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Fuzzy Regular Compact Space

Habeeb Kareem Abdullah and Worood Mohammed Hussein
University of Kufa
College of Education for Girls / Department of Mathematics

Abstract

The purpose of this paper is to construct the concept of fuzzy regular compact space in fuzzy topological spaces .We give some characterization of fuzzy compact space and fuzzy regular compact space . A comparison between these concepts and we obtained several properties .

الخلاصة

الهدف من هذا البحث هو بناء مفهوم الفضاء المتراص المنتظم الضبابي في الفضاء التوبولوجي الضبابي . ونعطي بعض خصائص الفضاء المتراص الضبابي والفضاء المتراص المنتظم الضبابي . والمقارنة بين هذه المفاهيم وحصول على العديد من الخصائص .

1. Introduction

C. L. Chang [3] in 1968 , introduced and developed the concept topological spaces based on the concept of fuzzy set introduced by L. A. Zadeh in this classical paper [5] . On the other hand [1] , introduced the notion fuzzy net and fuzzy filter base and some other related concepts . In this paper , we introduce the concepts of fuzzy compact and fuzzy regular compact in fuzzy topological spaces . We give some characterization . Moreover , the study also included the relationship between have been studied and basic properties for these concepts .

2. Preliminaries

First , we present the fundamental definitions .

Definition 2.1. [7] Let X be a non – empty set and let I be the unit interval , i.e., $I = [0,1]$. A fuzzy set in X is a function from X into the unit interval I (i.e., $A : X \longrightarrow [0,1]$ be a function) .

A fuzzy set A in X can be represented by the set of pairs : $A = \{(x, A(x)) : x \in X\}$. The family of all fuzzy sets in X is denoted by I^X .

Definition 2.2. [7] Let X and Y be two non – empty sets $f : X \longrightarrow Y$ be function . For a fuzzy set B in Y , the inverse image of B under f is the fuzzy set $f^{-1}(B)$ in X with membership function denoted by the rule :

$f^{-1}(B)(x) = B(f(x))$ for $x \in X$ (i.e., $f^{-1}(B) = B \circ f$).

For a fuzzy set A in X , the image of A under f is the fuzzy set $f(A)$ in Y with membership function $f(A)(y)$, $y \in Y$ defined by

$$f(A)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} A(x) & if\ f^{-1}(y) \neq 0 \\ 0 & if\ f^{-1}(y) = 0 \end{cases}$$

Where $f^{-1}(y) = \{x : f(x) = y\}$.

Definition 2.3. [2, 6] A fuzzy point x_α in X is fuzzy set defined as follows

$$x_\alpha(y) = \begin{cases} \alpha & if\ y = x \\ 0 & if\ y \neq x \end{cases}$$

Where $0 < \alpha \leq 1$; α is called its value and x is support of x_α .

The set of all fuzzy points in X will be denoted by $FP(X)$.

Definition 2.4. [2, 6] A fuzzy point x_α is said to belong to a fuzzy set A in X (denoted by: $x_\alpha \in A$) if and only if $\alpha \leq A(x)$.

Definition 2.5. [1, 6] A fuzzy set A in X is called quasi – coincident with a fuzzy set B in X , denoted by AqB if and only if $A(x)+B(x)>1$, for some $x\in X$. If A is not quasi –coincident with B , then $A(x)+B(x)\leq 1$, for every $x\in X$ and denoted by $A\tilde{q}B$.

Lemma 2.6. [2] Let A and B are fuzzy sets in X . Then :

(i) If $A\wedge B=0$, then $A\tilde{q}B$.

(ii) $A\tilde{q}B$ if and only if $A\leq B^c$.

Proposition 2.7. [2] If A is a fuzzy set in X, then $x_\alpha\in A$ if and only if $x_\alpha\tilde{q}A^c$.

Definition 2.8. [3] A fuzzy topology on a set X is a collection T of fuzzy sets in X satisfying :

(i) $0\in T$ and $1\in T$,

(ii) If A and B belong to T , then $A\wedge B\in T$,

(iii) If A_i belongs to T for each $i\in I$ then so does $\vee_{i\in I}A_i$.

If T is a fuzzy topology on X, then the pair (X,T) is called a fuzzy topological space . Members of T are called fuzzy open sets . Fuzzy sets of the forms $1-A$, where A is fuzzy open set are called fuzzy closed sets .

Definition 2.9. [6] A fuzzy set A in a fuzzy topological space (X,T) is called quasi-neighborhood of a fuzzy point x_α in X if and only if there exists $B\in T$ such that $x_\alpha qB$ and $B\leq A$.

Definition 2.10. [6] Let (X,T) be a fuzzy topological space and x_α be a fuzzy point in X . Then the family $N^Q_{x_\alpha}$ consisting of all quasi-neighborhood (q-neighborhood) of x_α is called the system of quasi-neighborhood of x_α .

Remark 2.11 Let (X,T) be a fuzzy topological space and $A\in FP(X)$. Then A is fuzzy open if and only if A is q – neighbourhood of each its fuzzy point .

Proof :

$\Rightarrow$ Clearly .

$\Leftarrow$ Let $x_\alpha\in A$, then $\alpha\leq A(x)$, hence $1-\alpha>A^c(x)$, thus $x_{1-\alpha}qA$, then there exists $B\in T$ such that $x_{1-\alpha}qB\leq A$ .Thus $x_\alpha\in B\leq A$, therefore $A\in T$.

Definition 2.12. [1] A fuzzy topological spaces (X,T) is called a fuzzy hausdorff (fuzzy T_2- space **)** if and only if for pair of fuzzy points x_r, y_s such that $x\neq y$ in X , there exists $A\in N^Q_{x_r}$,$B\in N^Q_{y_s}$ and $A\wedge B=0$

Definition 2.13. [4] Let A be a fuzzy set in X and T be a fuzzy topology on X . Then the induced fuzzy topology on A is the family of fuzzy subsets of A which are the intersection with A of fuzzy open set in X . The induced fuzzy topology is denoted by T_A , and the pair (A,T_A) is called **a fuzzy subspace** of X .

Proposition 2.14 Let $A\leq Y\leq X$ .Then :

(i) If A is a fuzzy open set in Y and Y is a fuzzy open set in X, then A is a fuzzy open set in X .

(ii) If A is a fuzzy closed set in Y and Y is a fuzzy closed set in X , then A is a fuzzy closed set in X .

Proof :

(i) Let A is a fuzzy open set in Y , then there exists fuzzy open set B in X such that $A=Y\wedge B$, since Y is a fuzzy open set in X. Then $Y\wedge B$ is a fuzzy open set in X . Thus A is a fuzzy open set in X.

(ii) clearly.

Definition 2.15. [9 , 7] Let (X,T) be a fuzzy topological space and $A \in I^X$. Then :

(i) The union of all fuzzy open sets contained in A is called the fuzzy interior of A and denoted by A°. i.e. , $A^\circ = sup\{B : B \leq A , B \in T\}$

(ii) The intersection of all fuzzy closed sets containing A is called the fuzzy closure of A and denoted by $\overline{A}$. i.e. , $\overline{A} = inf\{B : A \leq B , B^c \in T\}$.

Remarks 2.16. [7]

(i) The interior of a fuzzy set A is the largest open fuzzy set contained in A and trivially , a fuzzy set A is fuzzy open if and only if $A = A^\circ$.

(ii) The closure of a fuzzy set A is the smallest closed fuzzy set containing A and trivially , a fuzzy set A is a fuzzy closed if and only if $A = \overline{A}$.

Theorem 2.17 . [9 ,7] Let (X,T) be a fuzzy topological space and A, B are two fuzzy sets in X .Then :

(i) $0 = \overline{0}$, $1 = \overline{1}$.

(ii) $\overline{A \vee B} = \overline{A} \vee \overline{B}$, $\overline{A \wedge B} \leq \overline{A} \wedge \overline{B}$.

(iii) $(A \wedge B)^\circ = A^\circ \wedge B^\circ$,$A^\circ \vee B^\circ \leq (A \vee B)^\circ$.

(iv) $\overline{\overline{A}} = \overline{A}$, $(A^\circ)^\circ = A^\circ$.

(v) $A^\circ \leq A \leq \overline{A}$.

(vi) If $A \leq B$ then $A^\circ \leq B^\circ$.

(vii) If $A \leq B$ then $\overline{A} \leq \overline{B}$.

Proposition 2.18. Let (X,T) be a fuzzy topological space and A be a fuzzy set in X . A fuzzy point $x_\alpha \in \overline{A}$ if and only if for every fuzzy open set B in X , if $x_\alpha qB$ then AqB .

Proof : $\Longrightarrow$ Suppose that B be a fuzzy open set in X such that $x_\alpha qB$ and $A\tilde{q}B$. Then $A \leq B^c$. But $x_\alpha \notin B^c$ (since $x_\alpha qB$, then $\alpha > B^c(x)$) and B^c be a fuzzy closed set in X . Thus $x_\alpha \notin \overline{A}$.

$\Longleftarrow$ Let $x_\alpha \notin \overline{A}$, then there exists a fuzzy closet set B in X such that $A \leq B$ and $x_\alpha \notin B$, hence by proposition (2.7) , we have $x_\alpha qB^c$. Since $A \leq B$, then by lemma (2.6. ii) , $A\tilde{q}B^c$. This complete the proof .

Definition 2.19. [7] A fuzzy subset A of a fuzzy topological space X is called fuzzy regular open if $A = \overline{A}^\circ$ **.** The complement of fuzzy regular open is called fuzzy regular closed set .Then fuzzy subset of a fuzzy space X is fuzzy regular closed if $A = \overline{A^\circ}$.

Remark 2.20. Every fuzzy regular open set is a fuzzy open set and every fuzzy regular closed set is a fuzzy closed set .

The converse of remark (2.20) , is not true in general as the following example shows :

Example 2.21. Let $X = \{a, b\}$ be a set and $T = \{0, \{a_{0.3}, b_{0.5}\}, \{a_{0.5}, b_{0.5}\}, \{a_{0.3}, b_{0.7}\}, \{a_{0.5}, b_{0.7}\} ,1\}$ be a fuzzy topology on X.

Notice that $A = \{a_{0.3}, b_{0.5}\}$ is a fuzzy open set in X , but its not fuzzy regular open set and $A = \{a_{0.7}, b_{0.5}\}$ is a fuzzy closed set in X , but its not fuzzy regular closed set .

Proposition 2.22. Let $A \leq Y \leq X$. Then:

(i) If A is a fuzzy regular open set in Y and Y is a fuzzy regular open set in X , then A is a fuzzy regular open set in X .

(ii) If A is a fuzzy regular closed set in Y and Y is a fuzzy regular closed set in X , then A is a fuzzy regular closed set in X .

Definition 2.23. [8] The collection of all fuzzy regular open sets of the fuzzy space (X,T) forms a base for a fuzzy topology on X say T^r and its called the fuzzy semi – regularization of T.
Definition 2.24 [7] Let X and Y be fuzzy topological spaces . A map $f: X \longrightarrow Y$ is fuzzy continuous if and only if for every fuzzy point x_α in X and for every fuzzy open set A such that $x_\alpha \in A$, there exists fuzzy open set B of Y such that $f(x_\alpha) \in B$ and $f(B) \leq A$.
Theorem 2.25. [6] Let X ,Y are fuzzy topological spaces and let $f: X \longrightarrow Y$ be a mapping . Then the following statements are equivalent :
(i) f is fuzzy continuous .
(ii) For each fuzzy open set B in Y , $f^{-1}(B)$ is a fuzzy open set in X .
(ii) For each fuzzy closed set B in Y , then $f^{-1}(B)$ is a fuzzy closed set in X.
(iii) For each fuzzy set B in Y , $\overline{f^{-1}(B)} \leq f^{-1}(\overline{B})$.
(iv) For each fuzzy set A in X , $f(\overline{A}) \leq \overline{f(A)}$.
(v) For each fuzzy set B in Y , $f^{-1}(B^\circ) \leq \left(f^{-1}(B)\right)^\circ$.
Definition 2.26 Let $f: X \longrightarrow Y$ be a map from a fuzzy topological space X to a fuzzy topological space Y. Then f is called fuzzy regular irresolute mapping if $f^{-1}(A)$ is a fuzzy regular open set in X for every fuzzy regular open set A in Y .
Definition 2.27. [1] A fuzzy filter base on X is a nonempty subset $\mathcal{F}$ of I^X Such that
(i) $0 \notin \mathcal{F}$.
(ii) If $A_1, A_2 \in \mathcal{F}$, then $\exists\, A_3 \in \mathcal{F}$ such that $A_3 \leq A_1 \wedge A_2$.
Definition 2.28. A fuzzy point x_α in a fuzzy topological space X is said to be a fuzzy cluster point of a fuzzy filter base $\mathcal{F}$ on X if $x_\alpha \in \bar{B}$, for all $B \in \mathcal{F}$.
Definition 2. 29 . [1] A mapping $S: D \longrightarrow FP(X)$ is called a fuzzy net in X and is denoted by $\{S(n): n \in D\}$, where D is a directed set . If $S(n) = x^n_{\alpha_n}$ for each $n \in D$ where $x \in X$, $n \in D$ and $\alpha_n \in (0,1]$ then the fuzzy net S is denoted as $\{x^n_{\alpha_n}, n \in D\}$ or simply $\{x^n_{\alpha_n}\}$.
Definition 2.30. [1] A fuzzy net $\Im = \{y^m_{\alpha_m} : m \in E\}$ in X is called a fuzzy subnet of fuzzy net $S = \{x^n_{\alpha_n}, n \in D\}$ if and only if there is a mapping $f: E \longrightarrow D$ such that
(i) $\Im = S \circ f$,that is , $y^i_{\alpha_i} = x^{f(i)}_{\alpha_{f(i)}}$ for each $i \in E$.
(ii) For each $n \in D$ there exists some $m \in E$ such that $f(m) \geq n$.
We shall denote a fuzzy subnet of a fuzzy net $\{x^n_{\alpha_n}, n \in D\}$ by $\{x^{f(m)}_{\alpha_{f(m)}}, m \in E\}$.
Definition 2.31. [1] Let (X,T) be a fuzzy topological space and let $S = \{x^n_{\alpha_n}, n \in D\}$ be a fuzzy net in X and $A \in I^X$. Then S is said to be:
(i) Eventually with A if and only if $\exists\, m \in D$ such that $x^n_{\alpha_n} q A$, $\forall$ $n \geq m$.
(ii) Frequently with A if and only if $\forall\, n \in D$, $\exists\, m \in D$,$m \geq n$ and $x^m_{\alpha_m} q A$.
Definition 2.32. [1] Let (X,T) be a fuzzy topological space and $S = \{x^n_{\alpha_n} : n \in D\}$ be a fuzzy net in X and $x_\alpha \in FP(X)$. Then S is said to be :
(i) Convergent to x_α and denoted by $S \longrightarrow x_\alpha$, if S is eventually with A , $\forall\, A \in N^Q_{x_\alpha}$.
(ii) Has a cluster point x_α and denoted by $S \propto x_\alpha$, if S is frequently with A , $\forall\, A \in N^Q_{x_\alpha}$.

Proposition 2.33. A fuzzy point x_α is a cluster point of a fuzzy net $\{x_{\alpha_n}^n : n \in D\}$, where $(D, \geq)$ is a directed set , in a fuzzy topological space X if and only if it has a fuzzy subnet which converges to x_α.

Proof $\Longrightarrow$ Let x_α be a cluster point of the fuzzy net $\{x_{\alpha_n}^n : n \in D\}$, with the directed set $(D, \geq)$ as the domain . Then for any $U \in N_{x_\alpha}^Q$, there exists $n \in D$ such that $x_{\alpha_n}^n qU$. Let $E = \{(n, U) : n \in D , U \in N_{x_\alpha}^Q \ and \ x_{\alpha_n}^n qU\}$. Then $(E, \geq)$ is directed set where $(m, U) \geq (n, V)$ if and only if $m \geq n$ in D and $U \leq V$ in $N_{x_\alpha}^Q$. Then $\Im : E \longrightarrow FP(X)$ given by $\Im(m, U) = x_{\alpha_m}^m$ is a fuzzy subnet of fuzzy net $\{x_{\alpha_n}^n : n \in D\}$. To show that $\Im \longrightarrow x_\alpha$. Let $B \in N_{x_\alpha}^Q$. Then there exists $n \in D$ such that $(n, B) \in E$ and $x_{\alpha_n}^n qB$. Thus for any $(m, U) \in E$ such that $(n, U) \geq (n, B)$, we have $\Im(m, U) = x_{\alpha_m}^m qU \leq B$. Hence $\Im \longrightarrow x_\alpha$.

$\Longleftarrow$ If a fuzzy net $\{x_{\alpha_n}^n : n \in D\}$, has not a cluster point . Then for every fuzzy point x_α there is q-neighborhood of x_α and $n \in D$ such that $x_{\alpha_m}^m \tilde{q}\, U$, for all $m \geq n$. Then obviously no fuzzy net converge to x_α .

Theorem 2.34. Let (X, T) be a fuzzy topological space , $x_\alpha \in FP(X)$ and $A \in I^X$. Then $x_\alpha \in \overline{A}$ if and only if there exists a fuzzy net in A convergent to x_α .

Proof $\Rightarrow$ Let $x_\alpha \in \overline{A}$, then for every $B \in N_{x_\alpha}^Q$ there exists

$$x_B(y) = \begin{cases} A(x_\alpha) & if \quad y = x_B \\ 0 & if \quad y \neq x_B \end{cases}$$

Such that $B(x_B) + A(x_B) > 1$ notice that $(N_{x_\alpha}^Q, \geq)$ is a directed set , then $S : N_{x_\alpha}^Q \longrightarrow FP(X)$ is defined as $S(B) = x_B^A$ is a fuzzy net in A . To prove that $S \longrightarrow x_\alpha$. Let $D \in N_{x_\alpha}^Q$. Then there exists $F \in T$ such that $x_\alpha qF$ and $F \leq D$. Since $F(x_F^A) + x_F^A > 1$ and $F \leq D$. Then $D(x_F^A) + x_F^A > 1$. Thus $x_F^A qD$. Let $E \geq F$, then $E \leq F$. Since $E(x_E^A) + x_E^A > 1$ and $F \leq D$, then $D(x_F^A) + x_F^A > 1$. Thus $x_E^A qD$, $\forall\ E \geq F$. Therefore $S \longrightarrow x_\alpha$.

$\Longleftarrow$ Let $\{x_{\alpha_n}^n : n \in D\}$ be a fuzzy net in A where $(D, \geq)$ is a directed set such that $x_{\alpha_n}^n \longrightarrow x_\alpha$. Then for every $B \in N_{x_\alpha}^Q$, there exists $m \in D$ such that $x_{\alpha_n}^n qB$ for all $n \geq m$. Since $x_{\alpha_n}^n \in A$, then by proposition (2.7) , $x_{\alpha_n}^n \tilde{q}\, A^c$. Thus AqB . Therefore $x_\alpha \in \overline{A}$.

Proposition 2.35. If X is a fuzzy $T_2 -$ space , then convergent fuzzy net on X has a unique limit point .

Proof : Let $x_{\alpha_n}^n$ be a fuzzy net on X such that $x_{\alpha_n}^n \longrightarrow x_\alpha$, $x_{\alpha_n}^n \longrightarrow y_\beta$ and $x \neq y$. Since $x_{\alpha_n}^n \longrightarrow x_\alpha$, we have $\forall\ A \in N_{x_\alpha}^Q$,$\exists\ m_1 \in D$, such that $x_{\alpha_n}^n qA$, $\forall\ n \geq m_1$. Also , $x_{\alpha_n}^n \longrightarrow y_\beta$, we have $\forall\ B \in N_{x_\beta}^Q$,$\exists\ m_2 \in D$, such that $x_{\alpha_n}^n qB$, $\forall\ n \geq m_2$. Since D is a directed set , then there exists $m \in D$, such that $m_1 \geq m$ and $m_2 \geq m$, then $x_{\alpha_n}^n q(A \wedge B)$, $\forall n \geq m$. Thus $A \wedge B \neq 0$, a contradiction .

$\Longleftarrow$ Let X be a not fuzzy $T_2 -$ space , then there exists $x_\alpha , y_\beta \in FP(X)$ such that $x \neq y$ and $A \wedge B \neq 0$, $\forall A \in N_{x_\alpha}^Q$, $B \in N_{y_\beta}^Q$. Put $N_{x_\alpha y_\beta}^Q = \{A \wedge B / A \in N_{x_\alpha}^Q , B \in N_{y_\beta}^Q\}$. Thus $\forall D \in N_{x_\alpha, y_\beta}^Q$, there exists $x_D qD$, then $\{x_D\}_{D \in N_{x_\alpha y_\beta}^Q}$ is a fuzzy net in X .To prove that $x_D \longrightarrow x_\alpha$ and $x_D \longrightarrow y_\beta$. Let $E \in N_{x_\alpha}^Q$, then $E \in N^{Q}_{x_\alpha, y_\beta}$ (since $E = E \wedge X$) . Thus $x_D q E$, $\forall D \geq E$, thus $x_D \longrightarrow x_\alpha$. Also $x_D \longrightarrow y_\beta$, so $\{x_D\}_{D \in N^{Q}_{x_\alpha, y_\beta}}$ has two limit point .

3. Fuzzy compact space

Definition 3.1. [3] A family Λ of fuzzy sets is a cover of fuzzy set A if and only if $A \leq \vee\{B_i : B_i \in \Lambda\}$. It is called fuzzy open cover if each member B_i is a fuzzy open set . A sub cover of Λ is a subfamily of Λ which is also a cover of A .

Definition 3.2. [3] Let (X, T) be a fuzzy topological space and let $A \in I^X$. Then A is said to be a fuzzy compact set if for every fuzzy open cover of A has a finite sub cover of A . Let $A = X$, then X is called a fuzzy compact space that is $A_i \in T$ for every $i \in I$ and $\vee_{i \in I} A_i = 1$, then there are finitely many indices $i_1, i_2, \dots \dots, i_n \in I$ such that $\vee_{j=1}^{n} A_{i_j} = 1$.

Example 3.3.

(i) If (X, T) is a fuzzy topological space such that T is finite then X is fuzzy compact .

(ii) The indiscrete fuzzy topological space is fuzzy compact .

Proposition 3.4 Let Y be a fuzzy subspace of a fuzzy topological space X and let $A \in I^Y$. Then A is fuzzy compact relative to X if and only if A is fuzzy compact relative to Y .

Proof $\Longrightarrow$ Let A be a fuzzy compact relative to X and let $\{V_\lambda : \lambda \in \Lambda\}$ be a collection of fuzzy open sets relative to Y , which covers A so that $A \leq \vee_{\lambda \in \Lambda} V_\lambda$, then there exist G_λ fuzzy open relative to X , such that $V_\lambda = Y \wedge G_\lambda$ for any $\lambda \in \Lambda$. It then follows that $A \leq \vee_{\lambda \in \Lambda} G_\lambda$. So that $\{G_\lambda : \lambda \in \Lambda\}$ is fuzzy open cover of A relative to X . Since A is fuzzy compact relative to X , then there exists a finitely many indices $\lambda_1, \lambda_2, \dots., \lambda_n \in \Lambda$ such that $A \leq \vee_{i=1}^{n} G_{\lambda_i}$. since $A \leq Y$, we have $A = Y \wedge A \leq Y \wedge \left(G_{\lambda_1} \vee G_{\lambda_2} \vee \dots . \vee G_{\lambda_n}\right) = \left(Y \wedge G_{\lambda_1}\right) \dots . . (Y \wedge G_n)$, since $Y \wedge G_{\lambda_i} = V_{\lambda_i}$ $(i = 1,2, \dots . n)$ we obtain $A \leq \vee_{i=1}^{n} V_{\lambda_i}$. Thus show that A is fuzzy compact relative to Y .

$\Longleftarrow$ Let A be fuzzy compact relative to Y and let $\{G_\lambda : \lambda \in \Lambda\}$ be a collection of fuzzy open cover of X , so that $A \leq \vee_{\lambda \in \Lambda} G_\lambda$. Since $A \leq Y$, we have $A = Y \wedge A \leq Y \wedge (\vee_{\lambda \in \Lambda} G_\lambda) = \vee_{\lambda \in \Lambda} (Y \wedge G_\lambda)$. Since $Y \wedge G_\lambda$ is fuzzy open relative to Y , then the collection $\{Y \wedge G_\lambda : \lambda \in \Lambda\}$ is a fuzzy open cover relative to Y . Since A is fuzzy compact relative to Y , we must have $A \leq \left(Y \wedge G_{\lambda_1}\right) \vee \left(Y \wedge G_{\lambda_2}\right) \vee \dots \vee \left(Y \wedge G_{\lambda_n}\right)$ ……(*) for some choice of finitely many indices $\lambda_1, \lambda_2, \dots., \lambda_n$. But (*) implies that $A \leq \vee_{i=1}^{n} G_{\lambda_i}$. It follows that A is fuzzy compact relative to X .

Theorem 3.5 A fuzzy topological space (X, T) is fuzzy compact if and only if for every collection $\{A_j : j \in J\}$ of fuzzy closed sets of X having the finite intersection property , $\wedge_{j \in J} A_j \neq 0$.

Proof $\Longrightarrow$ Let $\{A_j : j \in J\}$ be a collection of fuzzy closed sets of X with the finite intersection property . Suppose that $\wedge_{j \in J} A_j = 0$, then $\vee_{j \in J} {A_j}^c = 1$. Since X is fuzzy compact , then there exists $j_1, j_2, \dots, j_n$ such that $\vee_{i=1}^{n} {A_{j_i}}^c = 1$ .Then $\wedge_{i=1}^{n} A_{j_i} = 0$. Which gives a contradiction and therefore $\wedge_{j \in J} A_j \neq 0$.

$\Longleftarrow$ let $\{A_j : j \in J\}$ be a fuzzy open cover of X . Suppose that for every finite $j_1, j_2, \dots, j_n$, we have $\vee_{i=1}^{n} A_{j_i} \neq 1$. then $\wedge_{i=1}^{n} {A_{j_i}}^c \neq 0$. Hence $\{{A_j}^c : j \in J\}$ satisfies the finite intersection property . Then from the hypothesis we have $\wedge_{j \in J} {A_j}^c \neq 0$. Which implies $\wedge_{i=1}^{n} A_{j_i} \neq 1$ and this contradicting that $\{A_j : j \in J\}$ is a fuzzy open cover of X . Thus X is fuzzy compact .

Theorem 3.6. A fuzzy closed subset of a fuzzy compact space is fuzzy compact.

Proof : Let A be a fuzzy closed subset of a fuzzy space X and let $\{B_i : i \in I\}$ be any family of fuzzy closed in A with finite intersection property , since A is fuzzy closed in X , then by proposition (2. 14 . ii) , B_i are also fuzzy closed in X , since X is fuzzy compact , then by proposition (3 . 5) , $\wedge_{i\in I} B_i \neq 0$. Therefore A is fuzzy compact .

Theorem 3.7. A fuzzy topological space (X, T) is a fuzzy compact if and only if every fuzzy filter base on X has a fuzzy cluster point .

Proof $\Longrightarrow$ Let X be fuzzy compact and let $\mathcal{F} = \{F_\alpha : \alpha \in \Lambda\}$ be a fuzzy filter base on X having no a fuzzy cluster point . Let $x \in X$. Corresponding to each $n \in N$ (N denoted the set of natural numbers), there exists q-neighbourhood U_x^n of the fuzzy point $x_{\frac{1}{n}}$ and an $F_x^n \in \mathcal{F}$ such that $U_x^n \tilde{q} F_x^n$. Since $1 - \frac{1}{n} < U_x^n(x)$, we have $U_x(x) = 1$, where $U_x = \vee\{U_{x_n} : n \in N\}$. Thus $\mathcal{U} = \{U_x^n : n \in N , x \in X\}$ is a fuzzy open cover of X . Since X is fuzzy compact , then there exists finitely many members $U_{x_1}^{n_1}, U_{x_2}^{n_2}, \ldots.., U_{x_k}^{nk}$ of $\mathcal{U}$ such that $\vee_{i=1}^{k} U_{x_i}^{n_i} = 1$. Since $\mathcal{F}$ is fuzzy filter base , then there exists $F \in \mathcal{F}$ such that $F \leq F_{x_{n1}} \wedge F_{x_{n2}} \wedge \ldots. \wedge F_{x_{nk}}$. But $U_{x_i}^{n_i} \tilde{q} F_{x_i}^{n_i}$, then $F \tilde{q}\, 1$. Consequently , $F = 0$ and this contradicts the definition of a fuzzy filter base .

$\Longleftarrow$ Let every fuzzy filter base on X have a fuzzy cluster point . We have to show that X is fuzzy compact . Let $\beta = \{F_\alpha : \alpha \in \Lambda\}$ be a family of fuzzy closed sets having finite intersection property .Then the set of finite intersections of members of β forms a fuzzy filter base $\mathcal{F}$ on X . So by the condition $\mathcal{F}$ has a fuzzy cluster point say x_s . Thus $x_s \in F_\alpha$. So $x_s \in \wedge_{\alpha\in\Lambda} \overline{F_\alpha} = \wedge_{\alpha\in\Lambda} F_\alpha$. Thus $\wedge\{F , F \in \mathcal{F}\} \neq 0$. Hence by theorem (3 . 5) , X is fuzzy compact .

Theorem 3.8. A fuzzy topological space (X, T) is fuzzy compact if and only if every fuzzy net in X has a cluster point .

Proof $\Longrightarrow$ Let X be fuzzy compact . Let $\{S(n) : n \in D\}$ be a fuzzy net in X which has no cluster point , then for each fuzzy point x_α , there is q – neighbourhood U_{x_α} of x_α and an $n_{U_{x\alpha}} \in D$ such that $S_m \tilde{q}\, U_{x_\alpha}$, for all $m \in D$ with $m \geq n_{U_{x\alpha}}$. Since $x_\alpha q U_{x_\alpha}$, then $S_m \neq 0$, $\forall\ m \geq n_{U_{x\alpha}}$. Let $\mathcal{U}$ denoted the collection of all U_{x_α} , where x_α runs over all fuzzy points in X . Now to prove that the collection $V = \{1 - U_{x_\alpha} : U_{x_\alpha} \in \mathcal{U}\}$ is a family of fuzzy closed sets in X possessing finite intersection property . First notice that there exists $k \geq n_{U_{\alpha_1}}, \ldots.., n_{U_{\alpha m}}$ such that $S_p \tilde{q}\, U_{x_{\alpha_i}}$ for i =1,2,….., m and for all $p \geq k$ $(p \in D)$, i.e. $S_p \in 1 - \vee_{i=1}^{m} U_{x_{\alpha_i}} = \wedge_{i=1}^{m}(1 - U_{x_{\alpha_i}})$ for all $p \geq k$. Hence $\wedge\left\{1 - U_{x_{\alpha_i}} : i = 1,2, \ldots., m\right\} \neq 0$. Since X is fuzzy compact , by theorem (3 . 5) , there exists a fuzzy point y_β in X such that $y_\beta \in \wedge\{1 - U_{x_\alpha} : U_{x_\alpha} \in \mathcal{U}\} = 1 - \vee\{U_{x_\alpha} : U_{x_\alpha} \in \mathcal{U}\}$. Thus $y_\beta \in 1 - U_{x_\alpha}$, for all $U_{x_\alpha} \in \mathcal{U}$ and hence in particular , $y_\beta \in 1 - U_{y_\beta}$, i.e., $y_\beta \tilde{q}\, U_{y_\beta}$. But by construction , for each fuzzy point x_α , there exists $U_{x_\alpha} \in \mathcal{U}$ Such that $x_\alpha q U_{x_\alpha}$, and we arrive at a contradiction .

$\Longleftarrow$ To prove that converse by theorem (3 . 7) , that every fuzzy filter base on X has a cluster point . Let $\mathcal{F}$ be a fuzzy filter base on X . Then each $F \in \mathcal{F}$ is non empty set , we choose a fuzzy point $x_F \in F$. Let $S = \{x_F : F \in \mathcal{F}\}$. Let a relation " $\geq$ " be defined in $\mathcal{F}$ as follows $F_\alpha \geq F_\beta$ if and only if $F_\alpha \leq F_\beta$ in X, for $F_\alpha, F_\beta \in \mathcal{F}$. Then $(\mathcal{F}, \geq)$ is directed set . Now S is a fuzzy net with the directed set $(\mathcal{F}, \geq)$ as domain . By hypothesis the fuzzy net S has a cluster point x_t . Then for every q –

neighourhood W of x_t and for each $F \in \mathcal{F}$, there exists $G \in \mathcal{F}$ with $G \geq F$ such that $x_G q W$. As $x_G \leq G \leq F$. It follows that FqW for each $F \in \mathcal{F}$, then by proposition (2 . 18) , $x_t \in \overline{F}$. Hence x_t is a cluster point of $\mathcal{F}$.

Corollary 3.9. A fuzzy topological space (X, T) is fuzzy compact if and only if every fuzzy net in X has a convergent fuzzy subnet .

Proof : By proposition (2.33) , and theorem (3.8) .

Theorem 3.10. Every fuzzy compact subset of a fuzzy Hausdroff topological space is fuzzy closed .

Proof : Let $x_\alpha \in \overline{A}$, then by theorem (2. 34) , there exists fuzzy net $x_{\alpha_n}^n$ such that $x_{\alpha_n}^n \longrightarrow x_\alpha$. Since A is fuzzy compact and X is fuzzy $T_2 -$ space , then by corollary (3. 9) and proposition (2 .35) , then $x_\alpha \in A$. Hence A is fuzzy closed set .

Theorem 3.11. In any fuzzy space , the intersection of a fuzzy compact set with a fuzzy closed set is fuzzy compact .

Proof : Let A be a fuzzy compact set and B be a fuzzy closed set . To prove that $A \wedge B$ is a fuzzy compact set . Let $x_{\alpha_n}^n$ be a fuzzy net in $A \wedge B$. Then $x_{\alpha_n}^n$ is fuzzy net in A , since A is fuzzy compact , then by corollary (3.9) , $x_{\alpha_n}^n \longrightarrow x_\alpha$ for some $x_\alpha \in FP(X)$ and by proposition (2.34) , $x_\alpha \in \overline{B}$. since B is fuzzy closed , then $x_\alpha \in B$. Hence $x_\alpha \in A \wedge B$ and $x_{\alpha_n}^n \longrightarrow x_\alpha$. Thus $A \wedge B$ is fuzzy compact .

Proposition 3.12 Let X and Y be fuzzy spaces and $f: X \rightarrow Y$ be a fuzzy continuous mapping . If U is a fuzzy compact set in (X, T) , then $f(U)$ is a fuzzy compact set in $(Y, \mathcal{T})$.

Proof : Let $\{V_i : i \in I\}$ be a fuzzy open cover of $f(U)$ in Y, i.e., $(f(U) \leq \vee_{i \in I} G_i)$. Since f is a fuzzy continuous , then $f^{-1}(G_i)$ is a fuzzy open set in X , $\forall i \in I$. Hence the collection $\{f^{-1}(G_i)$ $: i \in I\}$ be a fuzzy open cover of U in X ,i.e., $U \leq f^{-1}(f(U)) \leq f^{-1}(\vee_{i \in I} G_i) = \vee_{i \in I} f^{-1}(G_i)$. Since U is a fuzzy compact set in X , then there exists finitely many indices $i_1, i_2, \ldots \ldots, i_n$ Such that $U \leq \vee_{j=1}^{n} f^{-1}(G_{i_j})$, so that $f(U) \leq f(\vee_{j=1}^{n}(f^{-1}(G_{i_j})) = \vee_{j=1}^{n}(f(f^{-1}(G_{i_j}))) \leq \vee_{j=1}^{n} G_{i_j}$ .Hence $f(U)$ is a fuzzy compact set .

4. Fuzzy regular compact space

Definition 4.1. Let (X, T) be a fuzzy space . A family δ of fuzzy subset of X is called a fuzzy regular open cover of X if δ covers X and δ is subfamily of T^r.

Definition 4.2. A fuzzy space X is called fuzzy regular compact if every fuzzy regular open of cover X has a finite sub cover .

Example 4.3. The indiscrete fuzzy topological space is a fuzzy regular compact.

Proposition 4.4. Every fuzzy compact space is a fuzzy regular compact space.

Proof : Let $\{U_i , i \in I\}$ is a fuzzy regular open cover of fuzzy space X and $X = \vee_{i \in I} U_i$, since every fuzzy regular open set is fuzzy open and X is a fuzzy compact space , then there exists $i_1, i_2, \ldots \ldots \ldots, i_n$ such that $X = \vee_{j=1}^{n} U_{i_j}$, thus X is fuzzy regular compact space .

The converse of proposition (4.4), is not true in general as the following example shows :

Example 4.5. Let $X = \{a, b\}$ and $T = \{0, 1, f_n\}$ where $f_n: X \longrightarrow [0,1]$ such that $f_n(x) = 1 - \frac{1}{n}$, $\forall\ x \in X$, $n \in Z$.

Notice that the fuzzy topological space (X, T) is fuzzy regular compact , but its not fuzzy compact .

Remark 4.6. The fuzzy space (X,T) is fuzzy regular compact if and only if the fuzzy space (X,T^r) is fuzzy compact .

Proposition 4.7. Every fuzzy regular closed subset of a fuzzy regular compact space is fuzzy regular compact .

Proof : By remark (4.6) , and theorem (3.6) .

Remarks 4.8

(i) Every fuzzy regular closed subset of a fuzzy compact space is fuzzy regular compact .

(ii) Every fuzzy regular compact subset of a fuzzy $T_2 -$ space is fuzzy regular closed .

Proposition 4.9. Let X be a fuzzy compact set of a fuzzy T_2 $-$space and $A \in I^X$. Then :

(i) A is fuzzy closed if and only if A is fuzzy r- closed .

(ii) A is fuzzy compact if and only if A is fuzzy r- compact .

Proof : $(i) \Longrightarrow$ Let A be a fuzzy closed set in X . Since X is fuzzy compact , then by theorem (3.6) , A is a fuzzy compact set , so its fuzzy r- compact . Since X is a fuzzy T_2 $-$space , then by remark (4 . 8 . ii) , A is a fuzzy r- closed set .

$\Longleftarrow$ By remark (2 . 20) .

$(ii) \Longrightarrow$ By proposition (4 . 4) .

$\Longleftarrow$ Let A be a fuzzy r- compact set in X . Since X is a fuzzy T_2 $-$space , then by remark (4 . 8 . ii) , A is fuzzy r- closed in X , and then its fuzzy closed set . Since X is a fuzzy compact space , then by theorem (3.6) , A is a fuzzy compct set in X.

Proposition 4.10 Let X be a fuzzy space and Y be a fuzzy regular open sub space of X, $K \leq Y$. Then K is a fuzzy regular compact set in Y if and only if K is a fuzzy regular compact set in X .

Proof : $\Longrightarrow$ Let K be a fuzzy regular compact set in Y . To prove that K is a fuzzy regular compact set in X . Let $\{U_\lambda : \lambda \in \Delta\}$ be a fuzzy regular open cover in X of K , let $V_\lambda = U_\lambda \wedge Y$, $\forall \lambda \in \Delta$. Then V_λ is fuzzy regular open in X , $\forall \lambda \in \Delta$. But $V_\lambda \leq Y$, thus V_λ is fuzzy regular open in Y , $\forall \lambda \in \Delta$. Since $K \leq \vee_{\lambda \in \Delta} V_\lambda$, then $\{V_\lambda : \lambda \in \Delta\}$ is a fuzzy regular open cover in Y of K , and by hypothesis this cover has finite sub cover $\{V_{\lambda_1}, V_{\lambda_2, \ldots,} V_{\lambda_n}\}$ of K , thus the cover $\{U_\lambda : \lambda \in \Delta\}$ has a finite sub cover of K . Hence K is a fuzzy regular compact set in X.

$\Longleftarrow$ Let K be a fuzzy regular compact set in X . To prove that K is a fuzzy regular compact set in Y . Let $\{U_\lambda : \lambda \in \Delta\}$ be a fuzzy regular open cover in Y of K . Since Y is a fuzzy regular open subspace of X , then by proposition (2.22 . i) , $\{U_\lambda : \lambda \in \Delta\}$ is a fuzzy regular open cover in X of K . Then by hypothesis there exists $\{\lambda_1, \lambda_2, \ldots \ldots, \lambda_m\}$, such that $K \leq \vee_{\lambda=1}^{n} U_\lambda$, thus the cover $\{U_\lambda : \lambda \in \Delta\}$ has a finite sub cover of K . Hence K is a fuzzy r - compact set in Y .

Proposition 4.11 Let $f : X \rightarrow Y$ be a fuzzy regular irresolute mapping . If A is a fuzzy regular compact set in X , then $f(A)$ is a fuzzy regular compact set in Y .

Proof : Let $\{G_i : i \in I\}$ be a fuzzy regular open of $f(A)$ in Y (i.e., $f(A) \leq \vee_{i \in I} G_i$) . Since f is fuzzy regular irresolute , then $f^{-1}(G_i)$ is fuzzy regular open set in X , $\forall i \in I$. Hence the collection $\{f^{-1}(G_i) : i \in I\}$ be a fuzzy regular open cover of A in X . i.e., $A \leq f^{-1}(f(A)) \leq f^{-1}$ $(\vee_{i \in I} G_i) = \vee_{i \in I} f^{-1}(G_i)$, since A is fuzzy regular compact set in X, there exists finitely many indices $i_1, i_2, \ldots \ldots, i_n$

Such that $A \leq \vee_{j=1}^{n} f^{-1} (G_{i_j})$, so that $f(A) \leq f(\vee_{j=1}^{n} f^{-1} (G_{i_j})) = \vee_{j=1}^{n} f(f^{-1} (G_{i_j})) \leq \vee_{j=1}^{n} G_{i_j}$.

Hence $f(A)$ is a fuzzy regular compact set .

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