

## **The Construction of Complete (k,4)-Arcs in PG(2,8) on projective plane over Galois Field GF (8)**

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### **Abstract**

In this work, we construct by an algebraic method the projectively distinct (k,4)-arcs in PG(2,8), where  $k \geq 5$  and we prove that the complete (k,4)-arcs do not exist, when  $5 \leq k \leq 18$ . Also we find that complete (k,4)-arc in PG(2,8) is the (20,4)-arc and complete (k,3)-arc and PG(2,8) is the (19,4)-arc.

### **المستخلص**

في هذا البحث ، نبني بطريقة جبرية الأقواس (k,4)- المختلفة اسقاطيا في المستوي الاسقاطي PG(2,8) ، حيث  $k \geq 5$  ونبرهن إن الأقواس (k,4)- التامة غير موجودة عندما  $5 \leq k \leq 18$  . وجدنا قوس (k,4)- تام في PG(2,8) هو القوس (20,4)- وكذلك القوس (k,4)- تام في PG(2,8) هو القوس (19,4)-.

### **1. Projective geometry [2]**

The history of projective geometry is a remarkable instance of art and science feeding off one another. Based on the optics studies of the Arabic mathematician Alhazen (Ibu Ali al-Hasan ibn al-Haytham) (965–1040), several early Renaissance artists attempted to develop a style of visual depiction that presented the eye with a truer semblance of three dimensional space than did earlier, flatter styles. The discovery of the principle of linear perspective (the idea that all parallel lines appear to converge at a single point) is credited to Filippo Brunelleschi (1377–1446). This led to a flurry of activity, culminating in the work of Girard Desargues (1591–1661), which introduced projective geometry as we now it.

### **2. Introduction [2,5]**

The projective plane consists of the standard Euclidean plane, together with a set of points called points at infinity, one for each collection of parallel lines. We say that a line passes through the point at infinity corresponding to its direction (and no others), and that all of the points at infinity lie on a line at infinity. Note that three parallel lines now indeed have a common point at infinity, which retroactively justifies our calling such lines “concurrent”.

1. A point is that of which there is no part.
2. And a line is a length without breadth.
3. And the extremities of a line are points.
4. A straight-line is whatever lies evenly with points upon itself.
5. And a surface is that which has length and breadth alone.
6. And the extremities of a surface are lines.
7. A plane surface is whatever lies evenly with straight-lines upon itself.
8. And a plane angle is the inclination of the lines, when two lines in a plane meet one another, and are not laid down straight-on with respect to one another.
9. And when the lines containing the angle are straight then the angle is called rectilinear.
10. And when a straight-line stood upon (another) straight-line makes adjacent angles (which are) equal to one another, each of the equal angles is a right-angle, and the former straight-line is called perpendicular to that. upon which it stands.

A projective plane PG(2,q) over GF(q) is a two – dimensional projective space , which consists of points and lines with incidence relation between them.

PG(2,q) satisfying the following axioms :

- A . Any two distinct points are contained in a unique line.
- B . Any two distinct lines are intersected in a unique point.
- C . There exist at least four points such that no three of them are collinear.

Also , in  $PG(2,q)$  :There are

There are  $1+q+q^2$  points.

There are  $1+q+q^2$  lines.

There are  $1+q$  points on every line.

There are  $1+q$  lines through every point.

Finally , the points of  $PG(2,q)$  can be numerated as follows:

The number of the point  $(1,0,0)$  is 1, point  $(x,1,0)$  is numerated as  $x+2$ , the point  $(x,y,1)$  is numbered as  $x+(y.q) + q + 2$ .

### **3. Basic Definitions and Theorems**

#### **Definition 3. 1:-** [2]

A  $(k,n) - \text{arc } K$  in  $PG(2,q)$  is a set of  $k$  points , such that same  $n$ , but no  $n+1$  of which are collinear. A  $(k,4) - \text{arc}$  is a set of  $k$  points no five of them are collinear.

#### **Definition 3. 2:-** [2]

A  $(k,n) - \text{arc } K$  is complete if it is not contained in a  $(k+1,n) - \text{arc}$  .

#### **Theorem 3. 3:-** [2]

Let  $r_i$  be the total number of  $i - \text{secants}$  of a  $(k,n) - \text{arc } K$  in  $PG(2,q)$  , then the following equations are hold :-

$$\sum_{i=0}^n r_i = q^2 + q + 1 \quad \dots\dots\dots (1)$$

$$\sum_{i=1}^n i r_i = k(q + 1) \quad \dots\dots\dots (2)$$

$$\sum_{i=2}^n i(i - 1) r_i = k(k - 1) \quad \dots\dots\dots (3)$$

#### **Definition 3.4:-** [7]

The rays  $X=(x_1,x_2,x_3)$  and  $X = \lambda (x_1,x_2,x_3)$  are the same and are mapped to the same point  $m$  of the plane  $P$ ,  $X$  is the coordinate vector of  $m$ ,  $(x_1,x_2,x_3)$  are its homogeneous coordinates

#### **Definition 3.5:-** [6]

The matrix  $H$  can be multiplied by an arbitrary non-zero number without altering the projective transformation  
 $n$  Matrix  $H$  is called a ‘‘homogeneous matrix’’ (only ratios of terms are important)

**Theorem 3. 6:- [7]**

A mapping is a projectively if and only if the mapping consists of a linear transformation of homogeneous coordinates  $x'=Hx$  with  $H$  non singular

**The addition's and Multiplication's Operations Of GF(8).**

To find the addition and multiplication tables in PG(8) , we have the order triples  $(x_1,x_2,x_3)$  such that  $x_1,x_2,x_3$  in GF(2) , as follows :

$$\begin{aligned} 0 &\equiv (0,0,0) & , & & 1 &\equiv (1,0,0) & , & & 2 &\equiv (0,1,0) & , & & 3 &\equiv (1,1,0), \\ 4 &\equiv (0,0,1) & , & & 5 &\equiv (1,0,1) & , & & 6 &\equiv (0,1,1) & , & & 7 &\equiv (1,1,1). \end{aligned}$$

Put these points in one orbit  $(1,0,0)$  at the first point and by the principle  $(1,0,0) A_i$  ,  $i=$

$$0,1,2,\dots,7 \text{ and } A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$(1,0,0) \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{array}{l} (0,1,0) \\ (0,0,1) \\ (1,0,1) \\ (1,1,1) \\ (1,1,0) \\ (0,1,1) \end{array}$$

Now, in the left of this table the operation of multiplication  $m$ , and in the right the operation of addition  $n$ , in multiplication side we write the numeration of points as last and the addition side takes the normal sequence.

M(*)		(+)n = f(m)
1	(1,0,0)	0
2	(0,1,0)	1
3	(1,1,0)	5
4	(0,0,1)	2
5	(1,0,1)	3
6	(0,1,1)	6
7	(1,1,1)	4
Mod 7		

In the addition's table we have the following relation :

$$(x_1,x_2,x_3)+(y_1,y_2,y_3) = (z_1,z_2,z_3) \text{ where } z_i \equiv (y_i+x_i) \pmod{2}, \text{ for } i = 1,2,3$$

In multiplication table we have the following relation :

$$m_1 * m_2 = m_3 \Leftrightarrow ((1,0,0)A^{f(m_1)} A^{f(m_2)}) =$$

$$m_1 * m_2 = m_3 \Leftrightarrow ((1,0) A^{f(m_1)}) A^{f(m_2)} = (1,0,0) A^{f(m_1) + f(m_2)} \pmod{7} \\ = (x_1,x_2,x_3)$$

For example :  $3 * 7 = 4 \Leftrightarrow ((1,0,0)A^5)A^4 = (1,0,0) A^2 = (0,0,1)$

Where  $(0,0,1)$  equal to 4 in multiplication side.

Now, we have the addition's and multiplication's tables

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	1	2	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

*	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	2	4	6	5	7	1	3
3	3	6	5	1	2	7	4
4	4	5	1	7	3	2	6
5	5	7	2	3	6	4	1
6	6	1	7	2	4	3	5
7	7	3	4	6	1	5	2

**Definition 3.7:-** [1,3,4]

Let  $Q_1, Q_2$  in  $PG(2,q) \setminus K$  and let  $K_1 = K \cup \{Q_1\}$ ,  $K_2 = K \cup \{Q_2\}$ , then  $Q_1$  and  $Q_2$  are in the same set if and only if  $K_1$  and  $K_2$  are projectively equivalent under type of lines. The projective plane  $PG(2,8)$  contains 73 points, 73 lines, 9 points on every line and 9 lines through every point. Any line in  $PG(2,8)$

Table: Points and lines of  $PG(2,8)$

I	$P_i$	$L_i$								
1	1 0 0	2	10	18	26	34	42	50	58	66
2	0 1 0	1	10	11	12	13	14	15	16	17
3	1 1 0	3	10	19	28	37	46	55	64	73
4	2 1 0	8	10	24	27	41	44	54	61	71
5	3 1 0	6	10	22	31	35	49	53	60	72
6	4 1 0	5	10	21	32	39	43	52	65	70
7	5 1 0	9	10	25	29	38	48	51	63	68
8	6 1 0	4	10	20	30	40	47	57	59	69
9	7 1 0	7	10	23	33	36	45	56	62	67
10	0 0 1	1	2	3	4	5	6	7	8	9
11	1 0 1	2	11	19	27	35	43	51	59	67
12	2 0 1	2	16	24	32	40	48	56	64	72
13	3 0 1	2	14	22	30	38	46	54	62	70
14	4 0 1	2	13	21	29	37	45	53	61	69
15	5 0 1	2	17	25	33	41	49	57	65	73
16	6 0 1	2	12	20	28	36	44	52	60	68
17	7 0 1	2	15	23	31	39	47	55	63	71
18	0 1 1	1	18	19	20	21	22	23	24	25

I	P <sub>i</sub>	L <sub>i</sub>									
19	1 1 1	3	11	18	29	36	47	54	65	72	
20	2 1 1	8	16	18	33	35	46	52	63	69	
21	3 1 1	6	14	18	27	39	45	57	64	68	
22	4 1 1	5	13	18	31	40	44	51	62	73	
23	5 1 1	9	17	18	30	37	43	56	60	71	
24	6 1 1	4	12	18	32	38	49	55	61	67	
25	7 1 1	7	15	18	28	41	48	53	59	70	
26	0 2 1	1	58	59	60	61	62	63	64	65	
27	1 2 1	4	11	21	31	41	46	56	58	68	
28	2 2 1	3	16	25	30	39	44	53	58	67	
29	3 2 1	7	14	19	29	40	49	52	58	71	
30	4 2 1	8	13	23	28	38	43	57	58	72	
31	5 2 1	5	17	22	27	36	48	55	58	69	
32	6 2 1	6	12	24	35	37	47	51	58	70	
33	7 2 1	9	15	20	32	33	45	54	58	73	
34	0 3 1	1	42	43	44	45	46	47	48	49	
35	1 3 1	5	11	20	33	38	42	53	64	71	
36	2 3 1	9	16	19	31	36	42	57	61	70	
37	3 3 1	3	14	23	32	41	42	51	60	69	
38	4 3 1	7	13	24	30	35	42	55	65	68	
39	5 3 1	6	17	21	28	40	42	54	63	67	
40	6 3 1	8	12	22	29	39	42	56	59	73	
41	7 3 1	4	15	25	27	37	42	52	62	72	
42	0 4 1	1	34	35	36	37	38	39	40	41	
43	1 4 1	6	11	23	30	34	48	52	61	73	
44	2 4 1	4	16	22	28	34	45	51	65	71	
45	3 4 1	9	14	21	33	34	44	55	59	72	
46	4 4 1	3	13	20	27	34	49	56	63	70	
47	5 4 1	8	17	19	32	34	47	53	62	68	
48	6 4 1	7	12	25	31	34	43	54	64	69	
49	7 4 1	5	15	24	29	34	46	57	60	67	

i	P <sub>i</sub>	L <sub>i</sub>
50	0 5 1	1 66 67 68 69 70 71 72 73
51	1 5 1	7 11 22 32 37 44 57 63 66
52	2 5 1	6 16 20 29 41 43 55 62 66
53	3 5 1	5 14 25 28 35 47 56 61 66
54	4 5 1	4 13 19 33 39 48 54 60 66
55	5 5 1	3 17 24 31 38 45 52 59 66
56	6 5 1	9 12 23 27 40 46 53 65 66
57	7 5 1	8 15 21 30 36 49 51 64 66
58	0 6 1	1 26 27 28 29 30 31 32 33
59	1 6 1	8 11 25 26 40 45 55 60 70
60	2 6 1	5 16 23 26 37 49 54 59 68
61	3 6 1	4 14 24 26 36 43 53 63 73
62	4 6 1	9 13 22 26 41 47 52 64 67
63	5 6 1	7 17 20 26 39 46 51 61 72
64	6 6 1	3 12 21 26 35 48 57 62 71
65	7 6 1	6 15 19 26 38 44 56 65 69
66	0 7 1	1 50 51 52 53 54 55 56 57
67	1 7 1	9 11 24 28 39 49 50 62 69
68	2 7 1	7 16 21 27 38 47 50 60 73
69	3 7 1	8 14 20 31 37 48 50 65 67
70	4 7 1	6 13 25 32 36 46 50 59 71
71	5 7 1	4 17 23 29 35 44 50 64 70
72	6 7 1	5 12 19 30 41 45 50 63 72
73	7 7 1	3 15 22 33 40 43 50 61 68

**The Construction of the projectively Distinct (k,4) – Arcs in PG(2,8).**

Let  $A = \{1,2,3,10,19\}$  be the set of the reference and the unite points in  $PG(2,8)$  , no three of them are collinear.

The distinct (k,4) – arcs can be constructed by adding to A in each time one point from the remaining 68 points of the projective plane  $PG(2,8)$  as follows :

$$A_1 = A \cup \{4\} , A_2 = A \cup \{5\}, \dots, A_{68} = A \cup \{73\}.$$

						19	10	3	2	1					
Incomplete						19	10	4	3	2	1				
Incomplete						19	10	5	3	2	1				
Incomplete						19	10	5	4	2	1				
Incomplete						19	11	10	4	3	2	1			
Incomplete						19	12	10	4	3	2	1			
Incomplete						28	19	10	4	3	2	1			
Incomplete						29	19	10	4	3	2	1			
Incomplete						19	11	10	5	3	2	1			
Incomplete						29	19	10	5	4	2	1			
Incomplete						19	12	11	10	4	3	2	1		
Incomplete						19	18	11	10	4	3	2	1		
Incomplete						20	19	11	10	4	3	2	1		
Incomplete						22	19	11	10	4	3	2	1		
Incomplete						32	19	11	10	4	3	2	1		
Incomplete						29	19	12	10	4	3	2	1		
Incomplete						28	19	11	10	4	3	2	1		
Incomplete						28	19	14	10	4	3	2	1		
Incomplete						28	20	19	10	4	3	2	1		
Incomplete						38	28	19	10	4	3	2	1		
Incomplete						41	29	19	10	4	3	2	1		
Incomplete						19	12	11	10	5	3	2	1		
Incomplete						36	29	19	10	5	4	2	1		
Incomplete						19	18	12	11	10	4	3	2	1	
Incomplete						30	19	18	11	10	4	3	2	1	
Incomplete						44	19	18	11	10	4	3	2	1	
Incomplete						32	20	19	11	10	4	3	2	1	
Incomplete						52	22	19	11	10	4	3	2	1	
Incomplete						45	32	19	11	10	4	3	2	1	
Incomplete						28	19	18	11	10	4	3	2	1	
Incomplete						28	21	19	11	10	4	3	2	1	
Incomplete						28	23	19	11	10	4	3	2	1	
Incomplete						38	28	19	11	10	4	3	2	1	
Incomplete						72	28	19	14	10	4	3	2	1	
Incomplete						28	20	19	13	10	4	3	2	1	
Incomplete						53	38	28	19	10	4	3	2	1	
Incomplete						41	29	19	15	10	4	3	2	1	
Incomplete						19	18	12	11	10	5	3	2	1	
Incomplete						49	36	29	19	10	5	4	2	1	
Incomplete						26	19	18	12	11	10	4	3	2	1
Incomplete						31	30	19	18	11	10	4	3	2	1
Incomplete						44	30	19	18	11	10	4	3	2	1
Incomplete						44	33	19	18	11	10	4	3	2	1
Incomplete						45	32	20	19	11	10	4	3	2	1
Incomplete						63	52	22	19	11	10	4	3	2	1
Incomplete						71	45	32	19	11	10	4	3	2	1
Incomplete						30	28	19	18	11	10	4	3	2	1
Incomplete						38	28	19	18	11	10	4	3	2	1
Incomplete						63	28	19	18	11	10	4	3	2	1

Incomplete		38	28	21	19	11	10	4	3	2	1		
Incomplete		53	28	23	19	11	10	4	3	2	1		
Incomplete		72	39	28	19	14	10	4	3	2	1		
Incomplete		28	26	20	19	13	10	4	3	2	1		
Incomplete		20	19	18	12	11	10	5	3	2	1		
Incomplete		56	49	36	29	19	10	5	4	2	1		
Incomplete	27	26	19	18	12	11	10	4	3	2	1		
Incomplete	38	31	30	19	18	11	10	4	3	2	1		
Incomplete	44	31	30	19	18	11	10	4	3	2	1		
Incomplete	44	33	30	19	18	11	10	4	3	2	1		
Incomplete	45	44	33	19	18	11	10	4	3	2	1		
Incomplete	70	44	33	19	18	11	10	4	3	2	1		
Incomplete	63	52	42	22	19	11	10	4	3	2	1		
Incomplete	38	30	28	19	18	11	10	4	3	2	1		
Incomplete	39	30	28	19	18	11	10	4	3	2	1		
Incomplete	63	30	28	19	18	11	10	4	3	2	1		
Incomplete	63	38	28	19	18	11	10	4	3	2	1		
Incomplete	44	38	28	21	19	11	10	4	3	2	1		
Incomplete	53	38	28	23	19	11	10	4	3	2	1		
Incomplete	72	56	39	28	19	14	10	4	3	2	1		
Incomplete	43	28	26	20	19	13	10	4	3	2	1		
Incomplete	26	20	19	18	12	11	10	5	3	2	1		
Incomplete		56	49	36	29	19	11	10	5	4	2	1	
Incomplete	28	27	26	19	18	12	11	10	4	3	2	1	
Incomplete	39	38	31	30	19	18	11	10	4	3	2	1	
Incomplete	44	38	31	30	19	18	11	10	4	3	2	1	
Incomplete	45	44	31	30	19	18	11	10	4	3	2	1	
Incomplete	45	44	33	30	19	18	11	10	4	3	2	1	
Incomplete	70	44	33	30	19	18	11	10	4	3	2	1	
Incomplete	70	45	44	33	19	18	11	10	4	3	2	1	
Incomplete	69	63	52	42	22	19	11	10	4	3	2	1	
Incomplete	39	38	30	28	19	18	11	10	4	3	2	1	
Incomplete	44	38	30	28	19	18	11	10	4	3	2	1	
Incomplete	63	38	30	28	19	18	11	10	4	3	2	1	
Incomplete	41	39	30	28	19	18	11	10	4	3	2	1	
Incomplete	63	40	38	28	19	18	11	10	4	3	2	1	
Incomplete	44	40	38	28	21	19	11	10	4	3	2	1	
Incomplete	53	40	38	28	23	19	11	10	4	3	2	1	
Incomplete	27	26	20	19	18	12	11	10	5	3	2	1	
Incomplete	36	28	27	26	19	18	12	11	10	4	3	2	1
Incomplete	48	44	38	31	30	19	18	11	10	4	3	2	1
Incomplete	70	45	44	31	30	19	18	11	10	4	3	2	1
Incomplete	52	45	44	33	30	19	18	11	10	4	3	2	1
Incomplete	70	45	44	33	30	19	18	11	10	4	3	2	1
Incomplete	70	45	44	33	32	19	18	11	10	4	3	2	1
Incomplete	45	39	38	30	28	19	18	11	10	4	3	2	1
Incomplete	56	39	38	30	28	19	18	11	10	4	3	2	1
Incomplete	63	44	38	30	28	19	18	11	10	4	3	2	1
Incomplete	71	41	39	30	28	19	18	11	10	4	3	2	1
Incomplete	63	44	40	38	28	19	18	11	10	4	3	2	1



Incomplete						63	53	40	38	28	19	18	11	10	4	3	2	1							
Incomplete						53	44	40	38	28	21	19	11	10	4	3	2	1							
Incomplete						28	27	26	20	19	18	12	11	10	5	3	2	1							
Incomplete						38	36	28	27	26	19	18	12	11	10	4	3	2	1						
Incomplete						71	48	44	38	31	30	19	18	11	10	4	3	2	1						
Incomplete						71	70	45	44	31	30	19	18	11	10	4	3	2	1						
Incomplete						70	52	45	44	33	30	19	18	11	10	4	3	2	1						
Incomplete						71	52	45	44	33	30	19	18	11	10	4	3	2	1						
Incomplete						70	53	45	44	33	32	19	18	11	10	4	3	2	1						
Incomplete						71	70	45	44	33	32	19	18	11	10	4	3	2	1						
Incomplete						56	45	39	38	30	28	19	18	11	10	4	3	2	1						
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Incomplete						71	70	52	45	44	33	30	19	18	11	10	4	3	2	1					
Incomplete						63	56	52	39	38	30	28	19	18	11	10	4	3	2	1					
Incomplete						63	45	44	40	38	33	28	19	18	11	10	4	3	2	1					
Incomplete						70	63	44	40	38	33	28	19	18	11	10	4	3	2	1					
Incomplete						66	53	49	44	40	38	28	21	19	11	10	4	3	2	1					
Incomplete						40	39	38	28	27	26	20	19	18	12	11	10	5	3	2	1				
Incomplete						45	39	38	36	28	27	26	19	18	12	11	10	4	3	2	1				
Incomplete						70	66	53	49	44	40	38	28	21	19	11	10	4	3	2	1				
Incomplete						45	40	39	38	28	27	26	20	19	18	12	11	10	5	3	2	1			
Incomplete						53	45	39	38	36	28	27	26	19	18	12	11	10	4	3	2	1			
Incomplete						47	45	40	39	38	28	27	26	20	19	18	12	11	10	5	3	2	1		
Incomplete						56	53	45	39	38	36	28	27	26	19	18	12	11	10	4	3	2	1		
Incomplete						56	47	45	40	39	38	28	27	26	20	19	18	12	11	10	5	3	2	1	
Complete						63	56	53	45	39	38	36	28	27	26	19	18	12	11	10	4	3	2	1	
Complete						62	56	47	45	40	39	38	28	27	26	20	19	18	12	11	10	5	3	2	1

## **Results**

From the above results , the number of the distinct  $(k,4)$ -arcs ,  $k = 5,6,7,8,9,10,11, \dots, 18$  are All of these arcs are incomplete . Finally , we obtain two complete  $(19,4)$  – arcs and  $(20,4)$  – arcs.

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