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On Multi-Objective Geometric Programming Problems with a Negative Degree of Difficulty

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Abstract

Degree of difficulty is an important concept in the classical Geometric Programming (GP) theory. The dual problem is often infeasible when the degree of difficulty is negative and very little subjects have been published on this topic. This paper presents the basic concepts and principles of multiple-objective geometric programming model, and developed a numerical procedure to solve multi-objective Geometric Programming Problems (GPP) having a negative degree of difficulty using weighted method to obtain the non-inferior solution and using Brickers simple technique to ensure the dual feasibility; namely the addition of a constant term to the primal objective function.

البرمجة الهندسية متعددة الأهداف مع درجة الصعوبة السالبة

المستخلص

تعد درجة الصعوبة مبدأ مهماً في نظرية البرمجة الهندسية الكلاسيكية. والمسألة المقابلة غالبا ما تكون غير مقبولة عند درجة الصعوبة السالبة وبما أن البحوث المنشورة في هذا المجال قليلة جدا.فان هذا البحث سيقدم الأفكار والمبادئ الأساسية للبرمجة الهندسية

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أهداف؛ فقد تم تطوير إجراء عددي لحل مسألة البرمجة الهندسية متعددة الأهداف متعددة ذات درجة الصعوبة السالبة مستخدمين طريقة الأوزان للحصول على حل بمستوى مقبول واستخدام تقنية المبسطة للتأكد من معقولية المسألة المقابلة؛ وذلك بإضافة حد ثابت لدالـة الهدف الأساسية.

Keywords: *Geometric Programming, Multi-Objective (GP) Optimization, Negative Degree of Difficulty, Weighted Methods.*

MSC Classification: 49M37, 90C30, 90C29

1.Introduction:

Geometric Programming Problems (GPP) are smooth non-linear programs in which the objective and each constraint function is a posi-nomials; i.e. a linear combination of terms; with each term product of variables raised to real powers and each constraint function must be ≤ 1. The decision variables x_i are restricted to be positive, to ensure that terms involving variables raised to fractional powers are defined. If all the linear combination coefficients are positive, the functions are called posi-nomials and the problems is easily transformed to a convex programming problem in a new variable $y_i = \ln x_i$. Otherwise, the signomial problems are non-convex. Most of GPP application are posi-nomials type with zero or a few degree of difficulty [5]. The degree of difficulty of a GP problem is the number of dual variables minus the number of dual equality constraints. If zero (and assuming the system of linear equality constraints of the dual has full rank), then there is a unique dual feasible solution. If the degree of difficulty is positive, then the dual feasible region must be searched to maximize the dual objective, while if the degree of difficulty is negative, the dual constraints may be inconsistent. In linear programming, dual infeasibility does not imply that the primal objective is unbounded (since every posi-nomial is bounded below by zero). In the case of GP dual infeasibility, the primal minimum is not attained, but instead the objective approaches the infimum as one or more primal variables approach either zero (which is outside the primal feasible region) or infinity [3]. Generally, an engineering design problem has multiple objective-functions. In this case, it is not suitable to use any single-objective programming to find an optimal compromise solution. Ojha and Biswal [6] have developed the method to find the compromise optimal solution of certain multi-objective GPP by using weighting method.

2. The New Algorithm:

objective functions p are $g_{10}(x), g_{20}(x), \dots, g_{p0}(x)$,Step1- Define .$x = (x_1, x_2, \dots, x_n)^T$for any vector

where the primal multi-objective GP problem is

$$\left.\begin{aligned} &\min g_{k0}(x) = \sum_{t=1}^{T_{k0}} c_{k0t} \prod_{j=1}^{n} x_j^{a_{k0tj}} \quad , k = 1,2,\dots\dots p \\ &s.t. \\ &g_i(x) = \sum_{t=1}^{T_i} c_{it} \prod_{j=1}^{n} x_j^{d_{k0tj}} \le 1 \quad i = 1,2,\dots,q \\ &\qquad x_j > 0 \ , j = 1,2,\dots,n \end{aligned}\right\} \dots\dots\dots \dots (1)$$

c_{k0t} for all k and t are positive real numbers and d_{itj} and a_{k0tj} are real numbers for all i,k,t,j. T_{k0} = number of terms present in the k^{th} objective function. $T_i =$ number of terms present in the i^{th} constraint.

In the above multi-objective geometric program there are p number of minimization type objective function, q number of inequality type constraints and n number of strictly positive decision variables.

Step2- Start with the weighted sum method to find the non inferior optimal solution of a multi-objective GP problem (1):

Let: $w = \{w: w \in R^p, w_k > 0, \sum_{k=1}^{p} w_k = 1\}$ be the set of non-negative weights. Using weighting method, the above MOGPP (1) can be defined as:

$$min: Q(x) = \sum_{k=1}^{p} w_k\, g_{k0}(x) \quad \dots \dots \dots (2)$$
$$x \in X$$

s.t.

$$g_i(x) \le 1\,; i = 1,2,\dots,q \dots \dots \dots (3)$$
$$x_j > 0, j = 1,2,\dots,n$$

Step3- Add a constant term (α) to the MOGPP (2); to transfer the problem from negative degree of difficulty to positive degree of difficulty.

Step4- According to Duffin et al.[2] the model given by (2) & (3) can be transformed to the corresponding dual geometric programming:

$$v(\delta)=\prod_{i=1}^{m_0}\left(\frac{w_k c_{ki}}{\delta_i}\right)^{\delta_i}\prod_{i=1}^{q}\prod_{j=1}^{m_i}\left(\frac{\lambda_i c_{ij}}{\delta_{ij}}\right)^{\delta_{ij}}\prod_{i=1}^{q}\lambda_i^{\lambda_i}\ ,k=1,\dots,p \ldots\ldots(4)$$

where

$$\lambda_i=\sum_{j=m_i}^{m_{i+1}}\delta_j \ \ ,i=1,\dots,q \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

m_i is the number of the terms in the constraint i. The factors c_i are assumed to be positive constants and the vector variable is $\delta=(\delta_1,\dots,\delta_d)$ subject to the following constants:

$$\delta_1\geq 0,\delta_2\geq 0,\dots\ ,\ \delta_d\geq 0,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

$$\sum_{i=1}^{m_0}\delta_i=1,\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

and

$$\sum_{i=1}^{d}a_{ij}\delta_i=0,\ \ j=1,2,\dots\ ,q\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Here $d=m_0+m_1+\cdots+m_q$

Step5- Using ***(NLPSolve)*** Maple function to find the optimal dual variables $\delta^*=(\delta_1^*,\delta_2^*,\dots,\delta_d^*)\ and\ v(\delta^*)$ in the above [4,…,8] problem.

Step6- Each minimizing point x^* for problem (1) satisfies

$$u_i(x)=\begin{cases}\delta_i^* v(\delta^*) & for\ i\in m_0\\ \dfrac{\delta_i^*}{\lambda_k(\delta^*)} & for\ all\ i\in m_j\ ,j=1,2,\dots,q\end{cases}$$

Here $u_i(x)$ is the term j in the problem (1)

Step7- If the solution of (2) is Pareto optimal solution stop, else go to (step 2), reset the weights $w = \{w_k > 0, \sum_{k=1}^{p} w_k = 1\}$ and solve the problem again.

Step8- End

Note:

Pareto Optimality:

In contrast to single-objective optimization, a solution to a multi-objective problem is more of a concept than a definition. Typically, there is no single global solution, and it is often necessary to determine a set of points that all fit a predetermined definition for an optimum. The predominant concept in defining an optimal point is that of Pareto optimality (Pareto 1906), which is defined as follows Pareto Optimal: A point, $x^* \in X$, is Pareto optimal if and only if there does not exist another point, $x \in X$, such that $f_k(x) < f_k(x_k^*)$, and $f_k(x) < f_k(x_k^*)$ for at least one function, Marler et al.[4].

3. Numerical Examples:

Example (3.1):

$$\min f_1(x) = x_1 x_5$$

$$\min f_2(x) = x_1^{-1} x_3^2 x_4^4$$

s.t.

$$5x_1^{-1} x_2 \leq 1$$

$2.5x_2^{-1}x_3^2 + 1.5x_3^{-1}x_4^{-0.5}x_5^{-0.5} \leq 1$

$x_i \geq 0 \ where \ i = 1,2, \dots, 5.$

Solution:

$\min \quad w_1 x_1 x_5 + w_2 x_1^{-1} x_3^2 x_4^4$

s.t.

$5x_1^{-1}x_2 \leq 1$

$2.5x_2^{-1}x_3^2 + 1.5x_3^{-1}x_4^{-0.5}x_5^{-0.5} \leq 1$

$x_i \geq 0 \quad , i = 1,2, \dots, 5.$

Where $w_1 + w_2 = 1 \, , w_1, w_2 > 0$

Here the degree of difficulty is (-1). If the degree of difficulty is negative, the dual constraints could be inconsistent; hence in this case the dual problem presents a system of linear equations, and the number of these linear equations is greater than the number of dual variables. So the dual problem possesses no feasible solution. In general there is no solution vector for the dual variables in this case; therefore, we have to add α constant as follow:

$\min \ w_1 x_1 x_5 + w_2 x_1^{-1} x_3^2 x_4^4 + \alpha$

s.t.

$5x_1^{-1}x_2 \leq 1$

$2.5x_2^{-1}x_3^2 + 1.5x_3^{-1}x_4^{-0.5}x_5^{-0.5} \leq 1$

$x_i \geq 0 \, , \quad i = 1,2, \dots, 5.$

Where $w_1 + w_2 = 1 \, , w_1, w_2 > 0$

Now the problem is of zero degree of difficulty and is solved via dual programming due to Duffin [1].

$$v(\delta) = \left(\frac{w_1}{\delta_1}\right)^{\delta_1} * \left(\frac{w_2}{\delta_2}\right)^{\delta_2} * \left(\frac{1}{\delta_3}\right)^{\delta_3} * (5)^{\delta_4} * \left(\frac{2.5(\delta_5+\delta_6)}{\delta_5}\right)^{\delta_5} * \left(\frac{1.5(\delta_5+\delta_6)}{\delta_6}\right)^{\delta_6}$$

s.t.

$$\delta_1 + \delta_2 + \delta_3 = 1$$
$$\delta_1 - \delta_2 - \delta_4 = 0$$
$$\delta_4 - \delta_5 = 0$$
$$2\delta_2 + 2\delta_5 - \delta_6 = 0$$
$$4\delta_2 - 0.5\delta_6 = 0$$
$$\delta_1 - 0.5\delta_6 = 0$$
$$w_1 + w_2 = 1$$

The system of the above linear equations that form the feasible region for this new dual problem is now consistent and it is unique solution is:-

$\delta 1 = 8.88178419700125232 \times 10^{-16}$

$\delta 2 = 4.44089209850062616 \times 10^{-16}$

$\delta 3 = 0.9999999999999985$

$\delta 4 = 1.11022302462515654 \times 10^{-16}$

$\delta 5 = 1.11022302462515654 \times 10^{-16}$

$\delta 6 = 1.55431223447521916 \times 10^{-15}$

This indicates that the (first two terms of the objective function) and constraints terms converges to zero as $(x_1, x_2, x_3, x_4, x_5)$ approach to the infimum. Let $(\theta_{01}, \theta_{02}, \theta_{11}, \theta_{21}, \theta_{22})$ be five parameters converging to zero, corresponding to these five terms, we have used the primal-dual relationship (in log-linear form) to determine a path by which $(x_1, x_2, x_3, x_4, x_5)$ converges to the infimum:

Let

$$w_1 x_1 x_5 = \theta_{01} \quad \ldots\ldots (9)$$

$$w_2 x_1^{-1} x_3^2 x_4^4 = \theta_{02} \quad \ldots\ldots (10)$$

$$5 x_1^{-1} x_2 = e^{\theta_{11}} \quad \ldots\ldots (11)$$

$$2.5 x_2^{-1} x_3^2 = \frac{\theta_{21}}{\theta_{21} + \theta_{22}} \quad \ldots\ldots (12)$$

$$1.5 x_3^{-1} x_4^{-0.5} x_5^{-0.5} = \frac{\theta_{22}}{\theta_{21} + \theta_{22}} \quad \ldots\ldots (13)$$

The optimal solution to the MOGPP is the vector $x^* = (x_1, x_2, x_3, x_4, x_5)$ for which the constraint inequalities $g_j(x) \leq 1, j = 1,2 \; and \; x > 0$, become exact equalities. Therefore we suppose the first constraints $5x_1^{-1}x_2 = e^{\theta_{11}}$ where $\theta_{11} \to 0$.

Here

$$\ln 5x_1^{-1}x_2 = \ln e^{\theta_{11}} \to \ln 5x_1^{-1}x_2 = \theta_{11} \to e^{\ln 5x_1^{-1}x_2} = e^{\theta_{11}}$$

Now since

$$\theta_{11} \to 0, \quad \to \; 5x_1^{-1}x_2 = 1$$

at optimality, then each path for the parameters:

$$\theta = (\theta_{01}, \theta_{02}, \theta_{11}, \theta_{21}, \theta_{22}) \to 0,$$

This will define a path for $(x_1, x_2, x_3, x_4, x_5)$.

In particular, if $\theta_{01} = \theta_{02} = \theta_{11} = \theta_{21} = \theta_{22} \to 0$ and by solving equations (9)-(13), we will get:

$$x_1 = \frac{3.556893304\, w_1^{2/3}}{\theta_{01}^{5/6}\ e^{\frac{1}{6}\theta_{01}}}$$

$$x_2 = \frac{3.556893304\, w_1^{2/3}\ e^{\frac{5}{6}\theta_{01}}}{5\, \theta_{01}^{5/3}}$$

$$x_3 = 0.5034943245\, w_1^{1/3}\ e^{\frac{5}{12}\theta_{01}}$$

$$x_4 = \frac{2.236067977\, \theta_{01}}{w_2^{0.25}\ e^{\theta_{01}}}$$

$$x_5 = \frac{3.556893304\, \theta_{01}^{5/3}}{w_1^{1/3}\ e^{\frac{1}{6}\theta_{01}}}$$

And hence the infimum is approached as $(x_1\ \&\ x_2 \to \infty, x_4\ \&\ x_5 \to 0)$.

By considering different values of w_1, w_2 the primal variables, corresponding minimum value of the objective function are given in table (3.1).

Table (3.1)

The Compromise solution of Example (3.1)

w_1	w_2	**Obj. fun.**	x_1	x_2	x_3	x_4	x_5
0.1	0.9	3.6070	4.8885 $*10^9$	9.7771* 10^8	8.844* 10^3	3.0228	4.919* 10^{-9}
0.2	0.8	5.5053	4.0343 $*10^9$	8.0685* 10^8	8.0342* 10^3	3.4603	4.5488 $*10^{-9}$
0.3	0.7	6.8999	3.5506 $*10^9$	7.1012* 10^8	7.5372* 10^3	3.7855	4.3185 $*10^{-9}$
0.4	0.6	7.9400	9.5037	1.9007*	3.8995*	4.0748	1.3924

			$*10^{12}$	10^{12}	10^{5}		$*10^{-12}$
0.5	0.5	8.6703	2.2768 $*10^{9}$	4.5536* 10^{8}	6.0356* 10^{3}	4.3597	5.0775 $*10^{-9}$
0.6	0.4	9.0890	2.4603 $*10^{9}$	4.9205* 10^{8}	6.2741* 10^{3}	4.6645	4.1048 $*10^{-9}$
0.7	0.3	9.1517	2.3759 $*10^{9}$	4.7518* 10^{8}	6.1656* 10^{3}	5.0209	3.6685 $*10^{-9}$
0.8	0.2	8.7391	2.0907 $*10^{9}$	4.1814* 10^{8}	5.7837* 10^{3}	5.4929	3.4833 $*10^{-9}$
0.9	0.1	7.5028	4.5836 $*10^{8}$	9.1672* 10^{7}	2.7081* 10^{3}	6.2878	1.2125 $*10^{-9}$

Example (3.2):

$$\min f_1(x) = 30x_1x_2$$

$$\min f_2(x) = 18x_1x_3^{-2}x_4^{0.5}$$

s.t.

$$\frac{5}{3}x_1^{-1}x_2x_4^{-2} \le 1$$

$$0.5x_2^{-1}x_3x_4^{-0.5} \le 1$$

$$x_i \ge 0 \; where \; i = 1,2,\ldots,4.$$

$$\min \quad 30w_1x_1x_2 + 18w_2x_1x_3^{-2}x_4^{0.5}$$

s.t.

$$\frac{5}{3}x_1^{-1}x_2x_4^{-2} \le 1$$

$$0.5x_2^{-1}x_3x_4^{-0.5} \le 1$$

$w_1 + w_2 = 1\ , w_1, w_2 > 0$
$x_i \geq 0\ where\ i = 1,2, \ldots ,4.$

Similar to example (3.1), adding the parameter α to the above objective function to gain zero degree of difficulty. By solving this example, note that at $w_1 = 0.1\ \&\ w_2 = 0.9$, we will get the following compromise optimal solution:-

Objective function = 1.000000000544320e-018

solution =

'x1' [7.251390291090811e-015]

'x2' [6.928628846395429e-009]

'x3' [5.281837566041635e+025]

'x4' [7.148042031738008e+083]

while at $(w_1, w_2) = \{(0.2\ ,0.8), (0.3\ ,0.7), \ldots \ldots, (0.9\ ,0.1)\}$, we did not find large difference in the value of the objective function.

4. Conclusions:

Very few articles have been considered with negative degree of difficulties, with more linear equality constraints than the dual variables. Depending upon the rank of the dual linear equality constraints, the dual solution may be "over determined" i.e. the dual problem may be infeasible. There are several attempts to solve like problems. Ojha [5] tried to give a compromise optimal solution for multi-objective GPP having positive degree of difficulties. However, we have not find yet any numerical approach which deals with solving multi-objective GPP with negative degree of difficulties. In this paper, we have presented a new technique to solve multi-objective GPP with a negative degree of difficulty; namely (-1). Also, we have proposed special numerical examples which are suitable for designing an engineering problems.

5. Open Problems:

- This paper deals with solving multi-objective GPP with a negative degree of difficulty; namely (-1). As a first open problem, we may generalize this type of GPP problems by solving them for degree of difficulties starting from (-2), (-3), (-4) and so on.

- The general solutions of this paper are deals with compromise solutions. We may try to search further to get the optimal solutions.

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