Certain Properties of Fuzzy Subgroup

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Abstract

In this paper, we study certain properties of fuzzy subgroup many basic propertiesin group theory carried over on fuzzy group

الخلاصة

يهدف البحث الى دراسة بعض الخواص في نظرية الزمر و التي يمكن تطبيقها على الزمر الضبابية , ولقد وضعنا بعض الشروط اللازمة من اجل تحقيق ذلك

Introduction

Zadein in 1965 [9] *introduced the concept of Fuzzy set ,Anthony and Sherwood* [1] *introduced the concept of fuzzy group , in this paper, we study certain properties of fuzzy subgroup. We notice that many basic properties in group theory carried over on fuzzy group.*

We will use the symbol (*****) *to indicate the end of the proof.*

1- Fuzzy set :

In this section we shall start to introduce the concepts about fuzzy sets an the basic definitions with some examples, also we shall give some important definitions and properties with operation on fuzzy set which are used in the next .

Definition (1.1): [3]

let A be anon – empty set, the function $M : A \to [0,1]$ is called fuzzy set in A .

Example (1.2):

Let X be real number R , then the function $M : R \to [0,1]$ such that

$$M(x) = \begin{cases} 1 - \frac{1}{x} & if \quad x > 1 \\ 0 & if \quad x \le 1 \end{cases}$$

Is fuzzy set .

Remark (1.3) :

If we want to know the difference between fuzzy sets and ordinary sets we observe that when A is a set in ordinary sense of the term , so it membership function can take only two values 0 and 1 with a characteristic function :

$$A(x) = \begin{cases} 1 & if\ x \in A \\ 0 & if\ x \notin A \end{cases}$$

Then $A(x) \in \{0,1\}$ for all $x \in G$.

While if A is a fuzzy set in G then $0 \leq A(x) \leq 1$ for all $x \in G$ thus the ordinary sets become a special case of fuzzy sets .

Next we shall give some definitions and concepts related to fuzzy subsets of G .

Definition (1.4) : [7]

Let $Xt : A \to [0,1]$, $x \in A$ be a fuzzy set of A and $t \in [0,1]$ defined by $Xt(y)$ for all $y \in A$

$$Xt(y) = \begin{cases} t & if\ x = y \\ 0 & if\ x \neq y \end{cases}$$

Then Xt is called a fuzzy singletion .

Example (1.5) :

Let $x = 2 \in Z$, $t = \frac{1}{2} \in [0,1]$.

Let $Xt :\ Z \to [0,1]$

$2\frac{1}{2} : Z \to [0,1]$ then

$$2\frac{1}{2}(4) = 0 \qquad x \neq y \qquad x = 2 \qquad y = 4$$

$$2\frac{1}{2}(2) = \frac{1}{2} \qquad , \qquad x = y \neq z$$

Definition (1.6) : [6]

Let M_1, M_2 be two fuzzy subset of A then

i) $M_1 = M_2$ iff $M_1(x) = M_2(x), \forall x \in A$.

ii) $M_1 \subseteq M_2$ iff $M_1(x) \leq M_2(x), \forall x \in A$.

iii) if $M_1 \subseteq M_2$ and there exists, $x \in A$ such that $M_1(x) < M_2(x)$, then we write $M_1 \subset M_2$.

Example (1.7) :

Let $M_1 : Z \to [0,1]$ such that .

$$M_1(x) = \begin{cases} \frac{1}{3} & if x \in Z_e \\ \frac{1}{4} & if x \in Z_o \end{cases}$$

And

$$M_2(x) = \begin{cases} \frac{1}{2} & if\ x \in Z_e \\ \frac{1}{3} & if\ x \in Z_o \end{cases}$$

Then $M_1 \subset M_2$. Where Z_e is even integer number and Z_o is odd integer number

Definition (1.8) : [7]

Let M_1, M_2 be two fuzzy set in A then :

i) $(M_1 \cup M_2)(x) = Max\{M_1(x), M_2(x)\}$, for each $x \in A$.

ii) $(M_1 \cap M_2)(x) = Min\{M_1(x), M_2(x)\}$, for each $x \in A$.

Notice that $(M_1 \cup M_2)$, $(M_1 \cap M_2)$ are fuzzy set in A.

If we generalize this definition by a collection of fuzzy sets then :

$\left(\underset{\alpha\in\Omega}{\cup} M_1^{\alpha}\right)(x) = \sup\{M_1^{\alpha}(x) | \alpha \in \Omega\}$, for each $x \in A$.

$\left(\underset{\alpha\in\Omega}{\cap} M_1^{\alpha}\right)(x) = \inf\{M_1^{\alpha}(x) | \alpha \in \Omega\}$, for each $x \in A$.

Which are also fuzzy sets in A.

Example (1.9):

In the example (1.7)

If $x \in Ze$ then $(M_1 \cup M_2)(x) = \max\{M_1(x), M_2(x)\}$

$$= \max\left\{\frac{1}{3}, \frac{1}{2}\right\} = \frac{1}{2}$$

If $x \in Zo$ then $(M_1 \cup M_2)(x) = \max\{M_1(x), M_2(x)\}$

$$= \max\left\{\frac{1}{4}, \frac{1}{3}\right\} = \frac{1}{3}$$

Now

If $x \in Ze$ then $(M_1 \cap M_2)(x) = \min\{M_1(x), M_2(x)\}$

$$= \min\left\{\frac{1}{3}, \frac{1}{2}\right\} = \frac{1}{3}$$

If $x \in Zo$ then $(M_1 \cup M_2)(x) = \min\left\{\frac{1}{4}, \frac{1}{3}\right\} = \frac{1}{4}$

Hence $(M_1 \cap M_2)(x) = \begin{cases} \frac{1}{3} \ldots . if x \in Z_e \\ \frac{1}{4} \ldots . if x \in Z_o \end{cases}$

Notes (1.10) :

1) Denote to $[0,1]$ by I i.e $I = [0,1]$

2) $I^A = \{M : A \to I fuzzy.set\}$.

2- Fuzzy subgroups of a Group :

This section consists the concepts of the fuzzy groups which was coined by Rosenfeld [8], who found many basic properties in group theory carried over on fuzzy group and in the same way applied to another algebraic structures like rings, ideals, modules and so on (See [4] , [2]) .

Definition (2.1) : [4]

Let (G , $\bullet$) be a semi – group i.e. , $\bullet : G \times G \to G$

Such that $\mathrm{Im}(\bullet) \subseteq G$ and let $M_1, M_2 \in IG$.

Then, for each $x \in G$ we define :

$$(M_1.M_2)(x) = \begin{cases} \sup\{\min\{M_1(x_1), M_2(x_2)\} \mid x = x_1 x_2, x_1, x_2 \in G\}\} & if \;\; x \in \mathrm{Im}(\bullet) \\ 0 & otherwise \end{cases}$$

Its clear that, M_1, M_2 are fuzzy subsets of G. And if we take a collection $\{M \mid \alpha \in \Omega\}$ of fuzzy subsets then for each $x \in G$:

$$(\prod_{\alpha \in \Omega} M_\alpha)(x) = \begin{cases} \sup\{\inf\{M_\alpha(x_\alpha) \mid x = \prod_{\alpha \in \Omega} x_\alpha, x_\alpha \in G \quad \forall \alpha \in \Omega\}\} \\ 0 & otherwise \end{cases}$$

Proposition (2.2) : [4]

Let $(G,\bullet)$ be a semi – group Xt, Ys be two fuzzy singletons , where $X, Y \in G$ and $t, s \in [0,1]$. Then $xt., ys = (xy)_r$ where $r = \min\{t, s\}$.

Proof : For each $Z \in G$.

$$(x_t \cdot y_s)(z) = \begin{cases} \sup\{\min\{x_t(z_1), y_s(z_2) \mid z = z_1 z_2, and\ z_1 \cdot z_2 \in G\}\} & if\ z \in \operatorname{Im}(\bullet) \\ 0 & otherwise \end{cases}$$

$$= \begin{cases} \min\{t, s \mid x = z, and y = z_2\} = r & if\ z \in \operatorname{Im}(\bullet)\ and z = x.y \\ 0 & othewise \end{cases} \quad (1)$$

On the other hand,

$$((x.y)r)(z) = \begin{cases} r & if\ z = xy \\ 0 & otherwise \end{cases} \quad (2)$$

From (1) and (2) we have $Xt.Ys = (xy)_r$.

Where $r = \min\{t, s\}$

Definition (2.3) : [4]

Let G be a non – empty set and closed a binary operation ($\bullet$) and $M \in I^G$ such that $M \neq \phi$, where ϕ is the empty fuzzy set defined by $\phi(x) = 0$ for each $x \in G$. Then $(M,\bullet)$ is called closed if and only if $M.M \subseteq M$.

Proposition (2.4) : [4]

Let $M \in I^G$ and $M \neq \phi$. Then the following statements are equivalent :

i) $(M,.)$ is closed .

ii) For any $Xt, Yt \subseteq M$. Then $Xt.Yt \subseteq M$ for each $x, y \in G$.

iii) $M(xy) \geq \min\{M(x), M(y)\}$ for each $x, y \in G$.

Now we ready to define a fuzzy subgroup of a group :

Definition (2.5) : [2]

Let $(G,.)$ be a group and $M \in I^G$ such that $M \neq \phi$. $(M(x) \neq o \forall x \in G)$. Then M is called a fuzzy subgroup of G if and only if for each $X, Y \in G$.

1) $M(xy) \geq \min\{M(x), M(y)\}$.

2) $M(x) = M(x^{-1})$.

Remark (2.6):

Let e be the identity element of group $(G,.)$ then $M(e) \geq M(x) \forall x \in G$.

Proof : $M(e) = M(xx^{-1}) \geq \min\{M(x), M(x^{-1})\}$

$$= \min\{M(x), M(x)\}$$

$$= M(x)$$

$M(e) \geq M(x)$.

Example (2.7) :

The group (G,.) such that $G = \{1, -1, i, -i\}$ and $(.)$ multiplication ordinary and $M : G \rightarrow I$ such that .

1) $M(x) = \begin{cases} 1 & if\ x = 1, -1 \\ \frac{1}{2} & if\ x = i, -i \end{cases}$

Then M is not fuzzy subgroup .

2) $M(x) = \begin{cases} 1 & if\ x = 1, i \\ \frac{1}{2} & if\ x = -1, -i \end{cases}$

Then M is not fuzzy subgroup. Because $M(i) \neq M(i^{-1}), \left(M(i) = 1, M(i^{-1}) = M(-i) = \frac{1}{2} \right)$.

3) $M(x) = \begin{cases} \frac{1}{2} & if\ x = 1, -1 \\ \frac{3}{2} & if\ x = i, -i \end{cases}$

Then M is not fuzzy subgroup .

Because $M(e) = M(1) = \frac{1}{2} \neq M(i) = \frac{3}{2}$.

Definition (2.8):[5]

Let G be a group and M be a fuzzy subgroup of G. then we define the following:

1) $M^* = \{x \in G | M(x). > 0\}$ is called the support of M. also $M^* = \bigcup_{t \in (0,1]} M_t$

2) $M_* = \{x \in G | M(x) = M(e)\}$. It is easy to show that M^* and M_* are subgroups of G.

Definition (2.9) : [5]

M is called torsion fuzzy subgroup of G iff for each fuzzy singleton $Xt \leq M$ with $t > 0$, there exists $n \in N$ such that $n(Xt) = 0t$.

Now we explain the relation between torsion fuzzy subgroup and the subgroups Mt, M^* and M_* by the following proposition :

Proposition (2.10) : [5]

Let M be a fuzzy subgroup of G. Then :

1) M is torsion , iff M^* is torsion subgroup of G.

2) M is torsion , iff Mt is torsion subgroup of G.

3) If M is torsion , then M_* is torsion subgroup of G.

Now we are ready to introduce the concept of torsion – free fuzzy subgroup:

Definition (2.11):

A fuzzy subgroup M of G is called torsion – free iff for each $Xt \subseteq M$ with $t > 0$ and for each $n \in N$ such that $n(Xt) = 0t$, then $Xt = 0t$.

Proposition (2.12):

Let M be a fuzzy subgroup of G. Then :

1) M is torsion – free iff M^* is torsion - free subgroup of G.

2) M is torsion – free iff Mt is torsion - free subgroup of G for each $t \in (0,1]$.

Proof :

1) suppose that M is torsion – free fuzz subgroup iff $(\forall \chi_t \subseteq M)$, with $t > o$ and $(\forall n \in N)$ such that $n(\chi_t) = 0_t$, then $Xt = 0_t$.

Now $Xt = 0t \subseteq M$ iff $M(x) \geq (t > 0)$ iff $X = 0 \in M^*$.

Also $n(Xt) = Ot$ iff

$x_t + x_t + \ldots + x_t = o_t$ iff $(x + x + \ldots\ldots + x)_t = o_t$ iff $(nx)t = Ot$ iff $nx = o \forall n \in N$ iff M^* is torsion – free subgroup of G.

Similarly we can prove part (2) .

3- Some properties About fuzzy subgroup :

In this section we will give and prove some basic properties about fuzzy subgroup of an abelian group $(G,.)$

That we will need it in the next sections .

We start the following propositions :

Proposition (3.1) :

Let M_1, M_2 be two fuzzy subgroup of G then $M_1 \cap M_2$ is fuzzy subgroup .

Proof : Let $x, y \in G$.

$$(M_1 \cap M_2)(x.y) = \min\{M_1(xy), (M_2(xy)\}$$
$$\geq \min\{\min\{M_1(x), M_1(y)\}, \min\{M_2(x), M_2(y)\}\}$$
$$= \min\{M_1(x), M_1(y), M_2(x), M_2(y)\}$$
$$= \min\{\min\{M_1(x), M_2(y)\}, \min\{M_1(x), M_2(y)\}\}$$
$$= \min\{(M_1 \cap M_2)(x), (M_1 \cap M_2)(y)\}$$

$$(M_1 \cap M_2)(x) = \min\{M_1(x), M_2(x)\}$$
$$= \min\{M_1(-x), M_2(-x)\}$$
$$= (M_1 \cap M_2)(-x)$$

$(M_1 \cap M_2)$ fuzzy subgroup .

Proposition (3.2):

Let M_1 and M_2 be two fuzzy subgroup of G and $M_1 \subseteq M_2$ or $M_2 \subseteq M_1$ then $M_1 \cup M_2$ is fuzzy subgroup of G .

Proof : suppose $M_1 \subseteq M_2$.

Then $(M_1 \cup M_2)(x) = \max\{M_1(x), M_2(x)\} = M_2(x) \forall x \in G$.

Let $x, y \in G$.

$$(M_1 \cup M_2)(xy) = \max\{M_1(xy), M_2(xy)\}$$
$$= M_2(xy) \geq \min\{M_2(x), M_2(y)\} \qquad (1)$$

$$\min\{(M_1 \cup M_2)(x), (M_1 \cup M_2)(y)\} = \min\{\max\{M_1(x), M_2(x)\}, \max\{M_1(y), M_2(y)\}\}$$
$$= \min\{M_2(x), M_2(y)\} \qquad (2)$$

From (1) and (2)

$(M_1 \cup M_2)(xy) \geq \min\{(M_1 \cup M_2)(x),(M_1 \cup M_2)(y)\}$

$\therefore M_1 \cup M_2$ fuzzy subgroup .

Proposition (3.3) :

Let $\{M_\alpha\}_{\alpha\in\Omega}$ be a family of fuzzy subgroup of G .

Then $\bigcap_{\alpha\in\Omega} M_\alpha$ is a fuzzy subgroup of G .

Proof : for each $x, y \in G$ we have :

i) $\left(\bigcap_{\alpha\in\Omega} M_\alpha\right)(x) = \inf\{M_\alpha(x) | \alpha \in \Omega\} = \inf\{M_\alpha(-x) | \alpha \in \Omega\} = \left(\bigcap_{\alpha\in\Omega} M_\alpha\right)(-x)$

$$\geq \inf\{\min\{M_\alpha(x), M_\alpha(y)\} | \alpha \in \Omega\}$$

$$= \min\{\inf\{M_\alpha(x) | \alpha \in \Omega\}, \inf\{M_\alpha(y) | \alpha \in \Omega\}\}$$

$$= \min\left\{\left(\bigcap_{\alpha\in\Omega} M_\alpha\right)(x), \left(\bigcap_{\alpha\in\Omega} M_\alpha\right)(y)\right\}$$

Hence $\left(\bigcap_{\alpha\in\Omega} M_\alpha\right)(xy) \geq \min\left\{\left(\bigcap_{\alpha\in\Omega} M_\alpha\right)(x), \left(\bigcap_{\alpha\in\Omega} M_\alpha\right)(y)\right\}$.

Therefore $\bigcap_{\alpha\in\Omega} M_\alpha$ is fuzzy subgroup of G .

Proposition (3.4) :

Let $\{M_\alpha\}_{\alpha\in\Omega}$ be a family of fuzzy subgroup of G .

If $\{M_\alpha\}_{\alpha\in\Omega}$ are chains , then $\bigcup_{\alpha\in\Omega} M_\alpha$ is a fuzzy subgroup of G .

Proof : for each $x, y \in G$ we have :

i) $\left(\bigcup_{\alpha\in\Omega} M_\alpha\right)(x) = \sup\{M_\alpha(x) | \alpha \in \Omega\} = \sup\{M_\alpha(-x) | \alpha \in \Omega\}$

$$= \left(\bigcup_{\alpha\in\Omega} M_\alpha\right)(-x)$$

ii) $\left(\bigcup_{\alpha\in\Omega} M_\alpha\right)(xy) = \sup\{M_\alpha(xy) | \alpha \in \Omega\}$

$$\geq \sup\{\min\{M_\alpha(x), M_\alpha(y)\} | \alpha \in \Omega\}$$

$$= \min\{\sup\{M_\alpha(x) | \alpha \in \Omega\}, \sup\{M_\alpha(y) | \alpha \in \Omega\}\}$$

Because $\{M_\alpha\}_{\alpha\in\Omega}$ are chains .

$$= \min\left\{\left(\bigcup_{\alpha\in\Omega} M_\alpha\right)(x), \left(\bigcup_{\alpha\in\Omega} M_\alpha\right)(y)\right\}$$

Thus $\left(\bigcup_{\alpha\in\Omega} M_\alpha\right)(xy) \geq \min\left\{\left(\bigcup_{\alpha\in\Omega} M_\alpha\right)(x), \left(\bigcup_{\alpha\in\Omega} M_\alpha\right)(y)\right\}$

There for $\bigcup_{\alpha\in\Omega} M_\alpha$ is fuzzy subgroup of G .

Definition (3.5) : [5]

Let $G,+)$ be a group and M be a fuzzy subset of G and let $n \in N$. Define a fuzzy subset nM of G by :

For each $X \in G$.

$$(nM)(x) = \begin{cases} \sup\{M(y) \mid x = ny & if \ \ x \in nG \\ 0 & if \ \ x \notin nG \end{cases}$$

Where $nG = \{ny = y + y + ... + y | y \in G\} \subseteq G$.

Now we are ready to give the following propositions .

Proposition (3.6) :

Let M_1 and M_2 be two fuzzy subgroups of G and $n \in N$. Then the following holds :

1) $M_1 + M_2$ is a fuzzy subgroup of G .

2) nM is a fuzzy subgroup of G .

Proof : To begin our proof of (1) we must hold the following condifions :

i) $(M_1 + M_2)(x) = (M_1 + M_2)(-x)$ for each $x \in G$.

$$\begin{aligned}(M_1 + M_2)(x) &= \sup\{\min\{M_1(u), M_2(v)\} | x = u + v, (u, v \in G)\} \\ &= \sup\{\min\{M_1(-u), M_2(-v)\} | -x = -u - v, (-u, -v \in G)\} \\ &= \sup\{\min\{M_1(u_1), M_2(v_1)\} | -x = u_1 + v_1, u_1, v_1 \in G\} \\ &= (M_1 + M_2)(-x)\end{aligned}$$

ii) $(M_1 + M_2)(x + y) \geq \min\{(M_1 + M_2)(x), (M_1 + M_2)(y)\}$

for each $x, y \in G$. Let $x = u_1 + v_1 (\forall u_1 \in G), (\forall v_1 \in G)$.

$$y = u_2 + v_2 (\forall u_2 \in G), (\forall v_2 \in G)$$

Then $x + y = (u_1 + v_1) + (u_2 + v_2)$. since (G,+)is a commutative group ,

So $x + y = (u_1 + v_1) + (u_2 + v_2) = u + v$. Now we have

$$\begin{aligned}(M_1 + M_2)(x + y) &= \sup\{\min\{M_1(u), M_2(v)\} | x + y = u + v, (u, v \in G)\} \\ &\geq \min\{M_1(u_1 + u_2), M_2(v_1 + v_2 | x + y = (u_1 + u_2) + (v_1 + v_2)\}\end{aligned}$$

$$\geq \min\{\min\{M_1(u_1), M_1(u_2)\}, \min\{M_2(v_1), M_2(v_2)\}\}$$
$$= \min\{M_1(u_1), M_1(u_2), M_2(v_1), M_2(v_2)\}$$
$$= \min\{\min\{M_1(u_1), M_2(v_1)\} | x = u_1 + v_1\}, \min\{M_1(u_2), M_2(v_2) y = u_2 + v_2\}$$

Thus ,

$$(M_1 + M_2)(x+y) \geq \min\{\sup\{\min\{M_1(u_1), M_2(v_1)\} | x = u_1 + v_1\}, \sup\{\min\{M_1(u_2), M_2(v_2)\} | y = u_2 + v_2\}\}$$
$$= \min\{(M_1 + M_2)(x), (M_1 + M_2)(y)\}$$

Hence $(M_1 + M_2)(x+y) \geq \min\{(M_1 + M_2)(x), (M_1 + M_2)(y)\}$

Therefore , $M_1 + M_2$ is a fuzzy subgroup of G .

2. Let $n \in N$ to prove nM is fuzzy subgroup of G we must prove the following conditions :

For each $x, y \in G$.

i) $(nM)(x) = (nM)(-x)$

$$nM(x) = \begin{cases} \sup\{M(u) | x = nu\} & if\ x \in nG \\ 0 & if\ x \notin nG \end{cases}$$
$$= \begin{cases} \sup\{M(-u) | -x = n(-u) & if\ -x \in nG \\ 0 & if\ -x \in nG \end{cases}$$
$$= (nM)(-x)$$

Hence $(nM)(x) = (nM)(-x)$

ii) $(nM)(x+y) \geq \min\{(nM)(x), (nM)(y)\}$

$$(nM)(x) = \begin{cases} \sup\{M(u) | x = nu\} & if\ x \in nG \\ 0 & if\ x \notin nG \end{cases}$$

Also ,

$$(nM)(y) = \begin{cases} \sup\{M(v) \mid y = nv & if\ y \in nG \\ 0 & if\ y \notin nG \end{cases}$$

$x = nu$ and $y = nv$ implies that $x + y = nu + nv = n(u+v) = nw$

Therefore

$$(nM)(x+y) = \begin{cases} \sup\{M(w) \mid x + y = nw\} & if\ x + y \in nG \\ 0 & if\ x + y \notin nG \end{cases}$$

$$(nM)(x+y)=\begin{cases}\sup\{M(u+v)\mid x+y=n(u+v)\} & if\ x+y\in nG\\ 0 & if\ x+y\notin nG\end{cases}$$

If $x+y\in nG, x\in nG$ and $y\in nG$, then

$$(nM)(x+y)\geq \sup\{\min\{M(u),M(v)|x+y=n(u+v)\}$$

$$\geq \min\{M(u),M(v)|x+y=n(u+v)\}$$

$$\geq \min\{\sup\{M(u)|x=nu\},\sup\{M(v)|y=nv\}\}$$

$$=\min\{(nM)(x),(nM)(y)\}$$

If $x+y\notin nG, x\notin nG$ and $y\notin nG$, then :

$$(nM)(x+y)=\min\{(nM)(x),(nM)(y)\}$$

Hence $(nM)(x+y)\geq \min\{(nM)(x),(nM)(y)\}$

Therefore , nM is a fuzzy subgroup of G .

Definition (3.7) : [1]

Let M be a fuzzy subset of a group $(G,+)$ is called has the suprimum property iff $\sup\{M(x)|y=f(x)\}=\max\{M(x)|y=f(x)\}$ where f is a function from G to G .

i.e. there exists $x_0\in G$. Such that y=f(x_0) and $M(x_0)=\sup\{M(x)|\ y=f(x)\}$.

Now we will explain the relation between the suprimum property of a fuzzy subgroup M and the torsion – free support of M by the following proposition:

Proposition (3.8) :

Let M be a fuzzy subgroup of G and M^* is the support of M . Then the following holds :

1) If M^* is torsion – free subgroup of G , then M has the suprimum property.

2) If M is torsion – free fuzzy subgroup , then M has the suprimum property.

Proof :

1) suppose that M^* is torsion – free and $n\in N$. If we deal with n as a function from M^* to M^* by $X=n(u)=nu$ and $n(u_1+u_2)=nu_1+nu_2$ (n is a group homomorphism) .

$ker(n)=\{u\in M^*|n(u)=0\}=\{0\}$ –because M^* is torsion – free .

Thus n is injective iff $n(u)=n(v)$, then $u=v\in M^*$.

So $\sup\{M(u)|x = nu = M(u)\}$ where $x = nu$ because $\{u|x = nu\}$ is singleton . Thus we obtain the definition of suprimum property of M , i.e.

$\sup\{M(u)|x = nu\} = M(u) = Max\{M(u)|x = nu\}$.

Thus M has the suprimum property .

2) suppose that M is torsion – free, hence M^* is torsion – free by proposition (2.2.11(1)) . M has the suprimum property by part (1) .

Now we give the following proposition with prove in more details as mentioned in [5] .

Proposition (3.9):

Let M be a fuzzy subgroup of G (it is enough to say that M is a fuzzy subset) and $n \in N$. Then for each $t \in (0,1]$ we have :

1) $nMt \subseteq (nM)t$.

2) If M has the suprimum property , then $nMt = (nM)t$.

Proof :

1) Let $x \in nMt$. Then $x = nw$ for some $w \in Mt$ and $M(w) \geq t$. Thus $(nM)(x) = \sup\{M(w)|x = nw\} \geq t$. Then $x \in (nM)t$.

2) From part (1), $nMt \subseteq (nm)t$. To prove $(nM)t \subseteq nMt$. Let $x \in (nM)(x) = \sup\{M(y)|x = ny\} \geq t$ and since M has the suprimum property, So $\exists y_0 \in G$ such that $x = ny_0$ and $M(y_0) = \sup\{M(y)|x = ny\} \geq t$.

Therefore, $y_0 \in Mt$ and then $ny_0 = x \in nMt,$

i.e. $(nM)t \subseteq nMt$.

Hence $nMt = (nM)t$.

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