2,bx_3,t)$

$$N_1(x_1,x_2,cx_3,t) \le N_0(x_1,x_2,x_3,t) \le N_1(x_1,x_2,dx_3,t)$$

Then we have to show that there exist positive numbers e and f such that

$$N_1(x_1,x_2,ex_3,t) \le N\ (x_1,x_2,x_3,t) \le N_1(x_1,x_2,fx_3,t) \text{ for each } t \in R$$

Now $N_1(x_1,x_2,cx_3,t) \le N_0(x_1,x_2,x_3,t)$

$N_1(x_1,x_2,x_3,\frac{t}{c}) \le N_0(x_1,x_2,x_3,t)$

$N_1(x_1,x_2,ax_3,\frac{t}{c}) \le N_0(x_1,x_2,ax_3,t)$

$N_1(x_1,x_2,acx_3,t) \le N_0(x_1,x_2,ax_3,t)$

thus $N_1(x_1,x_2,acx_3,t) \le N(x_1,x_2,x_3,t) \le N_0(x_1,x_2,bx_3,t)$

Again $N_0(x_1,x_2,x_3,t) \le N_1(x_1,x_2,dx_3,t)$

$N_0(x_1,x_2,bx_3,t) \le N_1(x_1,x_2,bdx_3,t)$

So $N_1(x_1,x_2,acx_3,t) \le N(x_1,x_2,x_3,t) \le N_1(x_1,x_2,bdx_3,t)$

If we choose $ac = e$ and $bd = f$ then

$N_1(x_1,x_2,ex_3,t) \le N\ (x_1,x_2,x_3,t) \le N_1(x_1,x_2,fx_3,t)$

The following proposition shows the relation between convergent sequence in (X,N) and(X, $\|.,.,\|_\alpha$) for each $\alpha \in (0,1)$.

Proposition (4.16):-

Let (X,N) be a fuzzy 3-normed space satisfying the following conditions

(1) For each t>0, $N(x_1,x_2,x_3,t) > 0$ **implies** x_1,x_2,x_3 **are linearly dependent**

(2)For x_1,x_2,x_3 **are linearly independent,** $N(x_1,x_2,x_3,t)$ **is a continuous of** $t \in R$ **and strictly increasing in the subset** $\{t : 0 < N(x_1,x_2,x_3,t) < 1\}$ **of R.**

and $\{x_n\}$ **be sequence in X. Then** $\lim_{n\to\infty} N(x_1,x_2,x_n - x,t) = 1$ **for each** $x_1,x_2 \in X$ **and for each t>0 if and only if** $\lim_{n\to\infty} \|x_1,x_2,x_n - x\|_\alpha = 0$, for each $\alpha \in (0,1)$ **and for each** $x_1,x_2 \in X$.

Proof:-

Suppose $\lim_{n\to\infty} N(x_1,x_2,x_n - x,t) = 1$ **for each** $x_1,x_2 \in X$ **and for each t>0. Choose** $0 < \alpha < 1$, $x_1,x_2 \in X$ and $t > 0$, **Then exists K such that**

$N(x_1,x_2,x_n - x,t) > 1-\alpha$, for all $n \ge K$. **It follows that**

$\|x_1,x_2,x_n - x\|_{1-\alpha} \le t$, for each $n \ge K$. **Thus** $\lim_{n\to\infty} \|x_1,x_2,x_n - x\|_{1-\alpha} = 0$.

Conversely, choose $x_1,x_2 \in X$. **Let** $\lim_{n\to\infty} \|x_1,x_2,x_n - x\|_\alpha = 0$, for each $\alpha \in (0,1)$. **Fix** $\alpha \in (0,1)$ **and t>0. Then exists K such that**

$\|x_1,x_2,x_n - x\|_{1-\alpha} = \inf\{r : N(x_1,x_2,x_n - x,r) \ge 1-\alpha\} < t$, for all $n \ge K$

$N(x_1,x_2,x_n - x,t) \ge 1-\alpha$, for all $n \ge K$. **that is** $x_n \to x$ **in (X,N).**

Theorem (4.17):-

Let N_1 and N_2 **be two a fuzzy 3-norms on a linear space X, satisfying the following conditions**

(1) For each t>0, $N(x_1,x_2,x_3,t) > 0$ **implies** x_1,x_2,x_3 **are linearly dependent**

(2)For x_1, x_2, x_3 **are linearly independent,** $N(x_1, x_2, x_3, t)$ **is a continuous of** $t \in R$ **and strictly increasing in the subset** $\{t : 0 < N(x_1, x_2, x_3, t) < 1\}$ **of R.**

Then the two fuzzy 3-norm N_1 and N_2 **are equivalent if and only if their corresponding** α**-3-norms are equivalent for all** $\alpha \in (0,1)$.

<u>Proof:-</u>

First we suppose that N_1 and N_2 **are two equivalent fuzzy 3-norms in X. Thus there exist two positive constants a and b such that**

$N_2(x_1, x_2, ax_3, t) \le N_1(x_1, x_2, x_3, t) \le N_2(x_1, x_2, bx_3, t)$ **for each** $t \in R$. **Let** $\|.,.,.\|_\alpha^1$ **and** $\|.,.,.\|_\alpha^1$ **where** $\alpha \in (0,1)$ **are the corresponding** α**-3-norms of** N_1 and N_2 **respectively. First we have that**

$N_2(x_1, x_2, ax_3, t) \le N_1(x_1, x_2, x_3, t)$ for all $t \in R$

iff $\|x_1, x_2, x_3\|_\alpha^1 \le \|x_1, x_2, ax_3\|_\alpha^2$ **for all** $\alpha \in (0,1)$.

Suppose $N_2(x_1, x_2, ax_3, t) \le N_1(x_1, x_2, x_3, t)$ holds for each $t \in R$

Now,

$\|x_1, x_2, ax_3\|_\alpha^2 < t$, then, $\inf\{s : N_2(x_1, x_2, ax_3, s) \ge \alpha\} < t$

$\exists\, s_0 < t$ such that $N_2(x_1, x_2, ax_3, s_0) \ge \alpha$

$N_1(x_1, x_2, x_3, s_0) \ge \alpha$, $s_0 < t$ and $\alpha \in (0,1)$

$\|x_1, x_2, x_3\|_\alpha^1 \le s_0 < t$

$\|x_1, x_2, x_3\|_\alpha^1 \le \|x_1, x_2, ax_3\|_\alpha^2$ ....(4.3)

Next, we suppose that $\|x_1, x_2, x_3\|_\alpha^1 \le \|x_1, x_2, ax_3\|_\alpha^2$ **holds for each** $\alpha \in (0,1)$. **Now**

$r < N_2(x_1, x_2, ax_3, t)$

$r < \sup\left\{\alpha \in (0,1) \,\middle|\, \|x_1, x_2, ax_3\|_\alpha^2 \le t\right\}$

$\exists \alpha_0 \in (0,1)$ such that $r < \alpha_0$ and $\|x_1, x_2, ax_3\|_{\alpha_0}^2 \le t$

$\|x_1, x_2, x_3\|_{\alpha_0}^1 \le t$

$r < N_1(x_1, x_2, x_3, t)$

So,

$N_2(x_1, x_2, ax_3, t) \le N_1(x_1, x_2, x_3, t)$ **..(4.4)**

From (4. 3) and (4.4), it follows that

$N_2(x_1, x_2, ax_3, t) \le N_1(x_1, x_2, x_3, t)$ for all $t \in R$

iff $\|x_1, x_2, x_3\|_\alpha^1 \le \|x_1, x_2, ax_3\|_\alpha^2$ **for all** $\alpha \in (0,1)$

In similarly way we can verify that

$N_1(x_1, x_2, x_3, t) \le N_2(x_1, x_2, bx_3, t)$ for all $t \in R$

iff $\|x_1, x_2, bx_3\|_\alpha^2 \le \|x_1, x_2, x_3\|_\alpha^1$ **for all** $\alpha \in (0,1)$.

Suppose $N_2(x_1, x_2, ax_3, t) \le N_1(x_1, x_2, x_3, t)$ holds for each $t \in R$

Now,

$$\|x_1, x_2, x_3\|_\alpha^1 < t, \text{then, } \inf\{s : N_1(x_1, x_2, x_3, s) \geq \alpha\} < t$$

$$\exists\, s_0 < t \text{ such that } N_1(x_1, x_2, x_3, s_0) \geq \alpha$$

$$N_2(x_1, x_2, bx_3, s_0) \geq \alpha,\ s_0 < t \text{ and } \alpha \in (0,1)$$

$$\|x_1, x_2, bx_3\|_\alpha^2 \leq s_0 < t$$

$$\|x_1, x_2, bx_3\|_\alpha^2 \leq \|x_1, x_2, x_3\|_\alpha^1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.5)$$

Next, we suppose that $\|x_1, x_2, bx_3\|_\alpha^2 \leq \|x_1, x_2, x_3\|_\alpha^1$ **holds for each** $\alpha \in (0,1)$. **Now**

$$r < N_1(x_1, x_2, x_3, t), \text{then}, r < \operatorname{Sup}\left\{\alpha \in (0,1) \,\middle|\, \|x_1, x_2, x_3\|_\alpha^1 \leq t\right\}$$

$$\exists \alpha_0 \in (0,1) \text{ such that } r < \alpha_0 \text{ and } \|x_1, x_2, x_3\|_{\alpha_0}^1 \leq t$$

$$\|x_1, x_2, bx_3\|_{\alpha_0}^2 \leq t$$

$$r < N_2(x_1, x_2, bx_3, t)$$

$$N_1(x_1, x_2, x_3, t) \leq N_2(x_1, x_2, bx_3, t) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.6)$$

From (4.5) and (4.6), it follows that

$$N_1(x_1, x_2, x_3, t) \leq N_2(x_1, x_2, bx_3, t) \text{ for all } t \in R$$

iff $\|x_1, x_2, bx_3\|_\alpha^2 \leq \|x_1, x_2, x_3\|_\alpha^1$ **for all** $\alpha \in (0,1)$.

By combining the above results we have

$$N_2(x_1, x_2, ax_3, t) \leq N_1(x_1, x_2, x_3, t) \leq N_2(x_1, x_2, bx_3, t) \text{ for each } t \in R$$

if and only if $\|x_1, x_2, bx_3\|_\alpha^2 \leq \|x_1, x_2, x_3\|_\alpha^1 \leq \|x_1, x_2, ax_3\|_\alpha^2$ **for all** $\alpha \in (0,1)$

References:-

[1] A.Narayanan and S.Vijayabalaji, "Fuzzy n-normed linear space", International Journal of Mathematics and Mathematical Sciences, No.24,PP3963-3977,2005.

[2] Bag T. and Samanta S., "Finite Dimensional Fuzzy Normed Linear Spaces", J. Fuzzy Math., Vol.11, PP. 687-705, 2003.

[3] Felbin C., "Finite Dimensional Fuzzy Normed Linear Space", FSS, Vol. 48, PP. 239-248, 1992.

[4] H. Gunawan and M. Mashadi, "On n-normed spaces", Int. J. Math.Sci.Vol.27, PP.631- 639,2001.

[5] Jin-Xuan F. and Jun-Hua L. ,"Fuzzy Norm of a Linear Operator and Space of Fuzzy Bounded Linear Operator", J. Fuzzy, Math. Vol. 7, PP. 755-764, 1999.

[6] L. Fatemeh and N. Kourosh, "Compact operators defined on 2-Normed and 2-probabilistic Normed spaces", Hindaw Publishing Corporation Mathematical in Engineering, Vol.2009, Article ID 950234, 17 pages, 2009.

[7] Morsi N., "On Fuzzy Pseudo-Normed Vector Spaces", FSS, Vol. 27, PP. 351-372, 1988.

[8] S.Gahler, "Lineare 2-normierte Raume", Mathematische Nachrichten, Vol.28,PP.1-43, 1964.

[9] S.Vijayabalaji and N.Thillaigovindan, "Complete Fuzzy n-normed linear space", Journal of Fundamental Sciences, Vol.3, PP.119-126,2007.

[10] S.Vijayabalaji and N.Thillaigovindan, " Fuzzy n-inner product space", Bull. Kerean Math. Soc., Vol. 43, PP. 447-459, 2007.

[11] Xiao J. and Zhu X., "Fuzzy Normed Space of Operators and its Completeness", FSS, Vol. 133, PP. 389-399, 2003.