Bayesian Smoothing of Discrete -Time Signals with Application

Kurdistan I. Mawlood\College of Administration and Economic\ Salahaddin University-Erbil

Abstract

In this work we deal with Bayesian smoothing for a time varying system, to find smoothing estimators of signals in present of noise. The smoothing procedure is re-estimating a signal after adding new observations, or in the light of more new observations. This study began with modeling the fixed point smoothing process using Bayesian updating process, and then using this model to find smoothing estimators for the wind speed in Erbil city at the fixed point (t=15), for three month's (October, November, December) using (MATLAB 7). The results showed that the variances are reduced by adding any new observation; this demonstrates that the method works effectively.

المستخلص

في هذا البحث نتعامل مع تمهيد بيزلايجاد مقدرات تمهيد بيزللاشارة مع وجود ضوضاء، تمهيد هي عملية تخليص البيانات من الاخطاء المتبقي فيها من خلال اعادة النظر في عملية التقدير بالاعتماد على بيانات جديدة للحصول على قيم مقدرة أكثر دقة، أي أن التمهي د هي عملية تصحيح مقدرات الماضية. في البداية قمنا ببناء نموذج لعملية التمهيد للنقطة الثابتة بأستخدام طريقة بيز وبعد ذلك تطبيق النموذج على بيانات حقيقية تمثل سرعة الرياح لأ يجاد مقدرات تمهيد للنقطة الثابتة (١٥) بأستخدام برنامج (ماتلاب ٧). وأظهرت النتائج بان التميد يقلل التباين بعد اضافة كل مشاهدة جديدة، وهذا يدل على كفاءة النموذج.

Introduction

The problem of estimating the parameters of the signal is important in many applications and in most of the practical situations the signals are corrupted by additive noise and must be enhanced. The development of linear dynamic recursive filtering smoothing and prediction has been lead by Kalman (1960) and Kalman and Bucy (1961)[Kalman 1960],[Kalman and Bucy 1961], but in non Bayesian way. In (1993) Gamerman and Megon are disscused smoothing and filtering through the levels of Dynamic hierarchical models [Gamerman and Megon 1993]. In (1996) Jalil has derived a general fixed lag smoothing recursive relation [Jalil 1996] and recently the smoothing theory has been applied to several statistical problems[Wang 2003],[Zuccolo1, Maule and Gregori 2005][Thomas and Ghosal 2011]. Smoothing is kind of feed back filtering produces for signal values of the posterior distribution that supplant the current estimator it means that inference is more reasonably conducted by smoothing than by filtering, because smoothing procedure correct the past estimators. In this paper we introduce such an enhancement algorithm based on fixed point Bayesian smoothing with assumption that the noise process is white, because is an adequate situation in many signal processing applications, and the estimation is optimal and sufficient if normality holds.

Types of smoothing [Anderson 1979]

1- The Fixed-Point Smoothing

Estimate the system state x_t at a fixed point t based on measurement data X_k where $k = t+1, t+2, ...$, or the fixed point output is $\hat{x}_{t|t+1}, \hat{x}_{t|t+2}, ...$ for a fixed t. This type of smoothing is used when the state estimate is needed at only one time, such as for estimating the miss distance between two objects that are being traced by radar.

2- The Fixed-Lag Smoothing

Provides the estimate of signal or state x_{T-N} for a fixed lag N , and the fixed lag output is $\hat{x}_{T-N|T}$, for $N < T$.

Given a signal S_t and observation Y_t a dynamic linear system is simply a process that can be described by the following two equations [Harrison and Stevens 1976]

State equation

$$S_t = S_{t-1} + w_t$$

Observation equation

$$Y_t = S_t + \varepsilon_t$$

where the noises w_t and ε_t are given by:

$$w_t \sim N(0, W)$$

$$\varepsilon_t \sim N(0, V)$$

in the current time (t), $(s_t \mid D_t) \sim N(\hat{s}_t, C_t)$, where [Grewal and Andrews 2001]:

$$\begin{aligned} G_t &= P_t^-\left(P_t^- + V_t\right)^{-1} \\ \hat{s}_t &= \hat{s}_t^- + G_t\left(y_t - \hat{s}_t^-\right) \\ C_t &= \left(I - G_t\right)P_t^- \end{aligned} \quad ...(1)$$

Now we calculate the smoothing estimators and as follows:

One Step-smoothing

We mean by one step-smoothing, estimating the signal S_t after one new observation Y_{t+1} becomes available, using Bayes theorem [Box and Tiao 1973] the condition probability of one step smoothing is given by:

$$P(s_t \mid D_{t+1}) \propto P(y_{t+1} \mid s_t) P(s_t \mid D_t) \quad \ldots(2)$$

where:

$$D_{t+1} = y_1, y_2, y_3, ..., y_t, y_{t+1}$$

or the process is given by:

$$y_{t+1} = s_{t+1} + \varepsilon_{t+1}$$
$$s_{t+1} = s_t + w_{t+1} \qquad \ldots(3)$$

And we can get

$$E(y_{t+1}) = s_t \quad , V(y_{t+1}) = W + V = d_{t+1}$$

$$P(s_t \mid D_{t+1}) \propto Exp - \frac{1}{2}\left\{ \frac{(y_{t+1} - s_t)^2}{d_{t+1}} + \frac{(s_t - \hat{s}_t)^2}{C_t} \right\}$$
$$= Exp - \frac{1}{2}\left\{ \frac{(s_t - \hat{s}_{t(+1)})^2}{C_{t(+1)}} \right\} \qquad \ldots (4)$$

where:

$$\hat{s}_{t(+1)} = \hat{s}_t + C_t d^{-1}(y_{t+1} - \hat{s}_t) \qquad \ldots(5)$$

$$C_{t(+1)} = C_t - C_t^2 d^{-1} \qquad ...(6)$$

$$d^{-1} = (d_{t+1} + C_t)^{-1}$$

Or:

$$P(s_t \mid D_{t+1}) \sim N(\hat{s}_{t(+1)}, C_{t(+1)})$$

Two Step-smoothing

In the same way we find the distribution of two step-smoothing as follows, or after available two new observations:

$$E(y_{t+2}) = s_t \quad , V(y_{t+2}) = 2W + V = d_{t+2} \qquad \ldots(7)$$

$$P(s_t \mid D_{t+2}) \propto Exp - \frac{1}{2}\left\{\sum_{i=1}^{2}\frac{(y_{t+i}-s_t)^2}{d_{t+i}} + \frac{(s_t-\hat{s}_t)^2}{C_t}\right\}$$
$$= Exp - \frac{1}{2}\left\{\frac{(s_t-\hat{s}_{t(+2)})^2}{C_{t(+2)}}\right\} \qquad ..(8)$$

where:

$$\hat{s}_{t(+2)} = \hat{s}_t + C_t d^{-1}\sum_{i=1}^{2}(y_{t+i}-\hat{s}_t)$$
$$C_{t(+2)} = C_t - C_t^2 d^{-1}$$
$$d^{-1} = \left(\sum_{i=1}^{2} d_{t+i} + C_t\right)^{-1}$$
$$= \left(\sum_{i=1}^{2}(iW+V) + C_t\right)^{-1} \qquad \ldots(9)$$

Or:

$$P(s_t \mid D_{t+2}) \sim N(\hat{s}_{t(+2)}, C_{t(+2)})$$

K Step-smoothing

Then we can generalize this process for K step-smoothing as follows:

$$P(s_t \mid D_{t+k}) \propto Exp - \frac{1}{2}\left\{\sum_{i=1}^{k}\frac{(y_{t+i}-s_t)^2}{d_{t+i}} + \frac{(s_t-\hat{s}_t)^2}{C_t}\right\}$$
$$= Exp - \frac{1}{2}\left\{\frac{(s_t-\hat{s}_{t(+k)})^2}{C_{t(+k)}}\right\} \qquad ...(10)$$

where:

$$\hat{s}_{t(+k)} = \hat{s}_t + C_t d^{-1}\sum_{i=1}^{k}(y_{t+i}-\hat{s}_t) \qquad \ldots(11)$$

$$C_{t(+k)} = C_t - C_t^2 d^{-1} \qquad \ldots(12)$$

$$d^{-1} = \left(\sum_{i=1}^{k} d_{t+i} + C_t \right)^{-1}$$
$$= \left(\sum_{i=1}^{k} (iW + V) + C_t \right)^{-1}$$

Or:

$$P(s_t \mid D_{t+k}) \sim N(\hat{s}_{t(+k)}, C_{t(+k)})$$

We observe that the K-step smoothing posterior mean is combination of two terms, the first term is mean of the signal S_t at the time (t) and the second term is a correction term (or error smoothing) represent the difference between the observation and posterior mean multiplied by $\left(C_t \sum_{i=1}^{k} d^{-1} \right)$. And K-step smoothing posterior variance is equal to current posterior variance C_t minus small quantity $C_t^2 d^{-1}$; it means that the variance is reduced by smoothing. Smoothing will give better estimates more accurate than filtering or more reliable estimates with smaller error variance [Böhning and Kuhnert 2011].

Application

The data set consists of 92 observations represents the wind speed of three months (October, November December) in Erbil city sampled from Bahrka weather station during the period (1/10/2011-31/12/2011), after estimating the signals up to time (day) t=15 for each month using the initial values of wind speed and its variance of the first 15 observations, we find the smoothing estimators for the fixed time t=15 after utilization of the new observations $(y_{16}, y_{17}, ..., y_{31})$ or estimating $(s_{15|16}, s_{15|17}, ..., s_{15|31})$ and there variances, using the equations (11,12), and as follows:

smoothing estimations for a fixed point (15) in the October ($\hat{s}_0 = 6.5773, \hat{c}_0 = 1.0865, V = 0.05$)

Table (1); smoothing results for a fixed point (15) after adding the new observations $(y_{16}, y_{17}, ..., y_{31})$ in the October

t	$\hat{C}_t$	$\hat{S}_t$	k	$\hat{C}_{15(+k)}$	$\hat{S}_{15(+k)}$
1	0.0578	7.5502	16	0.0278	5.8442
2	0.036809	8.5903	17	0.0278	5.8669
3	0.031201	7.5242	18	0.0277	5.8802
4	0.029212	7.3554	19	0.0277	5.8842
5	0.028439	6.9883	20	0.0276	5.8907
6	0.028128	6.6256	21	0.0275	5.9044
7	0.028001	6.6499	22	0.0274	5.9586
8	0.027949	6.8485	23	0.0273	5.9669
9	0.027928	6.5435	24	0.0272	5.9351
10	0.027919	6.4147	25	0.0271	6.1395
11	0.027915	6.7111	26	0.0269	6.1327
12	0.027914	6.48	27	0.0268	6.1472
13	0.027913	6.0999	28	0.0267	6.095
14	0.027913	5.8735	29	0.0265	6.1116
15	0.027913	5.8443	30	0.0264	6.1664
			31	0.0262	6.1651

b- Smoothing estimations for a fixed point (15) in the November ($\hat{s}_0 = 5.0483, \hat{c}_0 = 2.4164, V = 0.045$)

Table (2); smoothing results for a fixed point (15) after adding the new observations $(y_{16}, y_{17}, ..., y_{30})$ in the November

t	$\hat{C}_t$	$\hat{S}_t$	k	$\hat{C}_{15(+k)}$	$\hat{S}_{15(+k)}$
1	0.053177	5.2591	16	0.0251	4.8142
2	0.033374	7.177	17	0.025	4.8174
3	0.028162	7.2507	18	0.025	4.8156
4	0.026322	6.7835	19	0.0249	4.8073
5	0.025608	5.8171	20	0.0249	4.8056
6	0.02532	5.1077	21	0.0248	4.7812
7	0.025203	5.1777	22	0.0247	4.7679
8	0.025155	5.3067	23	0.0247	4.7346
9	0.025135	4.6549	24	0.0246	4.6965
10	0.025127	4.6653	25	0.0245	4.6487
11	0.025124	4.2507	26	0.0244	4.6452
12	0.025123	4.0129	27	0.0243	4.6132
13	0.025122	3.7761	28	0.0242	4.565
14	0.025122	4.1365	29	0.0241	4.5072
15	0.025122	4.8066	30	0.024	4.4045

Smoothing estimations for a fixed point (15) in the December

($\hat{s}_0 = 5.0929, \hat{c}_0 = 6.8626, V = 0.035$)

Table (3); smoothing results for a fixed point (15) after adding the new observations $(y_{16}, y_{17}, \ldots\ldots, y_{31})$ in the December

t	$\hat{C}_t$	$\hat{S}_t$	k	$\hat{C}_{15(+k)}$	$\hat{S}_{15(+k)}$
1	0.026054	3.8228	16	0.0195	5.1656
2	0.021936	3.7512	17	0.0195	5.1693
3	0.020485	3.5142	18	0.0195	5.1678
4	0.019922	3.1106	19	0.0194	5.1607
5	0.019696	3.8805	20	0.0194	5.1486
6	0.019603	4.3607	21	0.0194	5.1283
7	0.019565	4.1461	22	0.0193	5.1025
8	0.01955	4.0227	23	0.0193	5.0636
9	0.019543	4.7622	24	0.0193	5.033
10	0.019541	6.5744	25	0.0192	4.9829
11	0.01954	8.2998	26	0.0192	4.944
12	0.019539	7.9806	27	0.0191	4.8718
13	0.019539	6.7329	28	0.0191	4.7951
14	0.019539	5.167	29	0.019	4.8933
15	0.026054	3.8228	30	0.019	5.0013
			31	0.0195	5.1656

Conclusion

We observe that due to smoothing the variances for three months are less than the variance at the fixed time t=15, and the variance reduced by adding every new observation, this means that the smoothing process can improve the accuracy of estimates generally even when k=1 or we have one new observation.

References

[1] Anderson B,D.O. and Moore J.B., Optimal Filtering, Englewood cliffs, NJ: Prentice Hall 1979.

[2] Böhning D. and Kuhnert R. "Capture–recapture estimation by means of empirical Bayesian smoothing with an application to the geographical distribution of hidden scrapie in Great Britain " Applied Statistics, Part 5, No. 60, pp. 723–741, 2011.

[3] Box G. E. and Tiao G. C., Bayesian Inference in Statistical Analysis, Addison-Wesley Publishing company 1973.

[4] Dwaning D. J., Pike D. H. and Morriso G. W. "Application of the Kalman Filter to Inventory Control" Techonometrics, Vol. 22, No. 1, pp 17-22, Feb. 1980.

[5] Gamerman D. and Megon H.S. "Dynamic Hierarchical Models" Journal of Royal Statistical Society, Series B, Vol. 55 No.3, pp. 629–642, 1993.

[6]Grewal M. S. and Andrews A. P. Kalman Filtering: Theory and Practice Using MATLAB, 2nd Edition, John Wiley & Sons, INC, 2001.

[7] Harrison P. J. and Stevens C. F. "Bayesian Forecasting" Journal of Royal Statistical Society, Series B, Vol. 38. pp 205-246, May 1976.

[8] Jalil T.S. "Bayesian Smoothing with Simulation" Natural and Engineering Sciences, Vol. 24, No. 1, pp. 75-78, 1996.

[9] John T. W. and Subhashis G. "Bayesian smoothing of photon-limited images with applications in astronomy" Journal of Royal Statistical Society, Series B, No.73, Part 4, pp. 579–599, 2011.

[10] Kalman R. E. "A New Approach to Linear Filtering and Prediction Problems" Journal of Basic Engineering", Vol. 82, pp 35-45, March 1960.

[11] Kalman R. E. and Bucy R. S. "New in Linear Filtering and Prediction Problems" Journal of Basic Engineering", Vol. 83, pp 95-108, March 1961.

[12] Schhick I. C. and Mitter S. K. "Robust Recursive Estimation in the Presence of Heavy-Tailed Observation Noise" The Annals of Statistics, Vol. 22, No. 2, pp 1045-1080, 1994.

[13] Wang G. and He S. "A Quantitative Study on Detection and Estimation of Weak Signal by Using Chaotic Doffing Oscillators" IEEE Trans Circuits Syst. I, Vol. 50. pp. 945-953, July 2003.

[14] Zuccolo1 L., Maule M. M. and Gregori D. "An Epidemiological Application of a Bayesian Nonparametric Smoother Based on a GLMM with an Autoregressive Error Component " Metodoloˇski zvezki, Vol. 2, No. 2, 259-270, 2005

البرنامج المستخدم

```
y=[];

y1=[];

y=y'

y1=y1'

duration=length(y);

dt=1;
```

```
V=
W=0.2*V; % initial estimation covariance
x=mean(y)
P=var(y)
xhat = x; % initial state estimate
Inn = zeros(size(V));
del = []; % true delition array
delhat = []; % estimated delition array
delmeas = []; % measured delition array
Counter = 0;
for t = 1 : dt: duration,
   Counter = Counter + 1;
   x = x + 0.2;
   % Innovation
   Inn = y(t) -xhat;
   % Covariance of Innovation
   s = P + V;
   % Gain matrix
   K = P * inv(s);
   kk(:,t)=K;
   % State estimate
   xhat = xhat + K * Inn;
   x_hat(:,t)=xhat;
   % Covariance of prediction error
   P = P + W - P  * inv(s) *  P;
  %P = (1-K)P
```

```
   PP(t)=P;
   del = [del; x(1)];
   delmeas = [delmeas; y(t)];
   delhat = [delhat; xhat(1)];
end
kk;
x_hat
PP
x_hat=x_hat'
PP=PP'
% Plot the results
t = 1 : dt : duration;
t = t';
plot(t,delhat,'g',t,delmeas,'b');
grid;
xlabel('Time (sec)');
ylabel('Time Delay (sec)');
title('Kalman Filter Performance');
for m =1:16
   d(m)=(W*m*(m+1)/2)+V*m+P;
   xx(m)=xhat*m;
end
xx=xx';
d=d';
C=P-(P^2)*d
m=1;
```

```
R(m)=y1(1);
while m <16
   R(m+1)=R(m)+y1(m+1);
   m=m+1;
end
R=R';
RR=R-xx;
for i=1:16
   q(i)=d(i)*RR(i);
end
Shat=xhat+P*q;
Shat=Shat'
```

References